

5

Weak Riemannian Metrics with Applications in Shape Analysis

In this chapter, we study in detail the (weak) L^2 -metric on spaces of smooth mappings. Its importance stems from the fact that this metric and its siblings, the Sobolev H^s -metrics, are prevalent in shape analysis. It will be essential for us that geodesics with respect to the L^2 -metric can be explicitly computed. Before we look into the specifics, let us clarify what we mean here by shape and shape analysis. Shape analysis seeks to classify, compare and analyse shapes. As a mathematical discipline, shape analysis goes back to the classical works by D'Arcy Thompson (1942) (originally published in 1917). In recent years there has been an explosion of applications in shape analysis to diverse areas such as computer vision (Celledoni et al., 2016), medical imaging, registration of radar images and many more (see Bauer et al., 2014, for an exposition). There are different mathematical settings as to what is meant by a shape and what kind of data describes it. Popular choices are, for example,

- points;
- curves (or surfaces) in Euclidean space of manifolds;
- level sets of functions; and
- images.

Another typical feature in (geometric) shape analysis is that one wants to remove superfluous information from the data. For example, in the comparison of shapes, rotations, translations, scalings and reflections are typically disregarded as being inessential differences. Conveniently, these inessential differences can mostly be described by actions of suitable Lie groups (such as the rotation and the diffeomorphism groups). This hints at the general process of constructing an (infinite-dimensional) manifold of shapes: One starts out with an (infinite-dimensional) manifold of data (e.g. smooth curves) called the pre-shape space. Then the undesirable information is removed by quotienting out

suitable group actions (e.g. if reparametrisation invariance of the shapes is desired, quotient out a suitable diffeomorphism group). The resulting quotient is then called shape space and one seeks to construct suitable tools (such as a Riemannian metric) to compare, classify and analyse the objects in shape space.

In the present chapter we will restrict our attention to shapes which arise as images of smooth curves which take their values in \mathbb{R}^2 . The pre-shape space will thus be the infinite-dimensional manifold of smooth immersions (from the circle) with values in the two-dimensional space. As the objective is to compare images of these curves, we need to remove the specific parametrisation of the immersion. Hence we pass to shape space by quotienting out an action of the diffeomorphism group on the immersions (modelling the reparametrisation). Our aim is then to construct a suitable Riemannian metric on shape space which will allow us to compare shapes using the geodesic distance induced by it.

5.1 The L^2 -metric and Its Cousins

Having now discussed ideas to remedy the problems in the (in general) ill-behaved weak Riemannian setting, we now consider several weak Riemannian metrics which admit metric derivatives, spray and so on. The metrics which we will consider are on one hand the L^2 -metric. We will see that the L^2 -metric admits a spray, a connector and a covariant derivative albeit being only a weak Riemannian metric. This metric will play a decisive role in the investigation of shape analysis in this chapter. The construction for the L^2 -metric follows the argument first presented in Bruveris (2018). Finally, we will briefly describe a Sobolev-type metric whose covariant derivative can explicitly be given and is of independent interest. Before we begin, let us set some conventions concerning the integration of functions on manifolds.

On Integration of Functions on \mathbb{S}^1 The unit circle \mathbb{S}^1 can be parametrised (up to the double endpoint) by $\theta: [0, 2\pi] \rightarrow \mathbb{S}^1, t \mapsto (\cos(t), \sin(t))$. We now abuse notation and denote by θ both the parameter and the parametrisation. If you have not seen integration theory on submanifolds of \mathbb{R}^d (e.g. Lang, 1999, XVI), this implies that for a continuous $f: \mathbb{S}^1 \rightarrow \mathbb{R}^d$, the integral on \mathbb{S}^1 satisfies

$$\int_{\mathbb{S}^1} f(\theta) d\theta = \int_0^{2\pi} f(\cos(t), \sin(t)) dt,$$

whence it can be computed as a usual one-dimensional integral. Further, for differentiable maps $c: \mathbb{S}^1 \rightarrow \mathbb{R}$ we write $c'(\theta) := T_{\theta}c(1)$, where $1 \in T_{\theta}\mathbb{S}^1 \cong \mathbb{R}$

(the notation was previously reserved for curves and we justify it as $T_{\theta(t)}c(1)$ is (up to the linear isomorphism $T_t\theta$) given by the curve differential $(c \circ \theta)'$).

5.1 Remark None of the techniques employed in this chapter depend on the compact manifold \mathbb{S}^1 , but we can thus skip a discussion of integration against a volume form on a compact manifold. However, we remark that all of the results in this section carry over if we replace \mathbb{S}^1 by an arbitrary compact manifold (we invite the reader to check this for themselves).

As in Example 4.3 consider the space $C^\infty(\mathbb{S}^1, M)$ for a strong Riemannian manifold (M, g) with the L^2 -metric

$$g_c^{L^2}(f, g) = \int_{\mathbb{S}^1} g_{c(\theta)}(f(\theta), g(\theta))d\theta, \quad f, g \in T_c C^\infty(\mathbb{S}^1, M). \quad (5.1)$$

We shall show that this weak Riemannian metric admits a metric spray and a metric derivative. It will turn out that all the relevant objects can be lifted from the target manifold to the manifold of mappings.

5.2 Let (M, g) be a strong Riemannian manifold with metric spray S , connector K and metric derivative ∇ . Since (M, g) is a strong Riemannian manifold, it admits a local addition (Michor, 1980, Lemma 10.2), whence the manifold structure on $C^\infty(\mathbb{S}^1, M)$ is canonical and, in addition, $TC^\infty(\mathbb{S}^1, M) \cong C^\infty(\mathbb{S}^1, TM)$ and $T^2C^\infty(\mathbb{S}^1, M) \cong C^\infty(\mathbb{S}^1, T^2M)$ (see Appendix C). Then the pushforward of the spray and the connector

$$\begin{aligned} K_* : C^\infty(\mathbb{S}^1, T^2M) &\rightarrow C^\infty(\mathbb{S}^1, TM), & q &\mapsto K \circ q, \\ S_* : C^\infty(\mathbb{S}^1, TM) &\rightarrow C^\infty(\mathbb{S}^1, T^2M), & v &\mapsto S \circ v \end{aligned}$$

are smooth mappings by Corollary 2.19. Moreover, the identification of the tangent bundles immediately shows that S_* is a spray on $C^\infty(\mathbb{S}^1, M)$ and that K_* is a connector.

5.3 Proposition *In the situation of 5.2, the connector associated to S_* is K_* .*

Proof By definition, the connector of S_* is uniquely determined by the associated map B_{S_*} . If we can show that B_{S_*} coincides with the pushforward B_* of the map B associated to S , then the definition of the connectors yields that the connector of S_* is K_* . To compute B_{S_*} we need to compute $d_2^2(S_*)_{O,2}((x, 0); \cdot)$, where $(S_*)_{O,2}$ is a local representative of S_* (see 4.17). Since the bundle trivialisations are complicated for manifolds of mappings, we apply instead the exponential law to see that we can just take partial derivatives of

$$(S_*)^\wedge : C^\infty(\mathbb{S}^1, TM) \times \mathbb{S}^1 \rightarrow T^2M, \quad (h, \theta) \mapsto S(h(\theta)).$$

Observe that we only need to take partial derivatives with respect to the first component; thus \mathbb{S}^1 is simply a parameter set and the formula holds since it can be checked pointwise (see Exercise 5.1.1). \square

We use the connector K of the covariant derivative on (M, g) to define a covariant derivative for vector fields along smooth maps.

5.4 Definition Let N be a smooth manifold and $F: N \rightarrow M$ be a smooth map to a Riemannian manifold (M, g) . A mapping $s \in C_F^\infty(N, TM) = \{s \in C^\infty(N, TM) \mid \pi_M \circ s = F\}$ is called a *vector field along F* . Assume that (M, g) admits a metric spray with associated connector K . For $X \in \mathcal{V}(N)$, we define

$$\nabla_X^g s := K \circ Ts \circ X: N \rightarrow TM. \tag{5.2}$$

Note that the construction in Definition 5.4 is a generalised version of Proposition 4.36. If ∇ is the metric derivative of a Riemannian manifold which admits a metric spray then the *covariant derivative along $F: N \rightarrow M$* also satisfies a version of 4.29 (see e.g. Klingenberg, 1995, Proposition 1.8.14):

$$\begin{aligned} X.g(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\ Y, Z \text{ vector field along } F, X &\in \mathcal{V}(N). \end{aligned} \tag{5.3}$$

We will exploit the exponential law for the canonical manifold $C^\infty(\mathbb{S}^1, M)$. To $s \in C^\infty(N, C^\infty(\mathbb{S}^1, TM))$ we associated the map $s^\wedge \in C^\infty(N \times \mathbb{S}^1, M)$.

5.5 Remark We face a problem as in Example 3.25: The exponential law suggests working with vector fields on $N \times \mathbb{S}^1$, while the covariant derivative in (5.2) is only defined for vector fields on N . Luckily, vector fields on product manifolds are products of vector fields on the parts. Hence we extend $X \in \mathcal{V}(N)$ to a vector field on $N \times \mathbb{S}^1$ by the zero vector field on \mathbb{S}^1 via $X \times \mathbf{0}_{\mathbb{S}^1} \in \mathcal{V}(N \times \mathbb{S}^1)$. This allows us to obtain vector fields on the correct manifold on which we can now extend the covariant derivative of N .

5.6 (A covariant derivative on $C^\infty(\mathbb{S}^1, M)$) Choosing $N = C^\infty(\mathbb{S}^1, M)$ we define via (5.2) a map $\nabla_{X \times \mathbf{0}_{\mathbb{S}^1}}^g s^\wedge \in C^\infty(C^\infty(\mathbb{S}^1, M) \times \mathbb{S}^1, TM)$ and set

$$\nabla_X s := (\nabla_{X \times \mathbf{0}_{\mathbb{S}^1}}^g s^\wedge)^\vee \in C^\infty(C^\infty(\mathbb{S}^1, M), C^\infty(\mathbb{S}^1, TM)). \tag{5.4}$$

Observe that the identification $TC^\infty(\mathbb{S}^1, M) \cong C^\infty(\mathbb{S}^1, TM)$ allows us to identify $s \in \mathcal{V}(C^\infty(\mathbb{S}^1, M)) \subseteq C^\infty(C^\infty(\mathbb{S}^1, M), C^\infty(\mathbb{S}^1, TM))$. Computing with the help of Exercises 5.1.2 and 2.2.2,

$$\begin{aligned} \pi_{C^\infty(\mathbb{S}^1, M)} \circ \nabla_X s &= (\pi_M)_* (\nabla_{X \times \mathbf{0}_{\mathbb{S}^1}}^g s^\wedge)^\vee = (\pi_M \circ (\nabla_{X \times \mathbf{0}_{\mathbb{S}^1}}^g s^\wedge)^\vee) \\ &= (\pi_M \circ K \circ T(s^\wedge) \circ (X \times \mathbf{0}_{\mathbb{S}^1}))^\vee = (\pi_M \circ K \circ (Ts \circ X)^\wedge)^\vee \\ &= (\pi_M \circ K)_*(Ts \circ X) = \text{id}_{C^\infty(\mathbb{S}^1, M)}. \end{aligned}$$

Hence (5.4) induces a bilinear map $\nabla: \mathcal{V}(C^\infty(\mathbb{S}^1, M))^2 \rightarrow \mathcal{V}(C^\infty(\mathbb{S}^1, M))$.

The construction 5.6 certainly looks quite messy; however, we remark that it is indeed a very natural covariant derivative we obtain in this way. Namely, for any smooth curve $c: [a, b] \rightarrow C^\infty(\mathbb{S}^1, M)$ and smooth lift $\alpha \in \text{Lift}(c)$ we apply the exponential law to obtain smooth maps $c^\wedge: [a, b] \times \mathbb{S}^1 \rightarrow M$ and $\alpha^\wedge: [a, b] \times \mathbb{S}^1 \rightarrow TM$. Now the covariant derivative ∇ from 5.6 is related to the covariant derivative ∇^g on M by the following formula:

$$\nabla_c \alpha(\cdot)(x) = \nabla_{c^\wedge(\cdot, x)}^g \alpha(\cdot, x) \quad \text{for all } x \in \mathbb{S}^1. \tag{5.5}$$

We relegate the verification of (5.5) to Exercise 5.1.2. However, the point is that, despite the technical difficulties in defining the covariant derivative, it can be viewed just as a lifting of the covariant derivative of the target manifold M .

5.7 Proposition *For the map $\nabla: \mathcal{V}(C^\infty(\mathbb{S}^1, M))^2 \rightarrow \mathcal{V}(C^\infty(\mathbb{S}^1, M))$ and $X, Y \in \mathcal{V}(C^\infty(\mathbb{S}^1, M))$, the formula*

$$\nabla_X Y = K_* \circ TY \circ X \tag{5.6}$$

holds. Since K_ is a linear connector, ∇ is a covariant derivative (with associated connector $K_*: C^\infty(\mathbb{S}^1, T^2M) \rightarrow C^\infty(\mathbb{S}^1, TM)$). In particular, (5.4) is the covariant derivative associated to the spray S_* .*

Proof By Proposition 5.3 the connector K_* is the associated connector to the spray S_* . Hence it suffices thus to prove (5.6). We have essentially done this already as the exponential law and Exercise 5.1.2 yield

$$\begin{aligned} (\nabla_X S)^\wedge &= \nabla_{X \times \mathbf{0}_{\mathbb{S}^1}}^g S^\wedge = K \circ Ts^\wedge \circ (X \times \mathbf{0}_{\mathbb{S}^1}) = K \circ (Ts \circ X)^\wedge \\ &= (K_* \circ Ts \circ X)^\wedge. \end{aligned} \quad \square$$

5.8 Proposition *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space with a strong Riemannian metric g (not necessarily $g = \langle \cdot, \cdot \rangle$). For the L^2 -metric (5.1) on $C^\infty(\mathbb{S}^1, H)$, the metric derivative is the covariant derivative ∇ from 5.6.*

Proof Recall that $TH \oplus TH = H \times H \times H$ and we can consider g as a smooth map of three variables. Let's agree that the first component represents the base point in the bundle. Since S is the spray of the metric g , the associated bilinear form B satisfies for all $X, Y, Z \in H$ the relation

$$-2g(p, (B(p; X, Y), Z)) = d_1g(p, Y, Z; X) + d_1g(p, Z, X; Y) - d_1g(p, X, Y; Z); \tag{5.7}$$

see Lang (1999, §VIII.4 Theorem 4.2) or Klingenberg (1995, Theorem 1.8.11). Hence if we can compute that an identity such as (5.7) holds for the bilinear

form associated to S_* (which is given as pushforward with B) with respect to g^{L^2} , this implies that g^{L^2} admits an associated bilinear form (aka Christoffel symbols). Thus if we know the Christoffel symbols, we can compute the connector giving the metric derivative. In this case, 5.6 shows that the connector is K_* , whence S_* is the metric spray and ∇ the metric derivative of the L^2 -metric (see also Theorem 4.30).

Let us now establish the desired analogue of (5.7) for the L^2 -metric. The integral operator $\int_{\mathbb{S}^1} : C^\infty(\mathbb{S}^1, \mathbb{R}) \rightarrow \mathbb{R}$ is continuous linear. Up to the identification C.13, the derivative of $g^{L^2} = \int_{\mathbb{S}^1} \circ g_*$ in a direction is thus given by the pointwise derivative, that is,

$$d_1 g^{L^2}(c, h, k; \xi) = \int_{\mathbb{S}^1} d_1 g(c(\theta), h(\theta), k(\theta); \xi(\theta)) d\theta.$$

We apply this observation to the right-hand side of (5.7) and recall from Proposition 5.3 that the associated bilinear form B_{S_*} of S_* is given by the pushforward of the associated bilinear form B of S . Together this yields

$$\begin{aligned} & d_1 g^{L^2}(c, Y, Z; X) + d_1 g^{L^2}(c, Z, X; Y) - d_1 g^{L^2}(c, X, Y; Z) \\ &= \int_{\mathbb{S}^1} d_1 g(c, Y, Z; X) + d_1 g(c, Z, X; Y) - d_1 g(c, X, Y; Z) d\theta \\ &= \int_{\mathbb{S}^1} -2g(c(\theta), B(c(\theta); X(\theta), Y(\theta), Z(\theta))) d\theta = -2g^{L^2}(c, B_{S_*}(c; X, Y), Z). \quad \square \end{aligned}$$

- 5.9 Remark** (a) Using the point evaluations of $TC^\infty(\mathbb{S}^1, M) \cong C^\infty(\mathbb{S}^1, TM)$, it is possible to directly describe the metric derivative of the L^2 -metric by looking at it pointwise evaluated. While this allows one to avoid the construction in 5.6, one is then left with lots of localisation arguments to establish Proposition 5.8. We refer to Maeda et al. (2015, Lemma 2.1) for more information.
- (b) All of the above computations for spray, connector and covariant derivative did not exploit that the source manifold was \mathbb{S}^1 . Thus via the same proof one can obtain a spray, connector and covariant derivative for the L^2 -metric on $C^\infty(K, M)$ for any compact manifold K .
- (c) Since (5.7) can be formulated in any local chart, a more involved argument works for any strong Riemannian manifold M . However, using the Nash embedding theorem, Proposition 5.8 generalises directly to all finite-dimensional Riemannian manifolds as Bruveris (2018, Theorem 4.1) shows.

We have identified the covariant derivative of the L^2 -metric, and thus we can describe geodesics of the L^2 -metric: A geodesic between $f, g \in C^\infty(\mathbb{S}^1, H)$ is

a path of curves which is pointwise for every $\theta \in \mathbb{S}^1$ a geodesic in H between $f(\theta)$ and $g(\theta)$. Let us explicitly compute this in the case where the inner product of the Hilbert space gives us the Riemannian metric.

5.10 Example Consider $(H, \langle \cdot, \cdot \rangle)$ as a strong Riemannian manifold. Recall that the metric spray and the covariant derivative were computed for this metric in Example 4.32: With the identification $T^k H = H^{2k}, k \in \mathbb{N}$ we obtain $S(x, v) = (x, v, v, 0)$ and $\nabla_X Y = X.Y$ for suitable paths and their lifts $\nabla_{\dot{c}} h = \dot{h}$. Now endow $C^\infty(\mathbb{S}^1, H)$ with the L^2 -metric and pick $p \in C^\infty(\mathbb{S}^1, H)$. We now compute the geodesics $c: J \rightarrow C^\infty(\mathbb{S}^1, H)$ starting at p with derivative $\dot{c} = X$. Now identify the first and second derivatives as $\dot{c}(t) = (c(t), c'(t)) \in C^\infty(\mathbb{S}^1, H \times H)$ and $\ddot{c}(t) = (c(t), c'(t), c''(t), c''(t))$. The exponential law allows us to compute the geodesic equation with respect to the spray as

$$(c(t), c'(t), c', c''(t))^\wedge = (\ddot{c}(t))^\wedge = (S_*(c(t), c'(t)))^\wedge = S\left(c^\wedge(t, \cdot), \frac{\partial}{\partial t} c^\wedge(t, \cdot)\right).$$

Evaluating in $\theta \in \mathbb{S}^1$ one immediately sees that $c''(t) = 0$, for all t . Hence, from the usual rules of calculus we deduce that $c(t)(\theta) = tX(\theta) + p(\theta)$. A geodesic between $f, g: \mathbb{S}^1 \rightarrow H$ does thus always exist and is described by a map $\gamma: [0, 1] \times \mathbb{S}^1 \rightarrow H$ such that for every $\theta \in \mathbb{S}^1$ we have $\gamma(t, \theta) = (1 - t)f(\theta) + tg(\theta)$. In other words, geodesics in this example are given by pointwise linear interpolation between the two functions.

We shall further investigate the geodesic equation of a generalised version of the L^2 -metric on the open submanifold $\text{Diff}(M) \subseteq C^\infty(M, M)$ in Chapter 7. We can also use our knowledge of the L^2 -metric to derive other interesting examples of metric derivatives for certain weak Riemannian metrics. Note, however, that the following example requires a more in-depth knowledge of Riemannian geometry (e.g. the Hodge Laplacian, Definition E.16 and curvature Definition 4.26).

5.11 Example (Maeda et al., 2015) Let (M, g) be a finite-dimensional Riemannian manifold with metric derivative ∇^g . We recall that every (finite-dimensional) Riemannian manifold has a (Hodge)Laplacian $\Delta = dd^* + d^*d$ associated to the metric (see Lang, 1999, p. 423) and a curvature tensor R (see Definition 4.26).

Now consider the loop space $LM := C^\infty(\mathbb{S}^1, M)$. By Remark 5.9(c) we know that the L^2 -metric admits a metric derivative ∇ . We will now use the notation of lifts and covariant derivative along curves on $[0, 2\pi]$ in the context of curves on \mathbb{S}^1 (implicitly identifying elements in LM with curves on $[0, 2\pi]$ by composing with $(\sin, \cos): [0, 2\pi] \rightarrow \mathbb{S}^1$). Hence for a smooth curve $\gamma \in LM$

we have $T_\gamma C^\infty(\mathbb{S}^1, M) = \text{Lift}(\gamma) \cap C^\infty(\mathbb{S}^1, TM)$. We can then endow every tangent space of $C^\infty(\mathbb{S}^1, M)$ with the H^1 -inner product:

$$g_\gamma^{H^1}(X, Y) := \frac{1}{2\pi} \int_0^{2\pi} g_{\gamma(\theta)}((1 + \Delta)X(\theta), Y(\theta)) \, d\theta. \tag{5.8}$$

Rewriting the codifferential in the above formula and exploiting duality with respect to the metric g , one can show that the inner product (5.8) describes the sum of the L^2 -inner products of the lift of γ and its first derivative. We can thus leverage Exercise 4.1.4(a), where we have seen that the L^2 -metric is a (weak) Riemannian metric. Differentiation is continuous linear (on the tangent space of $C^\infty(\mathbb{S}^1, M)$), whence the H^1 -metric is a weak Riemannian metric. Now a (non-trivial!) computation shows that the metric derivative of the H^1 -metric is intimately connected to the curvature tensor, the metric derivative of g and the metric derivative of the L^2 -metric. Namely, Maeda et al. (2015, Theorem 2.2) provide for $X, Y \in T_\gamma C^\infty(\mathbb{S}^1, M)$ the following formula for the metric derivative $\nabla_X^{H^1} Y(\gamma)$:

$$\begin{aligned} \nabla_X Y + \frac{1}{2}(1 + \Delta)^{-1} \left(-\nabla_c^g(R(X, \dot{c})Y) - R(X, \dot{c})\nabla_c^g Y - \nabla_c^g(R(Y, \dot{c}))X \right. \\ \left. - R(Y, \dot{c})\nabla_c^g X + R(X, \nabla_c^g Y)\dot{c} - R(\nabla_c^g X, Y)\dot{c} \right), \end{aligned}$$

where $\nabla_c^g(R(X, \dot{c})Y)$ denotes the lift of γ whose value at $\theta \in \mathbb{S}^1$ is given by the formula $-\nabla_{\dot{c}(\theta)}^g(R(X(\theta), \dot{c}(\theta))Y(\theta))$. While the above formula looks daunting and we do not attempt to unravel its meaning here, we would like to mention that it can be used to connect the Riemannian geometry of the H^1 -metric to pseudodifferential operators acting on a trivial bundle over the circle. This link then yields information on Chern–Simons classes on the tangent bundle of the loop space. We refer the interested reader to Maeda et al. (2015) for more information.

Exercises

5.1.1 Let $S : TM \rightarrow T^2M$ be a spray on M with $K : T^2M \rightarrow TM$ its associated connector.

- (a) Prove that S_* is a spray on $C^\infty(\mathbb{S}^1, M)$.
Hint: Use that $TC^\infty(\mathbb{S}^1, M) \cong C^\infty(\mathbb{S}^1, TM)$ identifies $T(p_*)$ with $(Tp)_*$ for smooth maps. For the quadratic condition review the effect of the diffeomorphism on the fibres.
- (b) Check that K_* is a connector.
- (c) Show that the second derivative of the vertical part of S_* is the pushforward of the second derivative of the vertical part of S (see Proposition 5.3).

5.1.2 Prove for $s \in C^\infty(N, C^\infty(\mathbb{S}^1, M))$ and $X \in \mathcal{V}(N)$ the identity

$$Ts \circ X = (Ts^\wedge \circ (X \times \mathbf{0}_{\mathbb{S}^1}))^\vee.$$

Furthermore, establish formula (5.5): $\nabla_{\dot{c}} \alpha(\cdot)(x) = \nabla_{\dot{c}^\wedge(\cdot, x)}^g \alpha^\wedge(\cdot, x)$, for all $x \in \mathbb{S}^1$.

5.1.3 Continue Example 5.10 and compute for $C^\infty(\mathbb{S}^1, H)$ with the L^2 -metric an explicit form of the geodesic equation $\nabla_{\dot{c}} \dot{c} = 0$. Deduce again that a geodesic $c: J \rightarrow C^\infty(\mathbb{S}^1, H)$ is given for each $\theta \in \mathbb{S}^1$ by the affine linear map $c(t)(\theta) = tc'(0)(\theta) + c(0)(\theta)$.

5.1.4 Let (M, g) be a Riemannian manifold with a metric spray and metric derivative ∇ . Work in a local chart to establish the identity $X.g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, (5.3), for the covariant derivative along a smooth map $F: N \rightarrow M$.

5.2 Shape Analysis via the Square Root Velocity Transform

We return to the announced application of the L^2 -metric to shape analysis. As we have seen in the introduction, we seek to construct a shape space together with a Riemannian structure which will allow us to compare its elements using geodesics and geodesic distance. We begin by defining the necessary spaces and metrics.

5.12 Define the *pre-shape space* of closed curves

$$\mathcal{P} := \text{Imm}(\mathbb{S}^1, \mathbb{R}^2) = \{c \in C^\infty(\mathbb{S}^1, \mathbb{R}^2) \mid \dot{c}(t) \neq 0, \text{ for all } t \in [0, 1]\}.$$

In Exercise 2.1.3 and Example 4.6 we have seen that \mathcal{P} is an open subset of $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ which becomes a weak Riemannian manifold with respect to the Riemannian metric

$$g_c(f, g) = \int_{\mathbb{S}^1} \langle f(\theta), g(\theta) \rangle \|\dot{c}(\theta)\| d\theta.$$

We are actually interested in the images of elements in \mathcal{P} and want to identify all curves which yield the same image up to a reparametrisation. To model the reparametrisation, consider the group

$$\text{Diff}(\mathbb{S}^1) := \{\varphi \in C^\infty(\mathbb{S}^1, \mathbb{S}^1) \mid \varphi \text{ is bijective with smooth inverse}\}$$

of diffeomorphisms of \mathbb{S}^1 . The group acts on \mathcal{P} via reparametrisation, that is,

$$\gamma: \text{Diff}(\mathbb{S}^1) \times \mathcal{P} \rightarrow \mathcal{P}, \quad (\varphi, c) \mapsto c \circ \varphi$$

is a Lie group action; see Example 3.5. We can thus define the *shape space* $\mathcal{S} = \mathcal{P} / \text{Diff}(\mathbb{S}^1)$ as the quotient of \mathcal{P} with respect to the Lie group action.

One can show that \mathcal{S} is almost a manifold¹ and since the Riemannian metric is invariant under the reparametrisation action, the Riemannian metric induces a Riemannian metric on \mathcal{S} . Unfortunately, Michor and Mumford (2006, 3.10) have shown that g has vanishing geodesic distance (see also Example 4.14), whence any attempt to compare shapes by computing their geodesic distance has to fail.

The defect of the weak Riemannian metric can be solved by incorporating derivatives in the definition of the Riemannian metric. This leads to the notion of a family of metric called H^1 -metrics (the name indicates that the associated strong Riemannian manifold consists of Sobolev H^1 -functions).

5.13 (An elastic inner product, Mio et al., 2007) Let $c \in \text{Imm}(\mathbb{S}^1, \mathbb{R}^2)$ with $\dot{c} = (c, c')$. Then we define $u_c(\theta) := c'(\theta)/\|\dot{c}(\theta)\|$ and the arc length derivative $D_{c,\theta}(h) = h'/\|\dot{c}\|$. We define an inner product on $T_c \text{Imm}(\mathbb{S}^1, \mathbb{R}^2) = C^\infty(\mathbb{S}^1, \mathbb{R}^2)$:

$$G_c(h, k) := \int_{\mathbb{S}^1} \frac{1}{4} \langle D_{c,\theta} h, u_c \rangle \langle D_{c,\theta} k, u_c \rangle + \langle D_{c,\theta} h - u_c \langle D_{c,\theta} h, u_c \rangle, D_{c,\theta} k - u_c \langle D_{c,\theta} k, u_c \rangle \rangle \|\dot{c}\| d\theta. \tag{5.9}$$

This inner product is called *elastic inner product* as the first term measures stretching in the direction of c , while the second term measures bending of the curve c . Note that due to its construction, the elastic inner products are invariant under the reparametrisation action of $\text{Diff}(\mathbb{S}^1)$ on \mathcal{P} .

It is not hard to see that these inner products yield a weak Riemannian metric on the pre-shape space \mathcal{P} . We will derive this only for a smaller submanifold using the so-called *square root velocity transform (SRVT)*:

5.14 Definition Define the mappings

$$\begin{aligned} \mathcal{R}: \mathcal{P} = \text{Imm}(\mathbb{S}^1, \mathbb{R}^2) &\rightarrow \{q \in C^\infty(\mathbb{S}^1, \mathbb{R}^2) \mid q(\theta) \neq 0 \text{ for all } \theta \in \mathbb{S}^1\} \\ &= C^\infty(\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\}) \\ c &\mapsto \mathcal{R}(c)(t) := c'(t) / \sqrt{\|\dot{c}(t)\|}, \\ \mathcal{R}^{-1}: C^\infty(\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\}) &\rightarrow \mathcal{P}, \\ q &\mapsto \left((\cos(t), \sin(t)) \mapsto \int_0^t q(\cos(s), \sin(s)) \cdot \|q(\cos(s), \sin(s))\| ds \right). \end{aligned}$$

¹ There exist singularities, turning \mathcal{S} into an orbifold. See Example 6.4. The existence of singularities is usually ignored in shape analysis, as an open dense subset of \mathcal{S} is a manifold such that the projection restricts on this set to a submersion.

We call \mathcal{R} the *square root velocity transform* (SRVT).

Using the idea that $\mathcal{P} \subseteq C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ and that pushforwards of smooth mappings are smooth maps between canonical manifolds of mappings (see Chapter 2), it is not hard to see that \mathcal{R} and \mathcal{R}^{-1} are smooth. Moreover, $\mathcal{R} \circ \mathcal{R}^{-1}(q) = q$, but since \mathcal{R} involves differentiation, we lose information on the starting point of the curve and have $\mathcal{R}^{-1} \circ \mathcal{R}(c) = c$ if and only if $c(\cos(0), \sin(0)) = 0$, that is, the curve starts at the origin. As we are interested in shapes, it will be irrelevant as to where in \mathbb{R}^2 the shape is located. In other words, we can restrict to the submanifold

$$\mathcal{P}_* := \{c \in \mathcal{P} \mid c(\cos(0), \sin(0)) = 0\}$$

of all immersions starting at the origin. We will see in Exercise 5.2.1 that the SRVT induces a diffeomorphism between \mathcal{P}_* and the manifold $C^\infty(\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\})$. Recall now the notion of pullback of a Riemannian metric.

5.15 Definition Let (M, g) be a (weak) Riemannian manifold and $\varphi: N \rightarrow M$ be an immersion. Then N can be made a (weak) Riemannian manifold with respect to the *pullback metric* defined via

$$(\varphi^*g)_m(v, w) := g_{\varphi(m)}(T_m\varphi(v), T_m\varphi(w)).$$

5.16 Example If $U \subseteq M$ and (M, g) is a weak Riemannian manifold, then the inclusion $\iota: U \rightarrow M$ is an immersion and the restriction of g to U coincides with the pullback metric obtained from ι . In particular, an open subset of a weak Riemannian manifold, such as $C^\infty(\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\}) \subseteq C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ (with the weak L^2 -metric), becomes a weak Riemannian manifold by restriction.

5.17 Proposition *The pullback metric of the non-invariant L^2 -metric on $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ from Example 4.3 with respect to the SRVT \mathcal{R} is the elastic metric described by (5.9) on each tangent space of \mathcal{P}_* .²*

Proof By construction, the square root velocity transform \mathcal{R} is the composition of the derivative operator $D: \text{Imm}(\mathbb{S}^1, \mathbb{R}^2) \rightarrow C^\infty(\mathbb{S}^2, \mathbb{R}^2 \setminus \{0\})$, $q \mapsto (\theta \mapsto q'(\theta) = T_\theta c(1))$, and the scaling map $\text{sc}: C^\infty(\mathbb{S}^1, \mathbb{R} \setminus \{0\}) \rightarrow C^\infty(\mathbb{S}^1, \mathbb{R} \setminus \{0\})$, $f \mapsto f / \sqrt{\|f\|}$. Thus by the chain rule we have $T_c\mathcal{R} = T_{\text{sc}} \circ T_c D$ and to arrive at the desired formula, we have to compute the derivatives of these two mappings. To this end, we exploit that $TC^\infty(\mathbb{S}^1, \mathbb{R}^2) = C^\infty(\mathbb{S}^1, T\mathbb{R}^2) = C^\infty(\mathbb{S}^1, \mathbb{R}^2 \times \mathbb{R}^2)$ and since $\text{Imm}(\mathbb{S}^1, \mathbb{R}^2) \subseteq C^\infty(\mathbb{S}^1, \mathbb{R}^2)$, we can similarly identify the tangent bundle of the immersions. Now arguments as in Exercise 1.2.2(c) show that the differential operator D is continuous linear, whence for an element (c, V)

² The statement remains valid if we replace \mathbb{R}^2 by an arbitrary Hilbert space of dimension ≥ 2 .

in $T_c C^\infty(\mathbb{S}^1, \mathbb{R}^2) = \{(c, V) \in C^\infty(\mathbb{S}^1, \mathbb{R}^2)^2\}$, we obtain $T_c D(V) = V'$. In Exercise 5.2.2 we shall show that the derivative of the scaling map is

$$T_q \text{sc}(Z) = \frac{Z}{\sqrt{\|q\|}} - \frac{1}{2\sqrt{\|q\|^5}} \langle Z, q \rangle q. \tag{5.10}$$

To obtain the derivative of the SRVT at (c, V) , we simply have to replace q with c' and Z with V' . This yields the following formula:

$$\begin{aligned} (\mathcal{R}^* \langle \cdot, \cdot \rangle_{L^2})_c (V, W) &= \langle T_c \mathcal{R}(V), T_c \mathcal{R}(W) \rangle_{L^2} = \int_{\mathbb{S}^1} \langle T_{c'} \text{sc}(V')(\theta), T_{c'} \text{sc}(W)(\theta) \rangle d\theta \\ &= \int_{\mathbb{S}^1} \left\langle \frac{V'}{\sqrt{\|c'\|}} - \frac{1}{2\sqrt{\|c'\|^5}} \langle V', c' \rangle c', \frac{W'}{\sqrt{\|c'\|}} - \frac{1}{2\sqrt{\|c'\|^5}} \langle W', c' \rangle c' \right\rangle (\theta) d\theta. \end{aligned} \tag{5.11}$$

Since the inner product is bilinear, we can factor out terms of the form $\|c'\|$ and replace V', W' and c' with their rescaled versions (see 5.13). Now an easy but tedious computation shows that the pullback metric (5.11) coincides with the elastic metric (5.9). □

Hence the elastic metric can be understood by studying the L^2 -metric on the manifold $C^\infty(\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\})$. However, as this is just an open subset of the (weak) Riemannian manifold $(C^\infty(\mathbb{S}^1, \mathbb{R}^2), \langle \cdot, \cdot \rangle_{L^2})$ we already know the spray, connector, covariant derivative and geodesics of this metric from our discussion of the L^2 -metric. It is important to observe that the elastic metric pulls back to the non-invariant L^2 -metric. For the non-invariant L^2 -metric, the geodesic distance does not vanish (compare this with the invariant L^2 -metric, Example 4.6) and the pullback metric (i.e. the elastic metric) is invariant under reparametrisation.

A natural extension for the vector space valued shape spaces discussed in Proposition 5.17 is Lie group valued shape spaces. Here a shape is (up to quotienting out the reparametrisation action) an element of the loop group $C^\infty(\mathbb{S}^1, G)$. If G is a Hilbert Lie group, the construction of the square root velocity transform can be adapted to this more general setting by using the logarithmic derivative of Lie group valued mappings. For details, we refer to Exercise 5.2.4. Similar techniques have been used in Celledoni et al. (2018) for shape analysis on homogeneous spaces.

5.18 Example (Sample application: Motion capturing; see e.g. Celledoni et al., 2016) Assume that we have motion capturing data of, for example, a human walking, given by a number of time-dependent datapoints in \mathbb{R}^3 . The associated virtual character is then modelled as a skeleton for which these datapoints represent positions of certain parts. Shifting the focus from the position to their relative position, we can interpret the positions of every part of the skeleton as

an angle between neighbouring parts. Now angles in \mathbb{R}^3 can be identified with rotations, that is, elements in the Lie group $\text{SO}(3)$ of rotations of \mathbb{R}^3 . Thus if we do not impose constraints on the allowed angles (which can lead to unnatural movements, but is generally fine when working with real motion-capturing data), we can think of motion-capturing data as a smooth curve with values in a product of copies of $\text{SO}(3)$ (the number depends on the number of data points which move relative to each other). In Celledoni et al. (2016) numerical algorithms for automatic interpolation and transformation of motion-capturing data have been constructed which exploit the above point of view.

5.19 Remark Note that the geodesic distance of the L^2 -metric on $C^\infty(\mathbb{S}^1, H)$ does not vanish and is indeed positive for any two shapes which are not equal (where H is a Hilbert space). Indeed the geodesic distance just coincides with the L^2 -distance of the curves. The situation gets more complicated in the space $C^\infty(\mathbb{S}^1, H \setminus \{0\})$ and, in particular, for $H = \mathbb{R}^2$ (which is not simply connected), so an element γ for which 0 is contained in a bounded connected component of $\mathbb{R}^2 \setminus \gamma(\mathbb{S}^1)$ cannot be connected by a continuous curve in $C^\infty(\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\})$ to an element for which 0 is not contained in such a component.

While this does not happen for open shapes (i.e. elements of $\text{Imm}([0, 1], \mathbb{R}^2)$), the following problem is even more significant when it comes to applications in shape analysis: As geodesics are not allowed to pass through 0, the L^2 -distance of two functions is *not* the geodesic distance even if there exist continuous paths between them. If the linear interpolation between two points $c(\theta)$ and $d(\theta)$ passes through 0, then the geodesic from the ambient space $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ does not exist in the smaller space. Instead, the geodesic distance then needs to be computed using curves which ‘move one shape around the hole’. This leads to a geodesic distance, which is strictly larger than the L^2 -distance. In numerical applications this is often just plainly ignored.

That the geodesic distance on $C^\infty(\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\})$ diverges from the easily computed L^2 -distance is a consequence of the space being incomplete. Indeed since we have ‘drilled a hole’ by excluding a point, geodesics passing through that point can only exist up to the time they enter. As the L^2 -distance moves points on the image of functions along straight lines interpolating between them, the L^2 -distance fails to give the geodesic distance for points lying on opposite sides of the excluded point.

Exercises

- 5.2.1 Establish some properties of the square root velocity transform \mathcal{R} , Definition 5.14.

Hint: Use the substitution rule for integrals on submanifolds; in the case at hand, this works just as in usual interval cases.

Show that:

- (a) \mathcal{R} and \mathcal{R}^{-1} are smooth maps with $\mathcal{R} \circ \mathcal{R}^{-1} = \text{id}_{C^\infty(\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\})}$.
- (b) $\mathcal{P}_* = \{c \in \mathcal{P} \mid c(\cos(0), \sin(0)) = 0\}$ is a closed submanifold of \mathcal{P} .
- (c) \mathcal{R} and \mathcal{R}^{-1} induce a diffeomorphism between the manifolds \mathcal{P}_* and $C^\infty(\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\})$.
- (d) If $\varphi \in \text{Diff}(\mathbb{S}^1)$ with $\varphi'(\theta) > 0$, for all $\theta \in \mathbb{S}^1$, then $\mathcal{R}(c \circ \varphi) = \varphi' \cdot \mathcal{R}(c) \circ \varphi$.

- 5.2.2 Let $(E, \|\cdot\|)$ be a Hilbert space with $\|v\|^2 = \langle v, v \rangle$ and K a compact manifold. Prove that the scaling map $\text{sc}: C^\infty(K, E \setminus \{0\}) \rightarrow C^\infty(K, E \setminus \{0\})$, $q \mapsto q/\sqrt{\|q\|}$ is smooth with the tangent map given by the formula (5.10):

$$T_q \text{sc}(W) = \frac{W}{\sqrt{\|q\|}} - \frac{1}{2\sqrt{\|q\|}^3} \langle W, q \rangle q.$$

Hint: Use the exponential law together with the canonical identification of the tangent bundles.

- 5.2.3 Show that the elastic metric (5.9) is invariant under reparametrisations with elements φ in $\text{Diff}(\mathbb{S}^1)$ which satisfy $T_\theta \varphi(1) > 0$, for all $\theta \in \mathbb{S}^1$.
- 5.2.4 Let G be a Hilbert Lie group, that is, $\mathbf{L}(G)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then we define a square root velocity transform on the subset of immersions of the loop group

$$\mathcal{R}: \text{Imm}(\mathbb{S}^1, G) \rightarrow C^\infty(\mathbb{S}^1, \mathbf{L}(G) \setminus \{0\}), \quad c \mapsto \delta^r(c) / \sqrt{\|\delta^r(c)\|},$$

where δ^r is the right logarithmic derivative.

- (a) Show that \mathcal{R} is a smooth diffeomorphism (what is its inverse?).
- (b) Compute a formula for the pullback of the L^2 -metric on $C^\infty(\mathbb{S}^1, \mathbf{L}(G) \setminus \{0\})$. This metric is known as the elastic metric on the Lie group valued immersions and it can be used in computer animation and motion-capturing applications. See Celledoni et al. (2016) for more information.