THE FACTORIZATION OF LOCALLY FINITE GRAPHS

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1. Introduction. A *graph* G consists of a set $V(G)$ of objects called *nodes* and a set $M(G)$ of objects called *links*, $V(G)$ and $M(G)$ having no members in common. With each link A there is associated just two nodes said to be the *ends* of A, or to be *incident* with A, or to be *joined* by A. The sets $V(G)$ and $M(G)$ may be finite or infinite. There may be nodes with which no link is incident. Such nodes are said to be *isolated.*

If $V(G)$ and $M(G)$ are finite, the graph G is said to be *finite*.

The *order* of a graph G is the cardinal of $V(G)$. The *degree* of a node a of G is the cardinal of the set of links of G with which a is incident. The graph G is said to be *locally finite* if the degree of each node of G is finite. If all the nodes of G have the same finite degree σ we may say that G is a *regular* graph of the σth *degree.*

Let x and y be any two nodes of a graph G. We say that they are *connected* in G if there exists a finite sequence

$$P = (b_1, B_1, b_2, B_2, \ldots, b_r, B_r, b_{r+1}) \tag{1}$$

satisfying the following conditions.

(i) $b_1 = x$; $b_{r+1} = y$.

(ii) The members of P are alternately nodes and links of G.

(iii) Consecutive members of P are incident.

If x and y are any nodes of a graph G, we define the relation $x \backsim y$ to mean that either $x = y$ or else x and y are connected in G. It is easily verified that the relation is an equivalence relation. It therefore partitions G into a set $\{G_\alpha\}$ of graphs such that G_α is connected, each node or link of G belongs to some G_α, and no two of the G_α have any node or link in common. We shall call the G_α the *components* of G.

A *subgraph* of a graph G is a graph G' such that $V(G') = V(G)$ and $M(G') \subseteq M(G)$, a link of G' having the same ends in G' as in G. A *factor* of G is a regular subgraph of G of the first degree. If G has no factor it is *prime.* Clearly all finite graphs of odd order are prime.

Let S be any finite subset of $V(G)$. Then we denote the number of members of S by $f(S)$. We denote by G_S the graph obtained from G by suppressing the members of S and all links of G having one or both ends in S. Let $h(S)$ be the cardinal of the set of components of G_S, and let $h_u(S)$ be the cardinal of the set of those components of G_S which are finite and of odd order. Clearly, if G is locally finite and is connected, then $h(S)$ and $h_u(S)$ are finite.

The object of this paper is to prove the following Theorem.

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THEOREM A. *A locally finite graph G is prime if and only if there is a finite subset S of $V(G)$ such that $h_u(S) > f(S)$.*

A proof of this Theorem for the case in which G is finite has already been given. (W. T. Tutte, "The Factorization of Linear Graphs," *J. London Math. Soc.*, vol. 22 (1947), 107-111). We refer to this paper below as Paper I. In the present paper we assume the truth of the Theorem for finite graphs and show how to extend it to the case in which G is infinite (but locally finite).

2. Preliminary results. We shall say that a graph G is *constricted* if there exists a finite subset S of $V(G)$ such that $h_u(S) > f(S)$.

THEOREM I. *A constricted graph has no factor.*

Let G be any graph such that $V(G)$ has a finite subset S such that $h_u(S) > f(S)$.

Suppose there exists a factor F of G. Then if C is any finite component of odd order of G_S it is clear that F must contain a link having one end in C and the other in S. Hence the cardinal of the set of links of F having ends in S is greater than the number of members of S. Hence some node of S must be incident with more than one link of F, which is absurd.

THEOREM II. *The truth of Theorem A for connected locally finite graphs implies its truth for all locally finite graphs.*

Let G be any locally finite graph, and let $\{G_\alpha\}$ be the set of its components. Let us assume that Theorem A has already been proved for connected locally finite graphs.

If G is prime, some component G_α of G must be prime. For if each G_α had a factor F_α we could clearly obtain a factor of G by combining the factors F_α.

But if G_α is prime there is a finite subset S of $V(G_\alpha)$ such that $f(S) < h_u(S)$ in G. For G_α is connected and so we can apply Theorem A to it. But the components of G_S are simply the components of $(G_\alpha)_S$ together with the components other than G_α of G. Hence the inequality $f(S) < h_u(S)$ is true also in G.

Hence if G is prime, it is constricted. But by Theorem I, if G is constricted, it is prime. Thus Theorem A is true for G.

We conclude from Theorems I and II that, in order to prove Theorem A, it now suffices to prove that any connected locally finite infinite graph which is not constricted has a factor.

3. Distance and n-factors. Let G be any connected locally finite graph, and let a be any node of G.

Suppose that b is any other node of G. Then because G is connected there exists a finite sequence P of the form (1) such that $b_1 = a$ and $b_{r+1} = b$. The least value of r for which such a sequence P exists will be called the *distance* $d(a, b)$ from a to b. We write $d(a, a) = 0$.

We define $V_n(G; a)$ to be the set of all nodes b of G such that $d(a, b) \leq n$. We define $M_n(G; a)$ to be the set of all links A of G such that both ends of A are in $V_n(G; a)$. It is clear from these definitions that if a link A of G has one

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end in $V_n(G; a)$, then $A \epsilon M_{n+1}(G; a)$. It is also evident, from the fact that G is locally finite, that $V_n(G; a)$ and $M_n(G; a)$ are finite for each non-negative integer n.

We define a *false factor* of G as a subgraph X of G which satisfies the following conditions.

(i) $M(X)$ is finite.

(ii) There is no node of G whose degree in X exceeds 1. If X is a false factor of G, we denote by $W(X)$ the set of all nodes of X which are incident with links of X.

If a is a node and X a false factor of G, we say that X is an *n-factor of G with respect to a* if

(2) $$V_{n+1}(G; a) \supseteq W(X) \supseteq V_n(G; a),$$

n being any non-negative integer.

THEOREM III. *Let G be any connected locally finite infinite graph which is not constricted, and let a be any node of G. Let n be any non-negative integer. Then there exists an n-factor of G with respect to a.*

Let Q denote the set $V_{n+1}(G; a) - V_n(G; a)$.

We define a graph H as follows. The nodes of H are the members of $V_{n+1}(G; a)$. We take each member of $M_{n+1}(G; a)$ as a link of H, assigning it the same ends in H as in G. In addition we join each pair of members of Q by a new link.

We next define a graph K. If the order of H is even, $K = H$. If the order of H is odd we construct K from H by adjoining to H a new node q and then joining q to each member of Q by a new link. By this construction the order $m(K)$ of K is always even. We write $Q' = Q$ or $Q \cup \{q\}$ according as the order of H is even or odd.

Suppose that K is constricted. Then there is a subset S, (possibly the null subset), of $V(K)$ such that

(3) $$h_u(S) > f(S) \text{ in } K.$$

But it is clear that

(4) $$m(K) \equiv h_u(S) + f(S) \pmod 2.$$

Consequently, since $m(K)$ is even, we must have

(5) $$h_u(S) \geq f(S) + 2.$$

At most one of the components of K_S contains a node of Q'. For any two nodes of Q' will be joined by a link of K. Two such nodes cannot therefore be in different components of K_S. Write $T = S$ or $S - \{q\}$ according as the order of H is even or odd. (If $q \bar{\epsilon} S$, $S - \{q\} = S$). Then it is clear that any component of K_S which contains no node of Q' is also a component of G_T. Hence, using (5), we deduce that the inequality $h_u(T) > f(T)$ holds in G. This is contrary to our hypothesis that G is not constricted. We conclude that K is not constricted.

Since Theorem A holds for finite graphs it follows that K has a factor F_1. Let X be the subgraph of G defined by

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(6) $$M(X) = M(F_1) \cap M_{n+1}(G; a).$$

Each node of G is incident with at most one member of $M(F_1)$. Hence X is a false factor of G. Now each link of K incident with any node $c \epsilon V_n(G; a)$ is a link of $M_{n+1}(G; a)$ incident with c in G. Hence c is incident in G with a member of $M(X)$. Thus

(7) $$W(X) \supseteq V_n(G; a).$$

Also each node of G incident with a member of $M_{n+1}(G; a)$ is a member of $V_{n+1}(G; a)$. Hence

(8) $$V_{n+1}(G; a) \supseteq W(X).$$

From (7) and (8) it follows that X is an n-factor of G with respect to a.

4. Proof of Theorem A. In this Section, G is any connected, locally finite, infinite graph, which is not constricted, and a is any node of G.

Let m and n be integers satisfying $m \geq n \geq 0$, and let X_m be any m-factor of G with respect to a. We denote by $C(X_m; n)$ the subgraph of G obtained from X_m by suppressing all links of X_m not in $M_{n+1}(G; a)$. It is clear that if $l \geq n$, then

(9) $$C(C(X_l; m); n) = C(X_l; n).$$

It is also evident that $C(X_m; n)$ is an n-factor of G with respect to a.

Suppose that the m-factor X_m and the n-factor X_n of G with respect to a are related by the equation

$$X_n = C(X_m; n).$$

Then we say that X_m is an *extension* of X_n to m. It may happen for a given n-factor X_n of G with respect to a, that there exists an integer $m > n$ such that X_n has no extension to m. In that case it follows by (9) that X_n has no extension to any integer $l > m$. There is thus a maximum integer $r(X_n) \geq n$ such that X_n has an extension to $r(X_n)$. We call $r(X_n)$ the *range* of X_n. The only other possibility is that the given n-factor X_n may have extensions to all integers $m \geq n$. We then say that X_n has *infinite range.*

THEOREM IV. *There exists a* 0*-factor of* G *with respect to* a *which has infinite range.*

If X_0 is any 0-factor of G with respect to a, it follows from (2) that $M(X_0) \subseteq M_1(G; a)$. As $M_1(G; a)$ is finite, it follows that the set of 0-factors of G with respect to a is finite.

Suppose that no one of them has infinite range. Then there exists an integer n greater than the range of any 0-factor of G with respect to a. By Theorem III there exists an n-factor X_n of G with respect to a. Then $C(X_n; 0)$ is a 0-factor of G with respect to a whose range is not less than n. This contradiction establishes the Theorem.

THEOREM V. *Let* n *be any non-negative integer, and let* X_n *be an* n*-factor of* G *with respect to* a *having infinite range. Then there exists an* $(n+1)$*-factor* X_{n+1} *of* G *with respect to* a *which has infinite range and which satisfies* $X_n = C(X_{n+1}; n)$.

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Let Z be the set of $(n+1)$-factors of G with respect to a which are extensions to $(n+1)$ of X_n. If X is any member of Z it follows from (2) that $M(X) \subseteq M_{n+1}(G; a)$. As $M_{n+1}(G; a)$ is finite it follows that Z is finite.

Suppose that no member of Z has infinite range. Then there is an integer $m > n+1$ which exceeds the range of each member of Z. Since X_n has infinite range there exists an m-factor X_m of G with respect to a such that $C(X_m; n) = X_n$. Then $C(X_m; n+1)$ is an $(n+1)$-factor of G with respect to a. By (9) it is a member of Z. Hence it can have no extension to m. This is absurd, since X_m is such an extension. The Theorem follows.

THEOREM VI. *There exists an infinite sequence* $(Y_0, Y_1, Y_2, Y_3, \ldots)$ *having the following properties.*

(i) *For each non-negative integer n, Y_n is an n-factor of G with respect to a having infinite range.*

(ii) *For each non-negative integer n,* $C(Y_{n+1}; n) = Y_n$.

The terms Y_r are defined successively as follows. Y_0 is defined to be a 0-factor of G with respect to a having infinite range. Such a Y_0 exists, by Theorem IV. For $r \geq 0$, Y_{r+1} is defined to be an extension to $(r+1)$ of Y_r having infinite range. If Y_r is fixed, such a Y_{r+1} exists, by Theorem V. The sequence of Y_r defined in this way satisfies (i) and (ii).

Let us consider some particular infinite sequence $(Y_0, Y_1, Y_2, \ldots)$ satisfying conditions (i) and (ii) of Theorem VI. Using (9) we can show, by an obvious induction, that if m and n are integers satisfying $m \geq n \geq 0$, then $C(Y_m); n) = Y_n$.

Let F be the subgraph of G defined by

$$M(F) = \bigcup_{r=0}^{\infty} M(Y_r). \tag{10}$$

Let b be any node of G. Write $d(a, b) = n$. Then $b \epsilon V_n(G; a) \subseteq W(Y_n)$, by (2). Hence b is incident in G with some link of F. Suppose that b is incident with two distinct links A_1 and A_2 of F. Then by (10) there exist integers s and t, ≥ 0, such that $A_1 \epsilon M(Y_s)$ and $A_2 \epsilon M(Y_t)$. Then $A_1 \epsilon M(Y_u)$ and $A_2 \epsilon M(Y_u)$, where u is any integer greater than s and t. For $C(Y_u; s) = Y_s$ and $C(Y_u; t) = Y_t$. But then the degree of b in the false factor Y_u of G exceeds 1. This contradicts the definition of a false factor of G.

From these considerations we conclude that each node of G has degree 1 in F. That is, F is a factor of G.

It now follows, from Theorem I, that Theorem A holds for all connected, locally finite, infinite graphs. Since, by Paper I, it holds for all finite graphs, we see that it holds for all connected locally finite graphs. Hence by Theorem II it holds for all locally finite graphs.

5. Regular graphs. If G is a connected locally finite graph we define an *isthmoid* of G as a finite subset S of $V(G)$ such that $h(S) > 1$. We then say that $f(S)$ is the *rank* of the isthmoid.

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We find that Theorem V of Paper I, and its Corollary, can be generalized as follows.

THEOREM VII. *Let G be any connected locally finite graph which is regular and of degree $\sigma > 0$, and which is either infinite or else of even order. Suppose further that G has no isthmoid of rank $< \sigma - 1$. Then G has a factor.*

COROLLARY. *Let A be any link of G. Then G has a factor which contains A.*

Here we shall only consider the case in which G is infinite, the finite case having been dealt with in Paper I. It will be found that the argument of Paper I remains valid in the infinite case as far as the Theorem is concerned, if we replace the appeal to Theorem IV of Paper I by an appeal to Theorem A. The proof of the Corollary in Paper I is not valid for the infinite case. We may replace it by the following argument (which is not valid for the finite case).

Let x and y be the ends of A. Suppose that the Corollary fails for some graph G. Then $G_{[x,\, y]}$ is prime. Hence, by Theorem A, there exists a finite subset S of $V(G) - \{x, y\}$ such that $h_u(S) > f(S)$ in $G_{[x,\, y]}$.

Let S' be the set formed by adjoining x and y to S. Hereafter functions of S will refer to $G_{[x,\, y]}$ and functions of S' to G. Clearly

$$f(S') = f(S) + 2 \tag{11}$$

and

$$h_u(S') = h_u(S). \tag{12}$$

Now if C is a finite component of $G_{S'}$ of odd order, the number of links of G having one end in C and the other in S' is at least σ. (Paper I, proof of Theorem V). Apart from any such components $G_{S'}$ has at least one infinite component C_∞. For G is infinite and connected, and each node of S' is of finite degree. The number of links of G having one end in C_∞ and the other in S' is at least $\sigma - 1$, since G has no isthmoid of rank less than $\sigma - 1$.

Let k be the number of links of G having just one end in S'. Using the above considerations and the fact that A has both ends in S' we find

$$(\sigma - 1) + \sigma.h_u(S') \leqslant k \leqslant \sigma.f(S') - 2.$$

Hence, since $\sigma > 0$,

$$f(S') \geq h_u(S') + 1 + 1/\sigma, \tag{13}$$

whence

$$f(S') \geq h_u(S') + 2, \tag{14}$$

since $f(S')$ and $h_u(S')$ are integers.

It follows from (11), (12) and (14) that $f(S) \geq h_u(S)$. This contradicts the definition of S. The Corollary follows.

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