SUBALGEBRAS OF C*-ALGEBRAS AND VON NEUMANN ALGEBRAS

by CHARLES A. AKEMANN†

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0. Introduction and notation. Recent work [**2**, **6**] on subalgebras of matrix algebras leads naturally to the following situation. Let A be a C*-subalgebra of the C*-algebra B and M be a weakly closed *-subalgebra of the von Neumann algebra N. Consider the following Conditions.

Condition 1. For every $b \neq 0$ in B there exists $a \in A$ such that $0 \neq ab \in A$.

Condition 2. For every $b \in B$ there exists $a \neq 0$ in A such that $ab \in A$.

If we replace A by M and B by N in Conditions 1 and 2 we get von Neumann algebra versions which we shall call *Condition* 1′ and *Condition* 2′. Clearly Condition 1 implies Condition 2, and both conditions suggest that A is some kind of weak ideal of B. This paper explores the extent to which this is true. The paper grew out of the author's attempts [**1**, **3**] to generalize the Stone-Weierstrass theorem to C*-algebras.

It follows immediately from [**2**] that if B is finite-dimensional, then Condition 1 is satisfied only when $B = A$. In this case every C*-algebra is a von Neumann algebra, so it is not surprising that in the infinite dimensional case this becomes a von Neumann algebra result. Specifically, Proposition 2.1 states that Condition 1′ holds only when $M = N$. The conclusions to be drawn from Condition 1 when B is not finite-dimensional are not known in general. When B is abelian and separable, we can conclude (Theorem 1.2) that A contains an ideal I of B which is essential in the sense that $\{b \in B : bI = 0\} = \{0\}$. It is clear that if A contains such an ideal (and B is abelian), then Condition 1 holds, so this is the limit of what can be expected. However, when B is not abelian, Condition 1 may hold even though A contains no non-trivial ideals and $B \neq A$. Specifically (Theorem 2.2), if A is an essential hereditary [**4**, p. 14] C*-subalgebra of B with open covering projection p of finite condimension in B^{**} [**4**, pp. 77–78], then Condition 1 holds.

Assuming Condition 2, it follows from [**2**] that if B is the algebra $B(H_n)$ of all bounded operators on the n-dimensional Hilbert space H_n, then there is a projection $p \in A$ with rank $(p) > n/2$ and $pBp \subset A$. Conversely, if p is as above and $pB(H_n)p \subset A \subset B(H_n) = B$, then Condition 2 holds. This can be generalized to infinite dimensional Hilbert space, with Condition 2 (Theorem 3.3), and to $N = B(H)$, with Condition 2′ (Theorem 3.1). The condition that rank $(p) > n/2$ becomes codimension $(p) < \infty$. If B is separable and abelian, then (Theorem 1.1) Condition 2 holds if and only if A contains a non-zero ideal of B. It is not clear what the appropriate conjecture should be in the general case.

1. The separable, abelian case; Conditions 1 and 2. In this section we shall assume

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that B is separable and abelian; thus B is isomorphic to $C_0(X)$, the continuous complex-valued functions vanishing at infinity on the locally compact metric space X. Since A is a C*-subalgebra of B, A defines an equivalence relation $\sim$ on X by $x \sim y$ if $a(x) = a(y)$ for all $a \in A$. Further, $b \in B$ actually is in A if $x \sim y$ implies $b(x) = b(y)$ for $x, y \in X$.

THEOREM 1.1. *Using the notation developed above, Condition 2 is satisfied if and only if A contains a non-zero ideal of B.*

Proof. If A contains a non-zero ideal I of B, then for any $b \in B$ we need only choose $a \neq 0$ in $I \subset A$ to ensure that $ab \in Ib \subset A$.

Now suppose that Condition 2 holds. Since the closed ideals I of B all have the form $I_U = \{b \in B\colon b$ vanishes outside of an open set $U \subset X\}$, by the Stone-Weierstrass theorem it suffices to find a non-void open subset U of X such that $A \cap I_U$ separates the points of U (and 0). We consider two cases. To distinguish them let Y contain the union of all of the equivalence classes under $\sim$ which contain more than one point, and let Y also contain the point $z_0 \in X$ such that $a(z_0) = 0$ for all $a \in A$, if there is a unique point with this property.

Case 1. Y is not dense in X. By Urysohn's lemma we may choose $c \in B$ with $c \neq 0$ and $c(y) = 0$ for all $y \in Y$. Thus $c \in A$ since $x \sim z$ means either $x = z$ or $x, z \in Y$, and so $c(x) = c(z) = 0$. Let $U = \{x \in X : c(x) \neq 0\}$. Note that for every distinct pair $x, z \in U$ we can find some $d \in A$ with $0 \neq (cd)(x) \neq (cd)(z) \neq 0$, so cA separates the points of U, and $I_U \subset A$ as required.

Case 2. Y is dense in X. We shall construct an element of B by induction. Choose a dense sequence $\{x_n\}_{n=1}^{\infty}$ in $Y - \{z_0\}$ and choose a sequence $\{y_n\}_{n=1}^{\infty}$ in Y such that $y_n \sim x_n$ but $y_n \neq x_n$. (In the finite dimensional case the sequence would be finite.) This is possible since X is a locally compact metric space, and hence X is second countable. By Urysohn's lemma choose $b_1 \in B$ such that $0 \leqq b_1(x) \leqq 1$ for all $x \in X$, $b_1(x_1) = 0$, $b_1(y_1) = 1$. Now suppose $\{b_i\}_{i<n}$ have been chosen so that b_1 is as above and for all $i < n$:

(a) $0 \leqq b_i(x) \leqq 2^{-i+1}$ for all $x \in X$;

(b) if $\sum_{j<i} (b_j(x_i) - b_j(y_i)) \geqq 0$, then $b_i(x_i) = 2^{-i+1}$ and $b_i(y_i) = 0$;

(c) if $\sum_{j<i} (b_j(x_i) - b_j(y_i)) < 0$, then $b_i(x_i) = 0$ and $b_i(y_i) = 2^{-i+1}$;

(d) $b_i(x_j) = b_i(y_j) = 0$ for all $j < i$.

By Urysohn's lemma we may choose $b_n \in B$ to continue the induction. Set $b = \sum_{n=1}^{\infty} b_n$. Since $\|b_n\| \leqq 2^{-n+1}$, the series converges in norm, and so $b \in B$. Note that $b(x_n) = \sum_{j=1}^{n} b_j(x_n)$ and $b(y_n) = \sum_{j=1}^{n} b_j(y_n)$ by induction hypothesis (d), so that

$$|b(x_n) - b(y_n)| = \left| \sum_{j=1}^{n-1} (b_j(x_n) - b_j(y_n)) + (b_n(x_n) - b_n(y_n)) \right|$$
$$\geqq |b_n(x_n) - b_n(y_n)| = 2^{-n+1},$$

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since $\sum_{j=1}^{n-1}(b_j(x_n)-b_j(y_n))$ and $(b_n(x_n)-b_n(y_n))$ have the same sign by induction hypotheses (b) and (c). In particular $b(x_n)\neq b(y_n)$.

By Condition 2 we can choose $a\in A$ with $a\neq 0$ and $ab\in A$. Thus for all i, $ab(x_i)=ab(y_i)$, so $a(x_i)b(x_i)=a(y_i)b(y_i)$. Since $a\in A$, $a(x_i)=a(y_i)$, but $b(x_i)\neq b(y_i)$, so $a(x_i)=a(y_i)=0$. Since $\{x_i\}_{i=1}^{\infty}$ is dense in $Y-\{z_0\}$, $a(y)=0$ for all $y\in Y$. Since Y is dense in X, $a(x)=0$ for all $x\in X$, contradicting $a\neq 0$. Thus Case 2 cannot occur, and the theorem follows from Case 1.

More should be expected from the stronger Condition 1 because it requires that for any $b\neq 0$ in B, no matter how "small" the support of b may be, there is an $a\in A$ with $0\neq ab\in A$. Thus for every non-void open subset U of X there is a non-zero $c\in A$ with support contained in U. This suggests that if Condition 1 is satisfied, then A must contain a "large" ideal of B. The following theorem makes this notion precise.

THEOREM 1.2. *Using the notation developed above, Condition* 1 *is satisfied if and only if A contains an essential ideal of B.*

Proof. If I is an essential ideal of B and $I\subset A$, then for every $b\neq 0$ in B, $Ib\neq\{0\}$, so Condition 1 holds.

Now suppose Condition 1 holds. Since the sum of ideals of B is again an ideal of B, there is an ideal I of B such that $I\subset A$, and, if J is any ideal of B with $J\subset A$, then $J\subset I$. Then $I=I_U$ for some open set U of X. If $\bar{U}\neq X$, let $X_0=X-\bar{U}$, $A_0=A\cap I_{X_0}$, $B_0=I_{X_0}$. If $b\in B_0$, then by Condition 1 there exists $a\in A$ such that $0\neq ab\in A$. Since $b\in B_0$, clearly $ab\in B_0\cap A=A_0$. Thus $0\neq(ab)^2=[a(ab)]b\in B_0$, i.e. the pair A_0, B_0 satisfies Condition 2; hence there is a non-zero ideal J of B_0 which is contained in A_0. Since B_0 is a closed ideal of B, J is an ideal of B, $J\subset A_0\subset A$ and $J\not\subset I$, contradicting the maximality of I. The theorem follows.

Theorems 1.1 and 1.2 are not always true if B is non-separable. For example, if $B=C(X)$, where X is the set of all ordinals less than or equal to the first uncountable ordinal, and $A=\{a\in B: a(\omega+2n)=a(\omega+2n+1)$ for all limit ordinals ω and positive integers $n\}$, then Theorem 1.1 fails. This example suggests that results in the non-separable case will require Conditions 1′ and 2′.

2. Results for Conditions 1 and 1′. The short proof of the first proposition is due to Uffe Haagerup.

PROPOSITION 2.1. *Condition* 1′ *holds only when* $N=M$.

Proof. Let $p\in N$ be a non-zero projection. It suffices to show that $p\in M$, and for this we need only show that there is a non-zero projection q in M with $q\leqq p$. (To see this let p_0 be the supremum of all projections q in M with $q\leqq p$. Since M is weakly closed, p_0 is a projection in M, and $(p-p_0)$ can't majorize a non-zero projection in M, so it must be 0.) By Condition 1′ let $a\in M$ with $0\neq ap\in M$. Then $0\neq pa*ap\in M$, so the range projection q of $pa*ap$ lies in M and $0\neq q\leqq p$ as required.

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In the last proof not only was it convenient to work with projections in N, but more important was the fact that M is weakly closed in N. This allowed us to reach a stronger conclusion from Condition 1′ than Condition 1 yielded even in the abelian case. In the non-abelian case we need to replace the ideals of $C_0(X)$ with something more general in order to see where Condition 1 leads. Hereditary C*-subalgebras are the obvious candidates. For any hereditary C*-subalgebra C of B there is an open projection $p \in B^{**}$ which is called the covering projection for C [**4**, pp. 77–78]. Comparing this with the abelian case, the open projection p covering C is analogous to the open set U within which an ideal I in $C_0(X)$ is supported. We can define C to be essential in B if $\{b \in B : bc = cb = 0 \text{ for all } c \in C\} = \{0\}$. In the non-abelian case, however, problems arise because Condition 1 is not right-left symmetric. The next theorem has hypotheses which are sufficient to imply Condition 1. We know from the abelian case that they are too strong to be necessary.

THEOREM 2.2. *If A is an essential hereditary* C*-*subalgebra of B with covering projection p and if p has finite codimension in B^{**}, then Condition* 1 *holds.*

Proof. If B is finite-dimensional, then A is essential if and only if $A = B$. Thus we can assume B is infinite-dimensional.

Let $b \in B$ and $p' = 1 - p$. For any $a \in A$ we have $ab \in A$ whenever $abp' = 0$. Let q be the range projection of bp' in B^{**}. Since p' has finite rank, so does q. Thus $q \vee p'$ has finite rank, and so, by [**1**, p. 28], $1 - (q \vee p') = p_0$ is open (and $p_0 \neq 0$ since B is infinite-dimensional). Consider two cases.

Case 1. $p_0 b \neq 0$. Then since $p_0 \leqq p$ and p_0 is open, there is a net $\{a_\alpha\} \subset \{c \in B : p_0 c p_0 = c\} \subset A$ such that $a_\alpha \to p_0$ weakly in B^{**} [**4**, p. 78]. Thus $a_\alpha b \neq 0$ for some α, and $a_\alpha b p' = 0$, since $a_\alpha q = 0$, so $a_\alpha b \in A$. Hence Condition 1 holds.

Case 2. $p_0 b = 0$. Then the range projection r of b is a finite rank projection in B^{**} which also lies in B, so $B \supset rBr = rB^{**}r$. However, by definition $q \leqq r \leqq p_0'$. By [**5**, p. 93] there is a unitary u in B such that $(uru^*) \wedge p' \neq 0$. But $(uru^*)B^{**}(uru^*) \subset B$, and this means that if $c = uru^*$, then $p'cp' = c$; hence $cA = Ac = \{0\}$. This contradicts the assumption that A is essential. The theorem then follows from Case 1.

It is reasonable to conjecture that (in the separable case) Condition 1 is equivalent to A containing an essential hereditary C*-subalgebra of B. The previous theorem supplies evidence in one direction and, of course, Theorem 1.2 suggests that both implications are valid. To get some more non-abelian evidence for the conjecture let H be an infinite-dimensional Hilbert space, $B(H)$ the algebra of all bounded operators on H and $K(H)$ the subalgebra of $B(H)$ consisting of all compact operators on H. Note that if $K(H) \subset B \subset B(H)$, then $K(H)$ is an essential, hereditary C*-subalgebra of B.

PROPOSITION 2.3. *If $K(H) \subset B \subset B(H)$, then Condition* 1 *implies that $K(H) \subset A$.*

Proof. Let $p \in K(H)$ be a rank-one projection. Since these projections generate $K(H)$, it suffices to prove that p lies in A. By Condition 1 we may choose $a \in A$ such that $0 \neq ap \in A$. Then $pa*ap \neq 0$ in A, and $pa*ap$ is a scalar multiple of p, so $p \in A$.

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Since the abelian case suggests that Condition 2 is the key to Condition 1, let us turn to that.

3. Results for Conditions 2 and 2′. Not surprisingly the results of this section are only suggestive, not definitive. First we look at Condition 2′ and try to find von Neumann algebras N for which the finite-dimensional result described in §0 can be generalized in a natural way. Theorem 3.1 is one example.

THEOREM 3.1. *If H is a Hilbert space and $N = B(H)$, then Condition 2′ holds if and only if $M \supset pB(H)p$ for some projection p with rank $(1-p) < \infty$.*

Proof. First assume that Condition 2′ holds. We first shall show that the commutant M' of M contains no projection which is infinite-dimensional in $B(H)$ and whose complement is infinite-dimensional in $B(H)$. If not, then M' will contain a projection q such that $vv^* = q$ and $v^*v = (1-q)$ for some partial isometry v in $B(H)$. Note that $(v+v^*) = u$ is a unitary operator. By Condition 2 there is some $0 \neq a \in M$ with $au \in M$. Thus $av^* = a(vq + v^*q) = (au)q = q(au) = aqu = aq(v+v^*) = av$. Thus $(av)(av)^* = (av)(av^*)^* = avva^* = 0$, and so $av = av^* = 0$. But this means $au = 0$, and so $a = 0$ since u is unitary. This contradiction means that M' is a finite-dimensional type 1 von Neumann algebra with center $\mathfrak{X}$, and M is also type 1 [**5**, p. 300]. Further, exactly one of the (non-unital) central projections (say p) in $\mathfrak{X}$ is infinite-dimensional. (If there are none but the identity, we already have $M = B(H)$.)

We claim that $pB(H)p = pMp$. This follows from the fact that $pM'p$ is a factor of type 1 (since it is finite-dimensional), and so, if it were not one-dimensional, it would contain a set $p_1, \ldots, p_n$ of orthogonal equivalent minimal projections with $p = \sum_{i=1}^{n} p_i$. As shown above each p_i must be finite-dimensional in $B(H)$ or have finite codimension in $B(H)$. This is clearly impossible if $n > 1$.

This reverse implication is more generally true as will be shown in Proposition 3.2.

PROPOSITION 3.2. *For any von Neumann algebra N, if p is a projection in N such that p is not equivalent to a subprojection of $(1-p)$ and $M \supset pNp$, then Condition 2′ holds.*

Proof. Fix $b \in N$. Let $pb(1-p) = v\,|pb(1-p)|$ be the polar decomposition of $pb(1-p)$. Since $v^*v \leqq (1-p)$, we cannot have $vv^* = p$ by hypothesis. Thus $0 \neq (p - vv^*) \in pNp \subset M$ and $(p - vv^*)pb(1-p) = 0$. If $a = p - vv^*$, we get

$$ab = (pap)(pbp + pb(1-p) + (1-p)bp + (1-p)b(1-p)) = papbp + papb(1-p) = papbp \in pNp \subset M.$$

Thus Condition 2 holds.

It is a reasonable conjecture that the converse of Proposition 3.2 holds in general, but the methods used in the special case of Theorem 3.1 will have to be replaced.

Now let us turn to Condition 2. If C is an hereditary C*-subalgebra of B with open covering projection p, then it is reasonable to conjecture that: if p is not equivalent to a

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sub-projection of $(1-p)$ and $C \subset A$, then Condition 2 holds. To prove this conjecture one could try to mimic the proof of Proposition 3.2, but a problem arises in the middle. Just because $(p-vv^*) \neq 0$ it does not follow that we can find $a \in C$ such that $a \neq 0$, and $apb(1-p)=0$ and $(p-vv^*)$ is not (in general) even in B but only in B^{**}. Similar, but greater difficulties stand in the way of the converse of the conjecture. However, there is one non-abelian case where both implications can be proved easily. This result is very close to Theorem 3.1.

THEOREM 3.3. *Let B be any C^*-subalgebra of $B(H)$, where H is a separable Hilbert space, such that $K(H) \subset B$. Then Condition 2 holds if and only if there is a projection p of finite codimension such that $pK(H)p \subset A$.*

Proof. If p exists as in the theorem and $pK(H)p \subset A$, let $b \in B$ and $pb(1-p) = v\,|pb(1-p)|$ be the polar decomposition of $pb(1-p)$. Since $(1-p)$ has finite rank, so does vv^*. Thus we may choose a non-zero projection $a \in K(H)$ with $a \leqq p - vv^*$, and, as in the proof of Proposition 3.2, $ab = papabp \in pK(H)p \subset A$; hence Condition 2 holds.

Now assume Condition 2 holds. Let $A_0 = A \cap K(H)$. We shall reduce the problem to the case $B = K(H)$ by showing that for any $b \in K(H)$ there is some $a \in A_0$ with $a \neq 0$ and $ab \in A_0$. Let $\{u_\alpha\}$ be the canonical approximate unit [**4**, p. 11] in A_0. We need only show that $u_\alpha c$ is eventually non-zero for every $c \neq 0$ in A, since Condition 2 gives, for every $b \in K(H)$, some $a_0 \in A$ with $a_0 \neq 0$ and $a_0 b \in A$; since $b \in K(H)$, if $a = u_\alpha a_0 \neq 0$, then $ab \in A_0$ and $a \in A_0$. Now suppose $0 \neq c \in A$ and $u_\alpha c = 0$ for all α. Thus $A_0 c = \{0\}$. Let $u_\alpha \nearrow x$ in $B(H)$. Since A_0 is an ideal of A, x is in the commutant A' of A, and $(1-x)c = c$. Since $c \notin A_0$, $(1-x)$ must have infinite rank, so we can, since H is separable, find $v \in K(H)$ such that the range of vv^* is dense in H and the range of v^*v is dense in $(1-x)H$. By Condition 2 there is some $0 \neq a \in A$ such that $av \in A$. Since the range of v is dense and $a \neq 0$, it follows that $0 \neq av \in A_0$ (since $v \in K(H)$). But $vx = 0$, and $0 = avx = \lim_\alpha avu_\alpha = av$, a contradiction.

We have now reduced to the case where $B = K(H)$, $A \subset B$ and Condition 2 holds. Mimicing the proof of Theorem 3.1 we can show that any projection in A' must be of finite rank or finite co-rank. (We must replace the partial isometry v of that proof with a compact operator as above.) Also (as in the proof of Theorem 3.1) A' contains a unique central projection p of infinite rank such that $pA'p$ is one-dimensional. Thus $pA''p = pB(H)p$. Since $A \subset K(H)$ and A acts irreducibly on pH, $pAp = pK(H)p$. Since $p \in A''$, $pAp \subset A$, and the theorem follows.

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