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DIFFERENCE EQUATIONS IN ABSTRACT SPACES

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Abstract

Existence results are presented for second order discrete boundary value problems in abstract spaces. Our analysis uses only Sadovskii's fixed point theorem.

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1. Introduction

This paper discusses the abstract discrete boundary value problem

$$(1.1)\qquad \begin{cases} \Delta^2 y(i-1) + f(i, y(i)) = 0, & i \in N \\ y(0) = 0, \qquad y(T+1) = 0. \end{cases}$$

Here $T \in \{1, 2, \ldots\}$, $N = \{1, 2, \ldots, T\}$, $N^+ = \{0, 1, \ldots, T+1\}$, E is a real Banach space with norm $\|.\|$, and $y : N^+ \to E$. We will assume throughout this paper that

$$(1.2)\qquad f : N \times E \to E \text{ is continuous.}$$

REMARK. Recall a map $f : N \times E \to E$ is continuous if it is continuous as a map of the topological space $N \times E$ into the topological space E (the topology on N will be the discrete topology).

Let $C(N^+, E)$ denote the class of maps w continuous on N^+ (discrete topology), with norm $\|w\|_0 = \max_{k \in N^+} \|w(k)\|$, that is, $C(N^+, E) = \{w; w : N^+ \to E\}$, which is a Banach space.

REMARK. Since N^+ is a discrete space then any mapping of N^+ to a topological space (in this case E) is continuous.

By a solution to (1.1) we mean a $w \in C(N^+, E)$ such that w satisfies (1.1) for $i \in N$ and w satisfies the boundary (Dirichlet) conditions. It is worth remarking here that in fact the Dirichlet boundary data could be replaced by Sturm Liouville boundary data and the results of this paper are again guaranteed; however since only minor adjustments are needed in the analysis we will as a result omit the details.

Agarwal in [1, 2] showed if $E = \mathbb{R}$ and if $\alpha \in C(N^+, E), \beta \in C(N^+, E)$ are respectively lower and upper solutions of (1.1) with $\alpha(i) \le \beta(i)$ for $i \in N$ then (1.1) has a solution y with $\alpha(i) \le y(i) \le \beta(i)$ for $i \in N$. In this paper by imposing a condition which coincides with the existence of lower and upper solutions in the scalar case we will show that the classical result in [1] can be extended to the Banach space setting. In fact our result will be new even in the finite dimensional setting, that is, when $E = \mathbb{R}^n, n > 1$. Some of the ideas in this paper were motivated by the papers of Frigon and O'Regan [5, 6] concerning initial value problems in the continuous case.

We now gather together some preliminaries which will be needed in Section 2. Let E be a Banach space and Ω_E the bounded subsets of E. Let $X \in \Omega_E$. The *diameter* of X is defined by

$$\mathrm{diam}(X) = \sup\{\|x - y\| : x, y \in X\}; \text{ here } \|.\| \text{ is the norm in } E.$$

The *Kuratowskii measure of non-compactness* is the map $\alpha : \Omega_E \to [0, \infty]$ defined by

$$\alpha(X) = \inf\{\epsilon > 0 : X \subseteq \bigcup_{i=1}^{n} X_i \text{ and } \mathrm{diam}(X_i) \le \epsilon\}; \text{ here } X \in \Omega_E.$$

Let E_1 and E_2 be two Banach spaces and let $F : Y \subseteq E_1 \to E_2$ be continuous and map bounded sets into bounded sets. We call such an F an *α-Lipschitzian map* if there is a constant $k \ge 0$ with $\alpha(F(X)) \le k\alpha(X)$ for all bounded sets $X \subseteq Y$. We also say F is a *Darbo* map if F is α-Lipschitzian with $k < 1$. Next we state a fixed point result due to Sadovskii [4].

THEOREM 1.1. *Let G be a closed convex subset of a Banach space E and let $F : G \to G$ be a bounded Darbo map. Then F has a fixed point in G.*

In [3] we proved the following result which will be needed in Section 2.

THEOREM 1.2. *Let $A \subseteq C(N^+, E)$ be bounded. Then*

(i) $\alpha(A) = \alpha(A(N^+))$;

(ii) $\alpha(A(N^+)) = \sup_{i \in N^+} \alpha(A(i))$

where

$$A(i) = \{\phi(i) : \phi \in A\} \quad \textit{and} \quad A(N^+) = \bigcup_{j \in N^+} \{\phi(j) : \phi \in A\}.$$

The semi-inner products on E are defined by

$$\langle x, y\rangle_+ = \|x\| \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t} \quad \text{and} \quad \langle x, y\rangle_- = \|x\| \lim_{t \to 0^-} \frac{\|x + ty\| - \|x\|}{t}.$$

The following properties are well known [7].

THEOREM 1.3. *Let E be a Banach space. Then*

(a) $|\langle x, y\rangle_+| \le \|x\|\|y\|$;
(b) $\langle y, x + \alpha y\rangle_+ = \langle y, x\rangle_+ + \alpha\|y\|^2$ *for all* $\alpha \in \mathbb{R}$;
(c) $\langle y, x\rangle_- \le \langle y, x\rangle_+$.

Finally we prove a very simple result which will be needed in Section 2.

THEOREM 1.4. *Let $x \in C(N^+, E)$ and $i \in \{0, \dots, T-1\}$. Then*

$$\|x(i+1)\|\Delta^2\|x(i)\| \ge \langle x(i+1), \Delta^2 x(i)\rangle_+.$$

PROOF. Now

$$\begin{aligned}
&\|x(i+1)\|\Delta^2\|x(i)\| - \langle x(i+1), \Delta^2 x(i)\rangle_+ \\
&\quad = (\|x(i+2)\| - 2\|x(i+1)\| + \|x(i)\|)\,\|x(i+1)\| \\
&\qquad - \langle x(i+1), x(i+2) - 2x(i+1) + x(i)\rangle_+ \\
&\quad = \|x(i+2)\|\|x(i+1)\| + \|x(i)\|\|x(i+1)\| \\
&\qquad - \langle x(i+1), x(i+2) + x(i)\rangle_+ \\
&\quad \ge \|x(i+2)\|\|x(i+1)\| + \|x(i)\|\|x(i+1)\| \\
&\qquad - \|x(i+1)\|\,(\|x(i+2)\| + \|x(i)\|) = 0.
\end{aligned}$$

2. Existence theory

In this section we use Theorem 1.1 to establish two existence results for the discrete boundary value problem (1.1).

THEOREM 2.1. *Suppose* (1.2) *holds. In addition assume*

$$\text{(2.1)} \qquad \begin{cases} \text{there exists } v \in C(N^+, E) \text{ and } M \in C(N^+, (0, \infty)) \text{ with} \\ \langle y - v(i), -f(i, y) - \Delta^2 v(i-1)\rangle_+ \ge M(i)\Delta^2 M(i-1) \\ \text{for all } i \in N \text{ and all } y \in E \text{ with } \|y - v(i)\| = M(i); \end{cases}$$

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$$\|v(0)\| \leq M(0) \text{ and } \|v(T+1)\| \leq M(T+1); \tag{2.2}$$

$$\begin{cases} \alpha(f(N \times A)) \leq k\alpha(A) \text{ for all bounded subsets } A \text{ of } E; \\ \text{where } k \geq 0 \text{ is a constant}; \end{cases} \tag{2.3}$$

$$rTk < 1; \text{ here } r = \sup_{i \in N^+} r_i \text{ and } r_i = \max_{j \in N} G(i,j); \tag{2.4}$$

and

$$\begin{cases} \text{for each } b > 0, \text{ there exists a constant } K_b \geq 0 \text{ with } \|f(j,u)\| \leq K_b, \\ \text{for all } j \in N \text{ and } u \in E \text{ with } \|u\| \leq b \end{cases} \tag{2.5}$$

are satisfied. Then (1.1) *has a solution* $y \in C(N^+, E)$ *with* $\|y(i) - v(i)\| \leq M(i)$ *for* $i \in N$.

REMARKS. (1) If E is finite dimensional then (2.3) (with $k = 0$), (2.4) and (2.5) are automatically satisfied.

(2) If $E = \mathbb{R}$ and if $\alpha, \beta \in C(N^+, \mathbb{R})$ are respectively lower and upper solutions of (1.1) (that is, $\Delta^2\alpha(i-1) + f(i, \alpha(i)) \geq 0$ for $i \in N, \alpha(0) \leq 0, \alpha(T+1) \leq 0$ and $\Delta^2\beta(i-1) + f(i, \beta(i)) \leq 0$ for $i \in N, \beta(0) \geq 0, \beta(T+1) \leq 0)$ and $\alpha(i) < \beta(i)$ for $i \in N$ then its easy to check that $v = (\alpha + \beta)/2$ and $M = (\beta - \alpha)/2$ satisfy (2.1) and (2.2) in Theorem 2.1.

PROOF. Consider the discrete boundary value problem

$$\begin{cases} \Delta^2 y(i-1) + f(i, p(i, y(i))) = 0, & i \in N \\ y(0) = 0, \quad y(T+1) = 0 \end{cases} \tag{2.6}$$

where

$$p(i,y) = \min\left\{1, \frac{M(i)}{\|y - v(i)\|}\right\} y + \left(1 - \left\{1, \frac{M(i)}{\|y - v(i)\|}\right\}\right) v(i),$$

that is,

$$p(i,y) = \begin{cases} y, & \text{if } \|y - v(i)\| \leq M(i) \\ M(i)\dfrac{y - v(i)}{\|y - v(i)\|} + v(i), & \text{if } \|y - v(i)\| > M(i) \end{cases}$$

is the radial retraction of E onto $\{y : \|y - v(i)\| \leq M(i)\}$.

REMARK. It is well known that $\|p(i, y_1) - p(i, y_2)\| \leq 2\|y_1 - y_2\|$ for all $y_1, y_2 \in E$ and 2 may be replaced by 1 if E is a Hilbert space.

We will now use Theorem 1.1 to show that (2.6) has a solution. Solving (2.6) is equivalent to finding a $y \in C(N^+, E)$ which satisfies

$$y(i) = \sum_{j=1}^{T} G(i, j) f(j, p(j, y(j))), i \in N^+ \tag{2.7}$$

where

$$G(i, j) = \begin{cases} \dfrac{j(T+1-i)}{T+1}, & 0 \leq j \leq i-1 \\ \dfrac{i(T+1-j)}{T+1}, & i \leq j \leq T+1. \end{cases}$$

Define the operator $S : C(N^+, E) \to C(N^+, E)$ by setting

$$Sy(i) = \sum_{j=1}^{T} G(i, j) f(j, p(j, y(j))).$$

Now (2.7) is equivalent to the fixed point problem $y = Sy$. We claim $S : C(N^+, E) \to C(N^+, E)$ is a Darbo map. To see this let Ω be a bounded subset of $C(N^+, E)$. Fix $i \in N^+$. Then

$$\begin{aligned} \alpha(S\Omega(i)) &= \alpha\left(\left\{\sum_{j=1}^{T} G(i, j) f(j, p(j, y(j))) : y \in \Omega\right\}\right) \\ &= \alpha\left(T\overline{\text{co}}\{G(i, j) f(j, p(j, y(j))) : y \in \Omega, j \in N\}\right) \\ &= T\alpha\left(\{G(i, j) f(j, p(j, y(j))) : y \in \Omega, j \in N\}\right) \\ &= [Tr_i]\alpha\left(\{f(j, p(j, y(j))) : y \in \Omega, j \in N\}\right) \\ &\leq [r_i T]\alpha\left(f\left(N \times \overline{\text{co}}\left(\Omega(N) \bigcup v(N)\right)\right)\right) \end{aligned}$$

since if $y \in \Omega$ and $j \in N$ we have

$$p(j, y(j)) = \lambda_j y(j) + (1 - \lambda_j) v(j) \in \overline{\text{co}}\left(\Omega(N) \bigcup v(N)\right)$$

where

$$\lambda_j = \min\{1, M(j)/\|y(j) - v(j)\|\}.$$

This together with (2.3) implies

$$\begin{aligned} \alpha(S\Omega(i)) &\leq [r_i Tk]\alpha\left(\overline{\text{co}}\left(\Omega(N) \bigcup v(N)\right)\right) = [r_i Tk]\alpha\left(\Omega(N) \bigcup v(N)\right) \\ &= [r_i Tk]\alpha(\Omega(N)) \leq [rTk]\alpha\left(\Omega(N^+)\right) = [rTk]\alpha(\Omega). \end{aligned}$$

Now Theorem 1.2 implies

$$\alpha(S\Omega) = \sup_{i\in N^+} \alpha(S\Omega(i)) \le [rTk]\alpha(\Omega).$$

Since $rkT < 1$ then $S : C(N^+, E) \to C(N^+, E)$ is a Darbo map. Also (2.5) implies S is a bounded map. Sadovskii's fixed point theorem (Theorem 1.1) guarantees that S has a fixed point. Consequently (2.6) has a solution $y \in C(N^+, E)$.

It remains to show $\|y(i) - v(i)\| \le M(i)$ for $i \in N$. If this is not true then

$$r(i) = \|y(i) - v(i)\| - M(i)$$

attains a positive global maximum at say $m \in N$, and we may assume without loss of generality that $r(m) > r(m-1)$. Thus $r(m) > r(m-1)$ and $r(m) \ge r(m+1)$ implies

$$r(m+1) - 2r(m) + r(m-1) < 0.$$

Consequently

$$\Delta^2 r(m-1) < 0. \tag{2.8}$$

On the other hand since $r(m) > 0$ we have, using Theorem 1.4 and assumption (2.1),

$$\begin{aligned}
\Delta^2 r(m-1) &= \Delta^2\|y(m-1) - v(m-1)\| - \Delta^2 M(m-1)\\
&\ge \frac{\langle y(m) - v(m), \Delta^2(y(m-1) - v(m-1))\rangle_+}{\|y(m) - v(m)\|} - \Delta^2 M(m-1)\\
&= \frac{\langle p(m, y(m)) - v(m), \Delta^2 y(m-1) - \Delta^2 v(m-1)\rangle_+}{M(m)} - \Delta^2 M(m-1)\\
&= \frac{\langle p(m, y(m)) - v(m), -f(m, p(m, y(m))) - \Delta^2 v(m-1)\rangle_+}{M(m)} - \Delta^2 M(m-1)\\
&\ge \Delta^2 M(m-1) - \Delta^2 M(m-1) = 0.
\end{aligned}$$

This contradicts (2.8). Thus $\|y(i) - v(i)\| \le M(i)$ for $i \in N$ and we are finished.

In fact it is also possible to discuss the case when $M(i)$, in (2.1), may take on the value zero.

THEOREM 2.2. *Suppose* (1.2) *holds. In addition assume*

$$\begin{cases} \text{there exists } v \in C(N^+, E) \text{ and } M \in C(N^+, [0,\infty)) \text{ with}\\ \langle y - v(i), -f(i, y) - \Delta^2 v(i-1)\rangle_+ \ge M(i)\Delta^2 M(i-1)\\ \text{for all } i \in N \text{ and all } y \in E \text{ with } \|y - v(i)\| = M(i) \text{ and } M(i) \ne 0 \end{cases} \tag{2.9}$$

and

$$(2.10)\quad \begin{cases} \textit{there exists } v \textit{ and } M \textit{ as in } (2.9) \textit{ with} \\ \dfrac{\langle y - v(i), -f(i, v(i)) - \Delta^2 v(i-1)\rangle_+}{\|y - v(i)\|} \geq \Delta^2 M(i-1) \\ \textit{for all } i \in N \textit{ and all } y \in E \textit{ with } \|y - v(i)\| > M(i) \textit{ and } M(i) = 0 \end{cases}$$

are satisfied. Also suppose (2.2), (2.3), (2.4) *and* (2.5) *hold. Then* (1.1) *has a solution* $y \in C(N^+, E)$ *with* $\|y(i) - v(i)\| \leq M(i)$ *for* $i \in N$.

REMARK. If $E = \mathbb{R}$ and if $\alpha, \beta \in C(N^+, \mathbb{R})$ are respectively lower and upper solutions of (1.1) with $\alpha(i) \leq \beta(i)$ for $i \in N$ then its easy to check that $v = (\alpha + \beta)/2$ and $M = (\beta - \alpha)/2$ satisfy (2.9) and (2.10) in Theorem 2.2. Consequently a special case of Theorem 2.2 is the classical result of Agarwal in [1].

PROOF. As in Theorem 2.1, (2.6) has a solution $y \in C(N^+, E)$. Let $r(i)$ and m be as in Theorem 2.1 and once again (2.8) is true. On the other hand if $M(m) > 0$ then exactly the same argument as in Theorem 2.1 establishes a contradiction. Next if $M(m) = 0$ then Theorem 1.4 and assumption (2.10) yield

$$\begin{aligned} \Delta^2 r(m-1) &\geq \frac{\langle y(m) - v(m), \Delta^2 y(m-1) - \Delta^2 v(m-1)\rangle_+}{\|y(m) - v(m)\|} - \Delta^2 M(m-1) \\ &= \frac{\langle y(m) - v(m), -f(m, p(m, y(m))) - \Delta^2 v(m-1)\rangle_+}{\|y(m) - v(m)\|} - \Delta^2 M(m-1) \\ &= \frac{\langle y(m) - v(m), -f(m, v(m)) - \Delta^2 v(m-1)\rangle_+}{\|y(m) - v(m)\|} - \Delta^2 M(m-1) \\ &\geq 0. \end{aligned}$$

In both cases we contradict (2.8) and so the proof is complete.

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