

THEOREMS OF LEGENDRE TYPE FOR OVERPARTITIONS

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Abstract

Formulas evaluating differences of integer partitions according to the parity of the parts are referred to as Legendre theorems. In this paper we give some formulas of Legendre type for overpartitions.

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1. Introduction

Throughout this paper, q denotes a complex number satisfying $|q| < 1$, m and n denote nonnegative integers, \mathbb{N} is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and \mathbb{Z} is the set of integers. We use the standard notation for q -series [2, 5]:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j),$$
$$(a_1, \dots, a_k; q)_n = \prod_{j=1}^k (a_j; q)_n \quad \text{and} \quad (a_1, \dots, a_k; q)_\infty = \prod_{j=1}^k (a_j; q)_\infty.$$

It is well known that

$$\frac{1}{(q; q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n$$

and

$$(q; q)_\infty = \sum_{n=0}^{\infty} (p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n))q^n,$$

where $p(n)$ denotes the number of partitions of n and $p_e(\mathcal{D}, n)$ and $p_o(\mathcal{D}, n)$ respectively denote the number of partitions of n into an even and an odd number of distinct parts (see [2]).

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An overpartition [4, 8] of n is a partition of n where the first occurrence of each part may be overlined. The number of overpartitions of n , written $\overline{p}(n)$, has the generating function

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}.$$

Note that overlined parts in overpartitions are distinct by definition. We say that an overpartition has distinct parts if its nonoverlined parts are also distinct. For example, letting $\overline{p}_d(n)$ denote the number of distinct overpartitions of n , we have $\overline{p}_d(4) = 9$ enumerating $4, \overline{4}, 3 + 1, \overline{3} + 1, 3 + \overline{1}, \overline{3} + \overline{1}, \overline{2} + 2, 2 + \overline{1} + 1, \overline{2} + \overline{1} + 1$.

Perhaps the most famous example of a partition difference is Legendre’s celebrated result [7] stating that

$$p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n) = \begin{cases} (-1)^k & \text{if } n = k(3k \pm 1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Formulas of type (1.1) are referred to as Legendre theorems. Recently, Andrews and Yee in [3] gave Legendre theorems for a variety of overpartition differences. For instance, letting $TH(n)$ denote the number of overpartitions of n in which there is both an overlined and a nonoverlined largest part and letting $THE(n)$ denote the number of TH -overpartitions with an even number of parts minus the number with an odd number of parts, the authors proved that

$$THE(n) = \begin{cases} (-1)^n(2k - 1) & \text{if } k^2 < n < (k + 1)^2 \text{ for some } k \in \mathbb{N}, \\ (-1)^n(2k - 2) & \text{if } n = k^2 \text{ for some } k \in \mathbb{N}. \end{cases} \tag{1.2}$$

For other overpartition differences and Legendre type results we refer to Kim *et al.* [6] and Lovejoy [9].

In this note we give Legendre theorems for overpartitions. Surprisingly, our formulas are similar to (1.2), with the conditions involving squares k^2 replaced by conditions involving triangular numbers $\binom{k}{2}$ (see, for instance, Corollary 2.6).

Our proofs combine basic facts from the theory of hypergeometric q -series with the Bailey pair machinery. From [1, 13], a pair of sequences $(\alpha_n, \beta_n)_{n \geq 0}$ is called a Bailey pair relative to a (or relative to (a, q) to avoid confusion) if

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}.$$

The paper is organised as follows. In Section 2 we gather together the definitions and state our main results. Sections 3–5 are devoted to the proofs of the main theorems and their corollaries. Finally, in Section 6 we close with some remarks and comments suggested by this work.

2. Definitions and statement of results

DEFINITION 2.1. For any positive integer n , let $A(n)$ denote the number of overpartitions π of n into distinct parts with a unique odd nonoverlined part $u(\pi)$ such that

$u(\pi)$ does not also occur overlined. Let $A_0(n)$ (respectively, $A_1(n)$) denote the number of overpartitions counted by $A(n)$ in which the number of even nonoverlined parts plus the number of overlined parts which are greater than $u(\pi)$ is even (respectively, odd) and let

$$A'(n) = A_0(n) - A_1(n).$$

By letting the term q^{2n-1} generate the unique nonoverlined odd part, the term $(q^2; q^2)_\infty$ generate the other nonoverlined parts and the term $(-q; q)_{2n-2}(q^{2n}; q)_\infty$ generate the overlined parts, it is directly verified that

$$\sum_{n=2}^\infty A'(n)q^n = (q^2; q^2)_\infty \sum_{n=1}^\infty q^{2n-1}(-q; q)_{2n-2}(q^{2n}; q)_\infty. \tag{2.1}$$

For example, we have $A(6) = 6$ with relevant partitions

$$5 + \bar{1}, \bar{5} + 1, 3 + 2 + \bar{1}, \bar{3} + 2 + 1, 3 + \bar{2} + \bar{1}, \bar{3} + \bar{2} + 1;$$

we have $A_0(6) = 4$ counting $5 + \bar{1}, \bar{3} + 2 + 1, 3 + \bar{2} + \bar{1}, \bar{3} + \bar{2} + 1$; we have $A_1(6) = 2$ counting $\bar{5} + 1, 3 + 2 + \bar{1}$; and thus $A'(6) = 4 - 2 = 2$.

THEOREM 2.2. *We have*

$$\sum_{n=1}^\infty A'(n)q^n = \sum_{n=0}^\infty (n+1) \left(q^{\binom{2n+2}{2}} - q^{\binom{2n+3}{2}} \right).$$

COROLLARY 2.3. *For any positive integer n we have*

$$A'(n) = \begin{cases} (-1)^k \left\lceil \frac{k-1}{2} \right\rceil & \text{if } n = \binom{k}{2} \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lceil x \rceil$ denotes the least integer which is greater than or equal to x .

We now introduce our second partition difference.

DEFINITION 2.4. For any positive integer $n > 1$, let $B(n)$ be the number of overpartitions π of n in which there is a smallest overlined part $\bar{s}(\pi)$ which is even and the nonoverlined parts are greater than or equal to $\bar{s}(\pi)/2$ such that the nonoverlined even parts which are greater than or equal to $\bar{s}(\pi)$ are distinct and the remaining nonoverlined parts occur zero times or twice. Let $B_0(n)$ (respectively, $B_1(n)$) denote the number of B -partitions in which the number of parts is even (respectively, odd) and let

$$B'(n) = B_1(n) - B_0(n).$$

Then one can easily see that

$$\sum_{n=2}^\infty B'(n)q^n = \sum_{n=1}^\infty q^{2n}(q^{2n+1}; q)_\infty (q^{2n}; q^2)_\infty (q^{2n}; q^2)_n, \tag{2.2}$$

where the term $q^{2n}(q^{2n+1}; q)_\infty$ generates the overlined parts and $(q^{2n}; q^2)_\infty (q^{2n}; q^2)_n$ generates the nonoverlined parts.

For example, we find $B(6) = 4$ enumerating $\bar{6}, 4 + \bar{2}, \bar{4} + \bar{2}, \bar{2} + 2 + 1 + 1$, from which we see that $B_0(6) = B_1(6) = 2$, and therefore $B'(6) = 0$.

THEOREM 2.5. *We have*

$$(1 - q) \sum_{n=2}^{\infty} B'(n)q^n = q^2 + \sum_{n=3}^{\infty} (-1)^n (q^{\binom{n}{2}+1} + q^{\binom{n}{2}}).$$

COROLLARY 2.6. *For any integer $n \geq 2$, we have*

$$B'(n) = \begin{cases} (-1)^k & \text{if } \binom{k}{2} < n < \binom{k+1}{2} \text{ for some } k \in \mathbb{N}, \\ 0 & \text{if } n = \binom{k}{2}. \end{cases}$$

We now deal with the third type of partition difference.

DEFINITION 2.7. For any positive integer $n > 1$, let $C(n)$ be the number of overpartitions π of n such that there is a smallest overlined part $\bar{s}(\pi)$ which is odd, any other overlined part is even, and the nonoverlined parts are greater than or equal to $(\bar{s}(\pi) + 1)/2$ such that the nonoverlined parts greater than or equal to $\bar{s}(\pi)$ are distinct and the remaining nonoverlined parts occur zero times or twice. Let $C_0(n)$ (respectively, $C_1(n)$) denote the number of C -partitions in which the number of parts is even (respectively, odd) and let

$$C'(n) = C_1(n) - C_0(n).$$

Then one can readily check that

$$\sum_{n=2}^{\infty} C'(n)q^n = \sum_{n=1}^{\infty} q^{2n-1} (q^{2n}; q^2)_{\infty} (q^{2n-1}; q)_{\infty} (q^{2n}; q^2)_{n-1}, \tag{2.3}$$

where the term $q^{2n-1} (q^{2n}; q^2)_{\infty}$ generates the overlined summands and the term $(q^{2n-1}; q)_{\infty} (q^{2n}; q^2)_{n-1}$ generates the nonoverlined summands.

For example, we have $C(6) = 7$ with the following relevant partitions:

$$5 + \bar{1}, 4 + \bar{1} + 1, \bar{4} + \bar{1} + 1, \bar{3} + 3, 3 + \bar{2} + \bar{1}, 3 + 2 + \bar{1}, 2 + \bar{2} + \bar{1} + 1.$$

We have $C_0(6) = 4$ counting $4 + \bar{1} + 1, \bar{4} + \bar{1} + 1, 3 + \bar{2} + \bar{1}, 3 + 2 + \bar{1}$ and $C_1(6) = 3$ counting $5 + \bar{1}, \bar{3} + 3, 2 + \bar{2} + \bar{1} + 1$, and therefore $C'(6) = 4 - 3 = 1$.

THEOREM 2.8. *We have*

$$(1 - q) \sum_{n=1}^{\infty} C'(n)q^n = q \left(1 + 2 \sum_{n=2}^{\infty} (-1)^{n+1} q^{\binom{n}{2}} \right).$$

COROLLARY 2.9. *Let $n \geq 1$ be a positive integer and let k be the unique positive integer such that $\binom{k}{2} < n - 1 \leq \binom{k+1}{2}$. Then we have*

$$C'(n) = (-1)^{k+1}.$$

3. Proof of Theorem 2.2

We shall require the following lemma which is a variant of a result given in Lovejoy [10, (1.7)].

LEMMA 3.1. *If (α_n, β_n) is a Bailey pair relative to (a^2q^2, q^2) , then*

$$\sum_{r,n=0}^{\infty} a^{2n} q^{2n^2+4nr+3n+2r} \frac{1 + aq^{2n+2r+2}}{1 - aq^{2r+1}} \alpha_r = \frac{(a^2q^4; q^2)_{\infty} (q^2; q^2)_{\infty}}{(aq; q)_{\infty}} \sum_{n=0}^{\infty} q^{2n} (aq; q)_{2n} \beta_n.$$

PROOF. Let (α_n, β_n) be a Bailey pair relative to (a^2q^2, q^2) . Then by [10, (1.7)],

$$\sum_{r,n=0}^{\infty} \frac{a^n q^{\binom{n+1}{2}+2nr+n+2r}}{1 - aq^{2r+1}} \alpha_r = \frac{(a^2q^4; q^2)_{\infty} (q^2; q^2)_{\infty}}{(aq; q)_{\infty}} \sum_{n=0}^{\infty} q^{2n} (aq; q)_{2n} \beta_n.$$

Separating the terms according to the parity of n , the left-hand side becomes

$$\begin{aligned} & \sum_{r,n=0}^{\infty} \frac{a^{2n} q^{n(2n+1)+4nr+2n+2r}}{1 - aq^{2r+1}} \alpha_r + \sum_{r,n=0}^{\infty} \frac{a^{2n+1} q^{(n+1)(2n+1)+4nr+2n+4r+1}}{1 - aq^{2r+1}} \alpha_r \\ &= \sum_{r,n=0}^{\infty} a^{2n} q^{2n^2+4nr+3n+2r} \frac{1 + aq^{2n+2r+2}}{1 - aq^{2r+1}} \alpha_r, \end{aligned}$$

which clearly gives the desired identity. □

We need the following Bailey pair relative to (q^2, q^2) (Slater [11, (F2)]):

$$\alpha_n = q^{2n^2+n} \frac{1 + q^{2n+1}}{1 + q}, \quad \beta_n = \frac{1}{(q^3; q^2)_n (q^2; q^2)_n}. \tag{3.1}$$

By Lemma 3.1 applied to the Bailey pair (3.1) with $a = -1$ we derive

$$\begin{aligned} & \sum_{r,n=0}^{\infty} q^{2n^2+4rn+3n+2r} \frac{1 - q^{2n+2r+2}}{1 + q^{2r+1}} q^{2r^2+r} \frac{1 + q^{2r+1}}{1 + q} \\ &= \frac{(q^4; q^2)_{\infty} (q^2; q^2)_{\infty}}{(-q; q)_{\infty}} \sum_{n=0}^{\infty} q^{2n} \frac{(-q; q)_{2n}}{(q^3; q^2)_n (q^2; q^2)_n}, \end{aligned}$$

which simplifies to

$$\begin{aligned} & \sum_{r,n=0}^{\infty} q^{2n^2+4rn+2r^2+3n+3r} (1 - q^{2r+2n+2}) \\ &= (1 + q) \frac{(-q^2; q)_{\infty} (q^2; q)_{\infty} (q^2; q^2)_{\infty}}{(-q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{2n} (-q; q)_{2n}}{(q^2; q)_{2n}} \end{aligned}$$

$$\begin{aligned}
 &= (q^2; q)_\infty (q^2; q^2)_\infty \sum_{n=0}^\infty \frac{q^{2n}(-q; q)_{2n}}{(q^2; q)_{2n}} \\
 &= (q^2; q^2)_\infty \sum_{n=0}^\infty q^{2n}(-q; q)_{2n}(q^{2n+2}; q)_\infty.
 \end{aligned}$$

This combined with (2.1) yields

$$\begin{aligned}
 \sum_{n=1}^\infty A'(n)q^n &= \sum_{r,n=0}^\infty (q^{2(n+r)^2+3(n+r)+1} - q^{2(n+r)^2+5(n+r)+3}) \\
 &= \sum_{m=0}^\infty (m+1)(q^{2m^2+3m+1} - q^{2m^2+5m+3}) = \sum_{m=0}^\infty (m+1)(q^{\binom{2m+1}{2}} - q^{\binom{2m+3}{2}}),
 \end{aligned}$$

which is the desired formula.

4. Proof of Theorem 2.5 and Corollary 2.6

We first prove a lemma.

LEMMA 4.1. *We have*

$$\begin{aligned}
 &\sum_{r,n=0}^\infty q^{2(n+r)^2+4(n+r)+n+2}(1 - q^{2(r+n)+3})(1 - q^{2r+2}) \\
 &= \frac{-1}{(1 - q)} \left(q + (1 + q) \sum_{m=1}^\infty (-1)^m q^{m(m+1)/2} \right).
 \end{aligned}$$

PROOF. We start by noting the following elementary fact:

$$\sum_{m=0}^\infty q^{2m^2+3m+1}(1 - q^{2m+2}) = \sum_{m=1}^\infty (-1)^{m+1} q^{m(m+1)/2}. \tag{4.1}$$

With the help of (4.1),

$$\begin{aligned}
 &\sum_{r,n=0}^\infty q^{2(n+r)^2+4(n+r)+n+2}(1 - q^{2(r+n)+3})(1 - q^{2r+2}) \\
 &= \sum_{m=0}^\infty q^{2m^2+4m+2}(1 - q^{2m+3}) \sum_{n=0}^m q^n(1 - q^{2(m-n)+2}) \\
 &= \frac{1}{1 - q} \left(\sum_{m=0}^\infty q^{2m^2+4m+2}(1 - q^{2m+3})(1 - q^{m+1}) - \sum_{m=0}^\infty q^{2m^2+5m+4}(1 - q^{2m+3})(1 - q^{m+1}) \right) \\
 &= \frac{1}{1 - q} \sum_{m=0}^\infty q^{2m^2+4m+2}(1 - q^{2m+3})(1 - q^{m+1})(1 - q^{m+2})
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1-q} \sum_{m=0}^{\infty} q^{2m^2+4m+2} (1-q^{4m+6}) - \frac{1+q}{1-q} \sum_{m=0}^{\infty} q^{2m^2+5m+3} (1-q^{2m+3}) \\
 &= \frac{q^2}{1-q} + \frac{1+q}{1-q} \sum_{m=0}^{\infty} q^{2m^2+7m+6} - \frac{1+q}{1-q} \left(\sum_{m=0}^{\infty} q^{2m^2+3m+1} + \sum_{m=1}^{\infty} (-1)^m q^{m(m+1)/2} \right) \\
 &= \frac{q^2}{1-q} - \frac{1+q}{1-q} \sum_{m=0}^{\infty} q^{2m^2+3m+1} (1-q^{4m+5}) - \frac{1+q}{1-q} \sum_{m=1}^{\infty} (-1)^m q^{m(m+1)/2} \\
 &= \frac{q^2}{1-q} - \frac{q(1+q)}{1-q} - \frac{1+q}{1-q} \sum_{m=1}^{\infty} (-1)^m q^{m(m+1)/2} \\
 &= \frac{-q}{1-q} - \frac{1+q}{1-q} \sum_{m=1}^{\infty} (-1)^m q^{m(m+1)/2}.
 \end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 2.5. We need the following Bailey pair relative to (q^4, q^2) which can be found in Lovejoy [10]:

$$\alpha_n = q^{2n^2+2n} \frac{1-q^{4n+4}}{1-q^4}, \quad \beta_n = \frac{1}{(q^4; q^2)_n (q^2; q^2)_n}. \tag{4.2}$$

We apply Lemma 3.1 to the Bailey pair (4.2) with $a = -q$. This yields, after basic simplification,

$$\begin{aligned}
 &\frac{1}{1-q^4} \sum_{r,n=0}^{\infty} q^{2n^2+4nr+2r^2+5n+4r} (1-q^{2r+2})(1-q^{2r+2n+3}) \\
 &= \frac{(q^6; q^2)_{\infty} (q^2; q^2)_{\infty}}{(-q^2; q)_{\infty}} \sum_{n=0}^{\infty} q^{2n} \frac{(-q^2; q)_{2n}}{(q^4; q^2)_n (q^2; q^2)_n}.
 \end{aligned}$$

Rearranging this identity, we successively get

$$\begin{aligned}
 &\sum_{r,n=0}^{\infty} q^{2n^2+4nr+2r^2+5n+4r} (1-q^{2r+2})(1-q^{2r+2n+3}) \\
 &= \frac{(q^4; q^2)_{\infty} (q^2; q^2)_{\infty}}{(-q^2; q)_{\infty}} \sum_{n=0}^{\infty} q^{2n} \frac{(-q^2; q)_{2n}}{(q^4; q^2)_n (q^2; q^2)_n} \\
 &= \sum_{n=0}^{\infty} q^{2n} \frac{(q^{2n+4}; q^2)_{\infty} (q^{2n+2}; q^2)_{\infty}}{(-q^{2n+2}; q)_{\infty}} \\
 &= \sum_{n=0}^{\infty} q^{2n} \frac{(q^{2n+2}; q)_{\infty} (q^{2n+4}; q^2)_{\infty} (q^{2n+2}; q^2)_{\infty}}{(q^{4n+4}; q^2)_{\infty}}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} q^{2n} (q^{2n+2}; q)_{\infty} (q^{2n+2}; q^2)_{\infty} (q^{2n+4}; q^2)_n \\
 &= \sum_{n=0}^{\infty} q^{2n} (q^{2n+3}; q)_{\infty} (q^{2n+2}; q^2)_{\infty} (q^{2n+2}; q^2)_{n+1} \\
 &= \frac{1}{q^2} \sum_{n=1}^{\infty} q^{2n} (q^{2n+1}; q)_{\infty} (q^{2n}; q^2)_{\infty} (q^{2n}; q^2)_n,
 \end{aligned}$$

which combined with (2.2) and Lemma 4.1 gives

$$(1 - q) \sum_{n=2}^{\infty} B'(n)q^n = \sum_{m=1}^{\infty} (-1)^{m+1} q^{m(m+1)/2+1} + \sum_{m=2}^{\infty} (-1)^{m+1} q^{m(m+1)/2},$$

which is equivalent to the desired formula. □

PROOF OF COROLLARY 2.6. It is obvious that $B'(2) = 1$. Furthermore, for $n \geq 3$ we derive from Theorem 2.5 that

$$\begin{aligned}
 B'(n) - B'(n - 1) &= \begin{cases} 1 & \text{if } n = \binom{2k}{2} \text{ or } n = 1 + \binom{2k}{2} \text{ for some } k \in \mathbb{N}_0, \\ -1 & \text{if } n = \binom{2k-1}{2} \text{ or } n = 1 + \binom{2k-1}{2} \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \\
 &= \begin{cases} (-1)^k & \text{if } n = \binom{k}{2} \text{ or } n = 1 + \binom{k}{2} \text{ for some } k \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Now the result follows by mathematical induction. □

5. Proof of Theorem 2.8 and Corollary 2.9

PROOF OF THEOREM 2.8. We need the following Bailey pair relative to (q^2, q^2) which is obtained in Warnaar [12]:

$$\alpha_n = q^{2n^2} \frac{1 - q^{4n+2}}{1 - q^2}, \quad \beta_n = \frac{1}{(q^2; q^2)_n^2}. \tag{5.1}$$

By Lemma 3.1 applied to the Bailey pair (5.1) with $a = -1$,

$$\begin{aligned}
 &\sum_{r,n=0}^{\infty} q^{2n^2+4rn+2r^2+3n+2r} (1 - q^{2r+1})(1 - q^{2r+2n+2}) \\
 &= (1 - q^2) \frac{(q^4; q^2)_{\infty} (q^2; q^2)_{\infty}}{(-q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{2n} (-q; q)_{2n}}{(q^2; q^2)_n^2} = \sum_{n=0}^{\infty} \frac{q^{2n} (q^{2n+2}; q^2)_{\infty}^2}{(-q^{2n+1}; q)_{\infty}} \\
 &= \sum_{n=0}^{\infty} \frac{q^{2n} (q^{2n+2}; q^2)_{\infty}^2 (q^{2n+1}; q)_{\infty}}{(q^{4n+2}; q^2)_{\infty}} = \sum_{n=0}^{\infty} q^{2n} (q^{2n+2}; q^2)_n (q^{2n+2}; q^2)_{\infty}^2 (q^{2n+1}; q^2)_{\infty},
 \end{aligned}$$

that is,

$$\begin{aligned} & \sum_{r,n=0}^{\infty} q^{2(n+r)^2+2(n+r)+n+1} (1 - q^{2r+1})(1 - q^{2r+2n+2}) \\ &= \sum_{n=1}^{\infty} q^{2n-1} (q^{2n-1}; q)_{\infty} (q^{2n}; q^2)_{\infty} (q^{2n}; q^2)_{n-1}. \end{aligned}$$

This combined with (2.3) implies

$$\begin{aligned} \sum_{n=1}^{\infty} C'(n)q^n &= \sum_{r,n=0}^{\infty} q^{2(n+r)^2+2(n+r)+n+1} (1 - q^{2(r+n)+2})(1 - q^{2r+1}) \\ &= \sum_{m=0}^{\infty} q^{2m^2+2m+1} (1 - q^{2m+2}) \sum_{n=0}^m q^n (1 - q^{2(m-n)+1}) \\ &= \frac{1}{1 - q} \sum_{m=0}^{\infty} q^{2m^2+2m+1} (1 - q^{2m+2})(1 - q^{m+1})^2 \\ &= \frac{1}{1 - q} \sum_{m=0}^{\infty} q^{2m^2+2m+1} (1 - q^{4m+4}) - \frac{2q}{1 - q} \sum_{m=0}^{\infty} q^{2m^2+3m+1} (1 - q^{2m+2}) \\ &= \frac{q}{1 - q} + \frac{2q}{1 - q} \sum_{m=1}^{\infty} (-1)^m q^{m(m+1)/2}, \end{aligned}$$

where the last equality follows from (4.1). Rearranging this result gives

$$(1 - q) \sum_{n=1}^{\infty} C'(n)q^n = q + 2 \sum_{m=1}^{\infty} (-1)^m q^{m(m+1)/2+1} = q + 2 \sum_{m=2}^{\infty} (-1)^{m+1} q^{\binom{m}{2}+1},$$

which completes the proof. □

PROOF OF COROLLARY 2.9. By Theorem 2.8,

$$C'(1)q + \sum_{n=2}^{\infty} (C'(n) - C'(n - 1))q^n = q + 2 \sum_{n=2}^{\infty} (-1)^{n+1} q^{\binom{n}{2}+1},$$

from which we get for $n > 1$,

$$C'(n) - C'(n - 1) = \begin{cases} 2(-1)^{k+1} & \text{if } n - 1 = \binom{k}{2} \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Combining this relation with the obvious fact that $C'(1) = 1$ we achieve the desired formula by mathematical induction. □

6. Concluding remarks

We first observe that one of the key features of this work is that our results of Legendre type do not use standard q -series formulas or the usual Bailey pair method to produce an identity between a q -series and a theta-type series.

The Bailey pair process is a strong tool to produce more Legendre theorems for integer partitions and overpartitions as double sums. In this work, we restricted ourselves to the application of a single Bailey lemma to only three Bailey pairs as we faced the challenge of rewriting double sums as single sums by employing other Bailey pairs. For instance, we are able to establish

$$\begin{aligned} & \sum_{n=1}^{\infty} q^{2n-1} (q^{2n}; q)_{\infty} (q^{2n}; q^2)_{\infty} (q^{2n}; q^2)_{n-1} \\ &= \sum_{r,n=0}^{\infty} q^{2(n+2r)^2+5n+4r+1} (1 - q^{2(n+2r)+3}) (1 - q^{4r+2}) \end{aligned} \quad (6.1)$$

whose left-hand side is similar to the right-hand side of (2.2). However, we did not succeed in finding a simpler formula for the double sum in (6.1).

Motivated by the formulas in Theorem 2.2 and Corollary 2.3 related to the series

$$(q^2; q^2)_{\infty} \sum_{n=1}^{\infty} q^{2n-1} (-q; q)_{2n-2} (q^{2n}; q)_{\infty},$$

it is natural to consider the similar q -series

$$(q; q)_{\infty} \sum_{n=1}^{\infty} q^{2n-1} (-q; q)_{2n-2} (q^{2n}; q)_{\infty} \quad \text{and} \quad (q; q^2)_{\infty} \sum_{n=1}^{\infty} q^{2n-1} (-q; q)_{2n-2} (q^{2n}; q)_{\infty}.$$

However, we were not able to find closed formulas for these two series. Furthermore, regarding the related series

$$\sum_{n=1}^{\infty} q^{2n-1} (-q; q)_{2n-2} (q^{2n}; q)_{\infty},$$

while we were able to prove by a combinatorial argument that its coefficients are nonnegative, we did not find such a proof based on hypergeometric q -series.

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