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Generalized k-Configurations

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Abstract. In this paper, we find configurations of points in n-dimensional projective space ($\mathbb{P}^n$) which simultaneously generalize both k-configurations and reduced 0-dimensional complete intersections. Recall that k-configurations in $\mathbb{P}^2$ are disjoint unions of distinct points on lines and in $\mathbb{P}^n$ are inductively disjoint unions of k-configurations on hyperplanes, subject to certain conditions. Furthermore, the Hilbert function of a k-configuration is determined from those of the smaller k-configurations. We call our generalized constructions k_D-configurations, where $D = \{d_1, \ldots, d_r\}$ (a set of r positive integers with repetition allowed) is the type of a given complete intersection in $\mathbb{P}^n$. We show that the Hilbert function of any k_D-configuration can be obtained from those of smaller k_D-configurations. We then provide applications of this result in two different directions, both of which are motivated by corresponding results about k-configurations.

1 Introduction

Recall that k-configurations in $\mathbb{P}^2$ were defined by Geramita, Harima and Shin as disjoint unions of points on lines and in $\mathbb{P}^n$ were inductively defined as disjoint unions of k-configurations in $\mathbb{P}^{n-1}$, (see [2]). The individual pieces in the union were connected by certain properties. Recall that $\alpha(\mathbb{X})$ is the smallest degree of a non-zero element in $I(\mathbb{X})$ and that $\sigma(\mathbb{X})$ is the smallest degree at which the Hilbert function of $\mathbb{X}$ stabilizes. Then, if $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2 \cup \cdots \cup \mathbb{X}_r$ where each $\mathbb{X}_i$ is contained in a hyperplane $\mathbb{H}_i$, we require $\sigma(\mathbb{X}_i) < \alpha(\mathbb{X}_{i+1})$. Intuitively, each piece must be small enough in relation to the piece in the next hyperplane. Geramita, Harima and Shin [2] have found a formula for the Hilbert function of a k-configuration in terms of those of the smaller k-configurations in the disjoint union.

In this paper, we generalize their construction by demanding that our configurations lie inside a given complete intersection $\mathbb{V}$ rather than only inside $\mathbb{P}^n$. Let $D = \{d_1, \ldots, d_r\}$ be the type of the given complete intersection. In order to mimic the construction of k-configurations, we define $\sigma_D(\mathbb{X})$ to be $\sigma(\mathbb{X})$ and $\alpha_D(\mathbb{X})$ to be the smallest degree of a form in $I(\mathbb{X})$ that is not in $\mathbb{V}$. We then demand at each stage that $\sigma_D(\mathbb{X}_i) < \alpha_D(\mathbb{X}_{i+1})$, where $\mathbb{X}_i$ and $\mathbb{X}_{i+1}$ are in two consecutive hyperplanes in the construction. We also define weak k_D-configurations in the same way, but with the condition $\sigma_D < \alpha_D$ replaced with $\sigma_D \leq \alpha_D$.

In Section 4, we provide two applications, one of which applies to weak k_D-configurations, while the other only applies to actual k_D-configurations. First, we determine the degree of each point in a weak k_D-configuration. This generalizes [7, Theorem 5.11]. Secondly, we consider sequences H which occur as the Hilbert function of some k_D-configuration. If $\mathbb{Y} \subseteq V(F_1, \ldots, F_r)$ is a set of points with Hilbert function H, and F is a hypersurface of degree d for which $(F_1, \ldots, F_r, F)$ is a radical

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ideal of height $r+1$, then we determine the maximal number of points on $F \cap \mathbb{Y}$ in terms only of $\mathcal{T} \leftrightarrow H$ and d. This generalizes [3, Theorem 3.15] and the proof follows closely that found there.

Section 3 was extracted from Section 4.2 of the author's Ph.D. thesis. The first application detailed in Section 4 was extracted from Section 4.6 of the thesis. The second application was not part of the thesis. This paper is the basis for two subsequent papers ([8] and [9]), both also based on the thesis, where we characterize the Hilbert functions of k_D-sequences by defining an analogue to O-sequences and where we study their minimal free resolutions.

2 Preliminary Results

Let k be a field of characteristic 0, $k = \overline{k}$. Let $R = k[x_0, \dots, x_n]$ be the polynomial ring in $n+1$ variables with the standard grading and let $\mathbb{P}^n$ denote n-dimensional projective space over k. All varieties will be reduced, although not necessarily irreducible.

Let $S_0 = k$, and let $S = R/I$ where I is a homogeneous ideal, so that S is a finitely generated $\mathbb{N}$-graded k-algebra. The sequence $H(S,i) := \dim_k S_i$ is called the *Hilbert function of S*. $H(S,i)$ will sometimes be denoted $H_S(i)$. If V is a variety, then the Hilbert function of V is the Hilbert function of $R/I(V)$, which we sometimes denote by H_V. If a sequence $a_1\ a_2\ a_3 \dots$ eventually becomes constant, say at the value a_j, we will denote this by $a_1\ a_2\ a_3 \dots a_j \longrightarrow$.

Definition 2.1 For any variety $\mathbb{X} \subseteq \mathbb{P}^n$, we put $\alpha_{\mathbb{X}} := \min\{i \mid H_{\mathbb{X}}(i) < H_{\mathbb{P}^n}(i)\}$. In addition, if $\mathbb{X}$ is a finite set of points, we put $\sigma_{\mathbb{X}} := \min\{i \mid \Delta H_{\mathbb{X}}(i) = 0\}$. We sometimes denote $\sigma_{\mathbb{X}}$ by σ_H, since σ depends only on H. Since α depends on H and n, we sometimes denote $\alpha_{\mathbb{X}}$ by $\alpha_{H,n}$ or, if n is understood, just α_H.

A complete intersection (CI) will always be reduced:

Definition 2.2 A *complete intersection* is a projective variety V such that the (radical) ideal of V is $I(V) = (F_1, \dots, F_r)$, where $F_1, \dots, F_r$ form a regular sequence. If we let $d_i := \deg F_i$, then $(d_1, \dots, d_r)$ is usually referred to as the *type* of the complete intersection, and we write $V := \mathrm{CI}(d_1, \dots, d_r)$ or $V := \mathrm{CI}(D)$, where D is the unordered set (repetition allowed) $\{d_1, \dots, d_r\}$. Furthermore, if $n \geq r$, we denote the Hilbert function of a $\mathrm{CI}(D)$ in $\mathbb{P}^n$ by $H_{D,n}$.

Remark 2.3 The notation $H_{D,n}$ makes sense because the Hilbert function of a $\mathrm{CI}(D)$ in $\mathbb{P}^n$ depends only on n and D (see for example the proof of [6, I Proposition 7.6] which proves that the Hilbert function of a hypersurface in $\mathbb{P}^n$ depends only on the degree of the hypersurface; one can then use induction on the length of the regular sequence).

Geramita, Maroscia and Roberts [4] have characterized the sequences which occur as the Hilbert function of a finite set of points in $\mathbb{P}^n$. The constructions they used in their proof came to be called k-configurations, which we now wish to define. Before we can do so, we need to define the notion of an n-type vector, defined in [2]. We simultaneously define the notion of a weak n-type vector.

Definition 2.4 ([2]) A 1-*type vector* is a vector of the form $\mathcal{T} = (d)$, where d is a positive integer. For such a 1-type vector $\mathcal{T}$, we define $\alpha(\mathcal{T}) = d = \sigma(\mathcal{T})$. A *weak 1-type vector* is a 1-type vector.

A *(weak) n-type vector* is a vector of the form $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_r)$, where $r \geq 1$, the $\mathcal{T}_i$ are (weak) $(n-1)$-type vectors, and $(\sigma(\mathcal{T}_i) \leq \alpha(\mathcal{T}_{i+1}))$ $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $1 \leq i \leq r-1$. Define $\alpha(\mathcal{T}) = r$, and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_r)$.

For convenience, we will denote the weak 2-type vector $((d_1), \dots, (d_m))$ by $(d_1, \dots, d_m)$. Thus, for example, the 3-type vector (((1),(2)),((1),(3),(4))) will be written as ((1,2),(1,3,4)). This does however create confusion since (d_1) could denote either the 2-type vector $((d_1))$ or the 1-type vector (d_1). If there is ever any confusion, we will explicitly state what we are referring to.

The importance of n-type vectors rests on the following result:

Theorem 2.5 ([2, Theorem 2.6]) *Let S_n denote the collection of Hilbert functions of all sets of points in $\mathbb{P}^n$. Then there is a* 1*-to-*1 *correspondence $S_n \leftrightarrow \{n\text{-type vectors}\}$ where if $H \in S_n$ and H corresponds to $\mathcal{T}$ (we write $H \leftrightarrow \mathcal{T}$) then $\alpha(H) = \alpha(\mathcal{T})$ and $\sigma(H) = \sigma(\mathcal{T})$.*

There is an inductive formula for obtaining a Hilbert function from its corresponding n-type vector, which we now state:

Theorem 2.6 ([2, Proof of Theorem 2.6]) *If $n = 1$ and $\mathcal{T} = (r)$, then $\mathcal{T} \leftrightarrow H = 1\ 2 \cdots r \rightarrow$. If $n > 1$ and $H \leftrightarrow \mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$ with $H_i \leftrightarrow T_i$, then $H(j) = H_r(j) + H_{r-1}(j-1) + \cdots + H_1(j-(r-1))$, where $H(t) = 0$ for $t < 0$.*

Remark 2.7 Let H be a Hilbert function of s points in $\mathbb{P}^n$. If $\mathcal{T} \leftrightarrow H$, then the sum of the 1-type vectors in $\mathcal{T}$ is s.

We are now ready to define the notions of k-configuration [2, Definition 4.1] and weak k-configuration. We caution the reader that the term "weak k-configuration" has been used in [5] to describe a slightly different object for points in $\mathbb{P}^2$; our notion is weaker than that of [5].

Definition 2.8 Let $\mathcal{T}$ be a (weak) n-type vector, $n \geq 1$. Then a (weak) k-configuration of type $\mathcal{T}$ is constructed in the following way:

$n = 1$: Then $\mathcal{T} = (d)$, and we choose any d distinct points of $\mathbb{P}^1$. We say that these d points form a (*weak*) *k-configuration of type* $\mathcal{T}$ in $\mathbb{P}^1$.

$n \geq 2$: Then $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$. Let $\mathbb{H}_1, \dots, \mathbb{H}_r$ be distinct hyperplanes in $\mathbb{P}^n$. By induction, we suppose we have a (weak) k-configuration $\mathbb{X}_i \subset \mathbb{H}_i$ of type $\mathcal{T}_i$ for each (weak) $(n-1)$-type vector $\mathcal{T}_i$. Suppose furthermore that $\mathbb{H}_i$ does not contain any point of $\mathbb{X}_j$ for any $j < i$. Then $\mathbb{X} = \bigcup_{i=1}^{r} \mathbb{X}_i$ is called a (*weak*) *k-configuration of type* $\mathcal{T}$.

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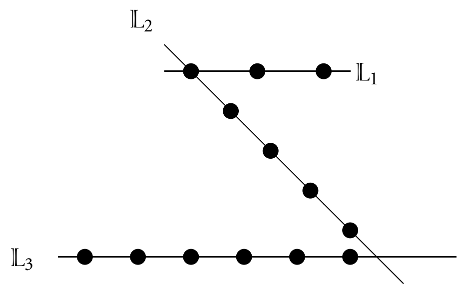
Example 2.9 In the diagram below, $\mathbb{X}_1$ consists of the two points of $\mathbb{L}_1$ that are not in $\mathbb{L}_2$, $\mathbb{X}_2$ consists of the five points of $\mathbb{L}_2$, and $\mathbb{X}_3$ consists of the six points of $\mathbb{L}_3$. Then $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2 \cup \mathbb{X}_3$ is a k-configuration of type $\mathcal{T} = (2, 5, 6)$. Notice that $\mathbb{L}_i$ does not contain a point of $\mathbb{X}_j$ for $j < i$, although $\mathbb{L}_1$ does contain a point of $\mathbb{X}_2$.



Notice that $\mathbb{X}$ is not a k-configuration of type $\mathcal{T} = (3,4,6)$ since $\mathbb{X}_1$ would have to consist of all 3 points on $\mathbb{L}_1$ and this includes a point of $\mathbb{L}_2$. This is not permitted.

Theorem 2.10 ([2, p. 21]) *If $\mathbb{X}$ is a k-configuration of type $\mathcal{T} \leftrightarrow H$, then $\mathbb{X}$ has Hilbert function H.*

A *separator* of a point $P \in \mathbb{X}$ from $\mathbb{X} \setminus P$ is a homogeneous polyonomial for which $f(P) \neq 0$ and $f(Q) = 0$ for all $Q \in \mathbb{X} \setminus P$. The *degree* of P in $\mathbb{X}$, denoted $\deg_{\mathbb{X}}(P)$, is the minimal $d \in \mathbb{N}$ for which there is a separator, homogeneous of degree d, of P from $\mathbb{X} \setminus P$. There is a formula for the degree of each point of a k-configuration. Before we can state this formula, we need to introduce some notation.

Definition 2.11 Let $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$. We define $\mathcal{T}^n := \mathcal{T}$ and $\mathcal{T}^{n-1} := \mathcal{T}_1$, the left-most $(n-1)$-type vector of $\mathcal{T}$. For $1 \leq j \leq n-2$, we define $\mathcal{T}^j := (\mathcal{T}^{j+1})_1$, where $\mathcal{T}^{j+1} = ((\mathcal{T}^{j+1})_1, \dots, (\mathcal{T}^{j+1})_{\alpha(\mathcal{T}^{j+1})})$. Thus, $\mathcal{T}^j$ is the left-most j-type vector of $\mathcal{T}$ for $1 \leq j \leq n$.

Example 2.12 Consider the two 4-type vectors $\mathcal{T}$ and $\mathcal{T}'$ where

$$\mathcal{T} = (((1)), ((1), (1, 2)))$$

and $\mathcal{T}' = (((2, 3), (1, 3, 4, 5)), ((1), (1, 2), \dots, (1, 2, 3, 4, 5, 6)))$. Then

$$\begin{array}{ll} \mathcal{T}^4 = \mathcal{T} & (\mathcal{T}')^4 = \mathcal{T}' \\ \mathcal{T}^3 = (((1))) & (\mathcal{T}')^3 = ((2, 3), (1, 3, 4, 5)) \\ \mathcal{T}^2 = ((1)) & (\mathcal{T}')^2 = (2, 3) \\ \mathcal{T}^1 = (1) & (\mathcal{T}')^1 = (2) \end{array}$$

Definition 2.13 Let $\mathcal{T}$ be an n-type vector. Define $t_n(\mathcal{T}) = 1$, and $t_{n-1}(\mathcal{T}) = \alpha(\mathcal{T})$. For $1 \leq k \leq n-2$, define $t_k(\mathcal{T}) := \alpha(\mathcal{T}^{k+1}) + \sum_{i=k+2}^{n} (\alpha(\mathcal{T}^i) - 1)$.

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Example 2.14 Let $\mathcal{T} = \big(\big((1,2,3),(1,2,3,4)\big),\big((1),(1,2),\ldots,(1,2,3,4,5)\big)\big)$. Then $t_4(\mathcal{T}) = 1$, $t_3(\mathcal{T}) = 2$, $t_2(\mathcal{T}) = 3$ and $t_1(\mathcal{T}) = 5$.

Definition 2.15 Each point P of a given k-configuration lies in a line which corresponds to a 1-type vector of the corresponding $\mathcal{T}$. We will denote α of this 1-type vector by $f(P)$. Remove from $\mathcal{T}$ everything to the left of the given 1-type vector $(f(P))$, and add the appropriate number of left brackets at the beginning. Call the resulting n-type vector $\mathcal{T}_P$.

Example 2.16 Let $\mathcal{T} = (\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3) = ((1),(1,2,4),(1,2,3,5,8))$. In a k-configuration of type $\mathcal{T}$, let P be any point on the second plane and on the second line in that plane. Then $f(P) = 2$, and $\mathcal{T}_P = ((2,4),(1,2,3,5,8))$.

Theorem 2.17 ([7, Theorem 5.11]) *Let $\mathbb{X}$ be a k-configuration of type $\mathcal{T}$, and $P \in \mathbb{X}$. Then $deg_{\mathbb{X}}(P) = f(P) + t_1(\mathcal{T}_P) - 2$.*

Remark 2.18 In Example 2.16, $f(P) = 2$, so $deg_{\mathbb{X}}(P) = t_1(\mathcal{T}_P) = 3$.

We will generalize Theorem 2.17 to determine the degree of each point in a weak k_D-configuration (Theorem 4.2).

Geramita, Harima and Shin have shown that among all sets of points in $\mathbb{P}^n$ with Hilbert function H, k-configurations have the most number of points on a hyperplane in [2] (or on a hypersurface of given degree in [3]). More precisely, for all sets of points $\mathbb{X}$ with $H_{\mathbb{X}} = H$, they consider all the subsets $\mathbb{Y}$ of $\mathbb{X}$ which lie on a hypersurface of $\mathbb{P}^n$ of degree $d \geq 1$ (assume that not all of $\mathbb{X}$ is in such a hypersurface, so that $d < \alpha(H)$). They referred to the set of all Hilbert functions of such subsets $\mathbb{Y}$ as $\mathrm{Sub}_d(H)$. They then partially ordered $\mathrm{Sub}_d(H)$ as follows: define $H_{\mathbb{Y}_1} \leq H_{\mathbb{Y}_2}$ if $H_{\mathbb{Y}_1}(i) \leq H_{\mathbb{Y}_2}(i)$ for every i. Under this partial ordering, $\mathrm{Sub}_d(H)$ has a unique maximal element.

Theorem 2.19 ([3, Theorem 3.15]) *Let $H \leftrightarrow \mathcal{T} = (\mathcal{T}_1,\ldots,\mathcal{T}_u)$. Let $d < u$ be a positive integer. Then $H' \leftrightarrow (\mathcal{T}_{u-d+1},\ldots,\mathcal{T}_u)$ is the unique maximal element of* $\mathrm{Sub}_d(H)$.

Furthermore, if $\mathbb{X}$ is any set of points having a subset $\mathbb{Y}$ with this extremal Hilbert function, they were able to determine the Hilbert function of $\mathbb{X} \setminus \mathbb{Y}$.

Theorem 2.20 ([3, Proposition 3.18]) *Let $\mathbb{X}$ be a set of points in $\mathbb{P}^n$ with $H_{\mathbb{X}} = H \leftrightarrow \mathcal{T} = (\mathcal{T}_1,\ldots,\mathcal{T}_u)$. Let $U \subset \mathbb{X}$ satisfy $H_U \leftrightarrow (\mathcal{T}_{u-d+1},\ldots,\mathcal{T}_u)$. Then $H_{\mathbb{X}\setminus U} \leftrightarrow (\mathcal{T}_1,\ldots,\mathcal{T}_{u-d})$.*

We generalize these results in Theorems 4.6 and 4.8.

3 Main Result

The main goal of this paper is to find configurations whose Hilbert functions can be built up from those of smaller such configurations. To do so, we will generalize both weak and actual k-configurations. We introduce some notation by attaching a number to each unordered set $D = \{d_1, \dots, d_r\}$ that will be used frequently.

Notation Let $D = \{d_1, \dots, d_r\}$. Then $\sigma(D) := d_1 + \cdots + d_r - r + 1$.

Just as k-configurations are constructed as unions of points on lines, our constructions will be unions of more general complete intersections. Letting

$$D = \{d_1, \dots, d_r\},$$

we begin by simultaneously defining n_D-type vectors and weak n_D-type vectors.

Definition 3.1 Let $D = \{d_1, \dots, d_r\}$ be a set of positive integers (repetition allowed). For $t \geq 1$, we define a (*weak*) t_D-*type vector* in the following way:

$t = 1$: A (weak) 1_D-type vector $\mathcal{T}$ is a vector of the form $\mathcal{T} = (e)$, where e is a positive integer. We define $\alpha_D(\mathcal{T}) := e$ and $\sigma_D(\mathcal{T}) := \sigma(D) + e - 1$.

$t > 1$: Let $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$. Then $\mathcal{T}$ is said to be a (weak) t_D-type vector if each $\mathcal{T}_i$ is a (weak) $(t-1)_D$-type vector and

$$(\sigma_D(\mathcal{T}_i) \leq \alpha_D(\mathcal{T}_{i+1})) \quad \sigma_D(\mathcal{T}_i) < \alpha_D(\mathcal{T}_{i+1}) \text{ for } 1 \leq i \leq u - 1.$$

We define $\alpha_D(\mathcal{T}) := u$ and $\sigma_D(\mathcal{T}) := \sigma_D(\mathcal{T}_u)$.

Example 3.2 Let $\mathcal{T}$ =((1,4),(2,6,9,12,15,19,23)) be a 3_D-type vector, where $D = \{2, 2\}$. Then $\alpha_D(\mathcal{T}) = 2$, and $\sigma_D(\mathcal{T}) = 25$.

Remark 3.3 If $D = \{1, 1, \dots, 1\}$, then $\sigma(D) = 1$ and a (weak) t_D-type vector is just a (weak) t-type vector.

Remark 3.4 Let $\mathcal{T} = ((e_1), \dots, (e_r))$ be a 2_D-type vector. As before, we write $\mathcal{T}$ as $(e_1, \dots, e_r)$ for simplicity.

Note that if we consider $\mathcal{T}$ as both a usual t-type vector and a t_D-type vector, we have $\sigma(\mathcal{T}) + \sigma(D) - 1 = \sigma_D(\mathcal{T})$. This is clear when $\mathcal{T}$ is a 1-type vector and then the general case follows from the inductive nature of $\sigma_D(\mathcal{T})$ and $\sigma(\mathcal{T})$ for $\mathcal{T}$ a t-type vector when $t > 1$.

Thus, we could have defined t_D-type vectors without defining α_D and σ_D, by just demanding at each stage that $\sigma(\mathcal{T}_i)+\sigma(D) \leq \alpha(\mathcal{T}_{i+1})$ rather than $\sigma_D(\mathcal{T}_i) < \alpha_D(\mathcal{T}_{i+1})$. We choose to define σ_D and α_D because it will be helpful when looking at our generalized notion of k-configurations.

Recall that a k-configuration $\mathbb{X}$ of type $\mathcal{T}$ satisfies $\sigma(\mathbb{X}) = \sigma(\mathcal{T})$ and $\alpha(\mathbb{X}) = \alpha(\mathcal{T})$. We would like to find an analogous statement for $\alpha_D(\mathcal{T})$ and $\sigma_D(\mathcal{T})$. In order to do so, we need to define a notion of α_D and σ_D for varieties contained in a fixed $\mathrm{CI}(D)$ similar to the notion of α and σ for varieties contained in $\mathbb{P}^n$.

Definition 3.5 Let $D = \{d_1, \dots, d_r\}$. Let $\mathbb{X}$ be contained in a $\mathrm{CI}(D)$ in $\mathbb{P}^n$. Then put $\alpha_D(\mathbb{X}) := \min\{i \mid H_{\mathbb{X}}(i) < H_{D,n}(i)\}$. If we wish to stress that $\mathbb{X}$ is being considered in $\mathbb{P}^n$, we sometimes write $\alpha_{D,n}(\mathbb{X})$. If $\mathbb{X}$ is a zero-dimensional subvariety, we put $\sigma_D(\mathbb{X}) := \min\{i \mid \Delta H_{\mathbb{X}}(i) = 0\}$, which is the usual σ.

We will sometimes use the notation $\alpha_D(H)$ and $\sigma_D(H)$ if H is the Hilbert function of $\mathbb{X}$, since the notions of $\alpha_D(\mathbb{X})$ and $\sigma_D(\mathbb{X})$ only depend on $H_{\mathbb{X}}$ and not on $\mathbb{X}$ itself.

Definition 3.6 Let $D = \{d_1, \dots, d_r\}$ and let $n \geq r + 1$ be an integer. Let $R = k[x_0, \dots, x_n]$. Let V be a fixed $\mathrm{CI}(D)$ in $\mathbb{P}^n$, so that $I(V) = (F_1, \dots, F_r) \subset R$ where $\deg F_i = d_i$.

We define a (weak) k-configuration with respect to V in $\mathbb{P}^n$ as follows:

$n = r+1$: Let $\mathcal{T} = (e)$ be a 1_D-type vector. A (weak) k-configuration $\mathbb{X}$ with respect to V of type $\mathcal{T}$ in $\mathbb{P}^n$ is $V(F_1, \dots, F_r, G)$ where G is a form of degree e and $(F_1, \dots, F_r, G)$ is a radical ideal of height $r + 1$ in R.

The requirement on the height guarantees that $\mathbb{X}$ is a complete intersection. The requirement that the ideal be radical guarantees that the type of the complete intersection is $(d_1, \dots, d_r, \deg G)$.

$n = r + t, t > 1$: Let $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$ be a (weak) t_D-type vector. Let $\mathbb{H}_1, \dots, \mathbb{H}_u$ be distinct hyperplanes in $\mathbb{P}^n$, where $\mathbb{H}_i$ is defined by the linear form H_i. Suppose that each $(F_1, \dots, F_r, H_i)$ is a radical ideal of height $r + 1$, so that $V_i := V(F_1, \dots, F_r, H_i)$ is a $\mathrm{CI}(D)$ in $\mathbb{H}_i$ for which $I(V_i) = (\overline{F_1}, \overline{F_2}, \dots, \overline{F_r})$ in R/H_i.

Let $\mathbb{X}_i$ be a (weak) k-configuration with respect to V_i in $\mathbb{H}_i$ of type $\mathcal{T}_i$. Suppose furthermore that $\mathbb{H}_i$ does not contain any point of $\mathbb{X}_j$ for $j < i$. Then $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ is a (weak) k-configuration with respect to V of type $\mathcal{T}$ in $\mathbb{P}^n$.

Notation Let $D = \{d_1, \dots, d_r\}$. Let $\mathbb{X}$ be a (weak) k-configuration of type $\mathcal{T}$ with respect to V, where V is a $\mathrm{CI}(d_1, \dots, d_r)$. Then we will say that $\mathbb{X}$ is a (weak) k_D-configuration.

Remark 3.7 While the notation "k_D-configuration" is very useful, it might suggest that $\mathbb{X}$ depends only on D and $\mathcal{T}$. In fact, $\mathbb{X}$ depends on the complete intersection V and it is crucial to the definition of a k_D-configuration that the same complete intersection be used throughout the construction.

The next result observes exactly how this new notion generalizes ordinary (weak) k-configurations.

Proposition 1 *Let $r \leq n$, $D = \{d_1, \dots, d_r\} = \{1, 1, \dots, 1\}$, so that $\sigma(D) = 1$. Let $\mathcal{T}$ be an $(n - r)_D$-type vector. A (weak) k_D-configuration $\mathbb{X}$ of type $\mathcal{T}$ in $\mathbb{P}^n$ is a usual (weak) k-configuration in $\mathbb{P}^{n-r}$ of type $\mathcal{T}$.*

Proof Since $D = \{1, 1, \dots, 1\}$, $\mathbb{X}$ is a k-configuration with respect to

$$V = V(F_1, \dots, F_r)$$

where each F_i is a linear form. So we have that $R/(F_1, \dots, F_r) \simeq k[x_0, \dots, x_{n-r}]$.

If $n = r + 1$, then $\mathcal{T}$ is a (weak) 1-type vector (e) and a (weak) k-configuration of type $\mathcal{T}$ with respect to V is $\mathbb{X} = V(F_1, \dots, F_r, G)$ where $\deg G = e$ and $(F_1, \dots, F_r, G)$ is a radical ideal of height $r + 1$. Let $\overline{G} = G \mod (F_1, \dots, F_r)$. Then

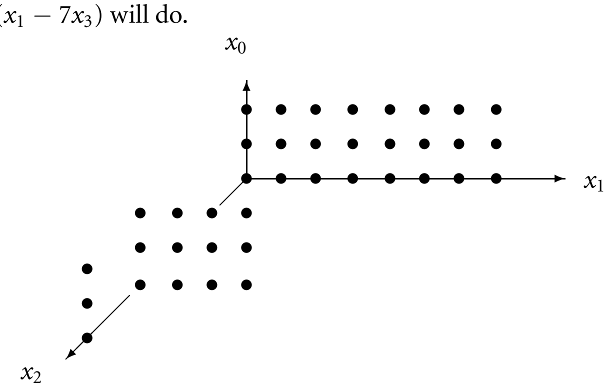
$$V(F_1, \dots, F_r, G) = V(\overline{G})$$

in $\mathbb{P}^{n-r} = \mathbb{P}^1$. But $\overline{G} \neq 0$ since $(F_1, \dots, F_r, G)$ has height $r + 1$. Thus, $\overline{G}$ is a non-zero form of degree e in $R/(F_1, \dots, F_r) \simeq k[x_0, x_1]$. But $\overline{G}$ does not have any repeated factors since $(F_1, \dots, F_r, G)$ is radical, so $V(\overline{G})$ consists of e distinct points in $\mathbb{P}^1$, which is a usual k-configuration of type (e).

If $n > r + 1$, then let $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$ be a (weak) $(n - r)$-type vector. Then $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ where $\mathbb{X}_i$ is a (weak) k_D-configuration in a hyperplane $\mathbb{H}_i$ with respect to $\overline{V} = V(F_1, \dots, F_r) \cap \mathbb{H}_i$ of type $\mathcal{T}_i$. By the induction hypothesis, $\mathbb{X}_i$ is a usual k-configuration of type $\mathcal{T}_i$ in $\mathbb{H}_i$. Furthermore, $\mathbb{H}_i$ does not contain any point of $\mathbb{X}_j$ for $j < i$, so $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ is a (weak) k-configuration. ■

Example 3.8 Let $R = k[x_0, x_1, x_2, x_3]$ (so that $n = 3$), and let $r = 1$. Let F be the degree 3 form $(x_0)(x_0 - x_3)(x_0 - 2x_3)$. We will construct a k-configuration with respect to $V = V(F)$. Let $\mathcal{T} = (1, 4, 8) = (e_1, e_2, e_3)$ be a 2_D-type vector with $D = \{3\}$. Let $\mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_3$ be three hyperplanes defined, respectively, by the linear forms $H_1 = x_2 - 2x_3$, $H_2 = x_2 - x_3$ and $H_3 = x_2$. Certainly, (F, H_i) is a radical ideal of height 2 for each $i = 1, 2, 3$. We construct $\mathbb{X}_i$, a k-configuration with respect to V_i in $\mathbb{H}_i$ of type $\mathcal{T}_i$, where $\mathcal{T}_i = (e_i)$, where $I(V_i) = \overline{F} = F \mod H_i$.

We need to find $\mathbb{X}_i = Z(F, G_i, H_i)$ where $(\overline{F}, \overline{G}_i)$ is a radical ideal of height 2 in $R = k[x_0, \dots, x_n]/(H_i)$. In particular, $\mathbb{X}_i$ is a complete intersection of type $(3, e_i)$ in $\mathbb{H}_i$. Letting $G_1 = x_1$, $G_2 = x_1(x_1 - x_3)(x_1 - 2x_3)(x_1 - 3x_3)$ and $G_3 = x_1(x_1 - x_3) \dots (x_1 - 7x_3)$ will do.



Note that we did not need to choose either F or the G_i as products of linear forms; we merely chose to do so for the purposes of this example. The H_i, of course, are always linear.

As in the case of k-configurations, the Hilbert function of a k_D-configuration of type $\mathcal{T}$ will depend only on $\mathcal{T}$ and D. In fact, our result will determine the Hilbert function of a *weak* k_D-configuration of type $\mathcal{T}$ as long as $\sigma(D) > 1$, *i.e.,* as long as $\mathbb{X}$ is not a k-configuration.

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Let V be a fixed CI(D) in $\mathbb{P}^n$ and let $\mathbb{X} \subseteq V$ be any subvariety. Let $\mathbb{H}$ be a hyperplane chosen so that $V \cap \mathbb{H}$ is a CI(D) in $\mathbb{H} \simeq \mathbb{P}^{n-1}$, not containing any irreducible component of $\mathbb{X}$. If we let H define $\mathbb{H}$, so that $(H) = I(\mathbb{H})$, then H is a non-zero-divisor mod $I(\mathbb{X})$. Let $\mathbb{W} \subseteq V \cap \mathbb{H}$ be any subvariety.

We have the following short exact sequence where b_i, c_i, d_i and e_i are the dimensions of the k-vector spaces indicated:

$$0 \to \underset{b_i}{(R/(I(\mathbb{X} \cup \mathbb{W}))_i} \to \underset{c_i}{(R/I(\mathbb{W}))_i} \oplus \underset{d_i}{(R/I(\mathbb{X}))_i} \to \underset{e_i}{(R/(I(\mathbb{X}) + I(\mathbb{W})))_i} \to 0$$

From linear algebra, we know that $b_i + e_i = c_i + d_i$ for all i. Our first goal is to show that if $\sigma(\mathbb{X}) \leq \alpha_{D,n-1}(\mathbb{W})$ and $\sigma(D) > 1$, then $b_i = c_i + d_{i-1}$ for all i. Then, once we prove this result, we will be able to obtain, given the Hilbert functions of $\mathbb{X}$ and $\mathbb{W}$, the Hilbert function of their union. We will then be able to obtain the Hilbert function of a weak k_D-configuration of type $\mathcal{T}$ as a special case. We will prove this result in several steps.

Theorem 3.9 *Let $\mathbb{W}, \mathbb{X}, b_i, c_i, d_i$ and e_i be as above. Then*

(1) *$e_i \leq \Delta d_i$ for all i;*
(2) *$e_i = \Delta d_i$ for $i < \alpha_{D,n-1}(\mathbb{W})$;*
(3) *for $i < \alpha_{D,n-1}(\mathbb{W})$, $b_i = c_i + d_{i-1}$;*
(4) *iIf $\sigma_{\mathbb{X}} \leq \alpha_{D,n-1}(\mathbb{W})$, then $b_i = c_i + d_{i-1}$ for all i.*

Proof (1) We have that $e_i = H_{R/(I(\mathbb{X})+I(\mathbb{W}))} \leq H_{R/(I(\mathbb{X})+(H))} = H_{R/I(\mathbb{X})/(I(\mathbb{X})+\mathbb{H})/I(\mathbb{X})} = \Delta H_{R/I(\mathbb{X})}$, since H is not a zero-divisor mod $I(\mathbb{X})$. By definition, this is just Δd_i.

(2) Since $\mathbb{W} \subseteq \mathbb{H}$, we know that $I(\mathbb{X}) + I(\mathbb{H}) \subseteq I(\mathbb{X}) + I(\mathbb{W})$. But in $\mathbb{P}^{n-1}$, $I(\mathbb{W})$ does not have any non-zero form of degree strictly less than $\alpha_{D,n-1}(\mathbb{W})$ that is not already in $I(V) \subseteq I(\mathbb{X})$. Hence, for $i < \alpha_{D,n-1}(\mathbb{W})$, $(I(\mathbb{X}) + I(\mathbb{W}))_i = (I(\mathbb{X}) + I(\mathbb{H}))_i$. Thus, $e_i = H_{R/(I(\mathbb{X})+I(\mathbb{H}))}(i) = \Delta H_{R/I(\mathbb{X})}(i) = \Delta d_i$.

(3)In general, $b_i + e_i = c_i + d_i$ for all i. But for $i < \alpha_{D,n-1}(\mathbb{W})$, we can, by (2), rewrite this as $b_i + d_i - d_{i-1} = c_i + d_i$. So, $b_i = c_i + d_{i-1}$, as required.

(4)For $i < \alpha_{D,n-1}(\mathbb{W})$, we are done, by (3). Now, $e_i = 0$ for $i \geq \sigma_{\mathbb{X}}$, from (1), so for $i \geq \sigma_{\mathbb{X}}$, we have $b_i = c_i + d_i = c_i + d_{i-1}$. Thus, for all i, $b_i = c_i + d_{i-1}$. ∎

Before proving our main result, we need the following lemma:

Lemma 1 *Let $\mathbb{W}, \mathbb{X}, b_i, c_i, d_i$ and e_i be as above. If $\sigma_D(\mathbb{X}) < \sigma_D(\mathbb{W})$, then $\sigma_D(\mathbb{X} \cup \mathbb{W}) = \sigma_D(\mathbb{W})$.*

Proof From 3.9(1), $e_i \leq \Delta d_i$ for all i. Thus, $e_i = 0$ for all $i \geq \sigma_{\mathbb{X}}$ and so $b_i = c_i + d_i$ for all $i \geq \sigma_{\mathbb{X}}$. It follows that $\Delta b_i = \Delta c_i + \Delta d_i = \Delta c_i$ for all $i > \sigma_{\mathbb{X}}$. But $\sigma(\mathbb{W}) > \sigma(\mathbb{X})$, so $\Delta b_{\sigma(\mathbb{W})} = \Delta c_{\sigma(\mathbb{W})} = 0$. Recalling that the b_i's represent $\mathbb{X} \cup \mathbb{W}$, we conclude that $\sigma_{\mathbb{X}\cup\mathbb{W}} \leq \sigma_{\mathbb{W}}$. But certainly, $\sigma_{\mathbb{X}\cup\mathbb{W}} \geq \sigma_{\mathbb{W}}$, so we have the desired equality. ∎

We are now ready to prove the main result of this paper.

Theorem 3.10 *Let $D = \{d_1, \dots, d_r\}$ be a set of positive integers with $\sigma(D) > 1$. Let V be a $\mathrm{CI}(D)$ in $\mathbb{P}^n$, so that $I(V) = (F_1, \dots, F_r)$ where $\deg F_i = d_i$. Let $\mathbb{X}$ be a weak k-configuration with respect to V in $\mathbb{P}^n$ of type $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$, where $\mathcal{T}$ is a weak t_D-type vector, where $t = n - r$. Then $\sigma_D(H_{\mathbb{X}}) = \sigma_D(\mathcal{T}), \alpha_D(H_{\mathbb{X}}) = \alpha_D(\mathcal{T})$ and if $t \geq 2$, then*

$$H_{\mathbb{X}}(j) = \sum_{i=1}^{u} H_{\mathbb{X}_i}(j - u + i),$$

where $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ with each $\mathbb{X}_i$ a weak k_D-configuration of type $\mathcal{T}_i$ in the hyperplane $\mathbb{H}_i$. Furthermore, there is a 1-1 correspondence between (weak) n_D-type vectors and Hilbert functions of (weak) k_D-configurations.

Proof We first prove by induction on t that $\alpha_D(\mathbb{X}) < \sigma_D(\mathbb{X})$. If $t = 1$, let $\mathcal{T} = (e)$ be a 1_D-type vector. Then $H_{\mathbb{X}} = H_{\mathrm{CI}(d_1,\dots,d_r,e)}$, so $\sigma(H_{\mathbb{X}}) = \sigma(D)+e-1$, and $\alpha_D(H_{\mathbb{X}}) = e$. Note that $\alpha_D(\mathbb{X}) < \sigma_D(\mathbb{X})$ since $\sigma(D) > 1$. If $t > 1$, then by induction on t, we know that

$$\alpha_{D,n-1}(\mathbb{X}_i) < \sigma_D(\mathbb{X}_i) \leq \alpha_{D,n-1}(\mathbb{X}_{i+1}) \quad \text{for } 1 \leq i \leq u - 1.$$

Letting H_i be the linear form defining $\mathbb{H}_i$, we know that each H_i is a non-zero-divisor modulo $I(V)$. Hence $H_1 H_2 \cdots H_u$ is in $I(\mathbb{X})$, but not in $I(V)$. So, $\alpha_{D,n}(\mathbb{X}) \leq u \leq \alpha_{D,n-1}(\mathbb{X}_1) + u - 1 \leq \alpha_{D,n-1}(\mathbb{X}_u) < \sigma_D(\mathbb{X}_u) \leq \sigma_D(\mathbb{X})$.

We now prove, by induction on k, that if $t \geq 2$, then $\sigma(\bigcup_{i=1}^{k} \mathbb{X}) = \sigma(\mathbb{X}_k)$ for $1 \leq k \leq u$. If $k = 1$, this is trivial, so we assume that $k > 1$. Then by induction on k, $\sigma(\bigcup_{i=1}^{k-1} \mathbb{X}_i) = \sigma(\mathbb{X}_{k-1}) \leq \alpha_{D,n-1}(\mathbb{X}_k) < \sigma(\mathbb{X}_k)$. So from Lemma 1, $\sigma(\bigcup_{i=1}^{k-1} \mathbb{X}_i \cup \mathbb{X}_k) = \sigma(\mathbb{X}_k)$.

We can now show that $\sigma(\mathbb{X}) = \sigma(\mathcal{T})$, by induction on t, the case $t = 1$ being clear: $\sigma(\mathbb{X}) = \sigma(\mathbb{X}_u) = \sigma(\mathcal{T}_u) = \sigma(\mathcal{T})$.

Also, $\sigma(\bigcup_{i=1}^{u-1} \mathbb{X}_i) = \sigma(\mathbb{X}_{u-1}) \leq \alpha_D(\mathbb{X}_u)$. Thus, from Theorem 3.9(4), $H_{\mathbb{X}}(i) = H_{\mathbb{X}_u}(i) + H_{\mathbb{Y}}(i-1)$. Since $\mathbb{Y} = \bigcup_{i=1}^{u-1} \mathbb{X}_i$ is also a weak k-configuration with respect to V (and the result is trivial for $u = 1$), we use induction to obtain that

$$H_{\mathbb{X}}(i) = H_{\mathbb{X}_u}(i) + H_{\mathbb{X}_{u-1}}(i-1) + \cdots + H_{\mathbb{X}_1}(i - u + 1).$$

We next claim that $\alpha_{D,n}(\bigcup_{i=1}^{k} \mathbb{X}_k) = \alpha_{D,n}(\bigcup_{i=1}^{k-1} \mathbb{X}_i) + 1$. Notice that

$$\alpha_{D,n}\Big(\bigcup_{i=1}^{k} \mathbb{X}_i\Big) \leq \alpha_{D,n}\Big(\bigcup_{i=1}^{k-1} \mathbb{X}_i\Big) + 1$$

since if $F \in I(\bigcup_{i=1}^{k-1} \mathbb{X}_i) \setminus I(V)$, then $FH_k \in I(\bigcup_{i=1}^{k} \mathbb{X}_i) \setminus I(V)$. But then

$$\alpha_{D,n}\Big(\bigcup_{i=1}^{k-1} \mathbb{X}_i\Big) < \sigma\Big(\bigcup_{i=1}^{k-1} \mathbb{X}_i\Big) \leq \alpha_{D,n}\Big(\bigcup_{i=1}^{k} \mathbb{X}_i\Big) \leq \alpha_{D,n}\Big(\bigcup_{i=1}^{k-1} \mathbb{X}_i\Big) + 1.$$

So in fact $\alpha_{D,n}(\bigcup_{i=1}^{k} \mathbb{X}_i) = \alpha_{D,n}(\bigcup_{i=1}^{k-1} \mathbb{X}_i) + 1$. Since $\mathbb{X}_1 \subseteq \mathbb{P}^{n-1}$, we have

$$\alpha_{D,n}(\mathbb{X}_1) = 1.$$

So, $\alpha_{D,n}(\mathbb{X}) = u = \alpha(\mathcal{T})$, as required.

Also, notice that the Hilbert function H_D of a weak k_D-configuration of type $\mathcal{T}$ is completely determined from $\mathcal{T}$ and D, since if $H \leftrightarrow \mathcal{T}$ as a usual $(n-r)$-type vector, we can obtain H_D from H in the same way in which the Hilbert function of a $\mathrm{CI}(D)$ in $\mathbb{P}^n$ is obtained from the Hilbert function of $\mathbb{P}^{n-r}$. Similarly, we can recover H from H_D. Thus, we have the following 1-to-1 correspondences: $\mathcal{T} \leftrightarrow H \leftrightarrow H_D$. ■

4 Some Applications

In this section, we provide two applications of Theorem 3.10. The first application will apply to any weak k_D-configuration (with $\sigma(D) > 1$), while the second application will only apply to k_D-configurations.

4.1 The Degree of Each Point in a Weak k_D-Configuration

Lemma 2 *Let $D = \{d_1, \dots, d_r\}$, with $\sigma(D) > 1$. Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a weak k_D-configuration, and let $P \in \mathbb{X}_i$. Then $\deg_{\mathbb{X}} P = \deg_{\mathbb{X}_1 \cup \cdots \cup \mathbb{X}_i} P + u - i$.*

Proof If $i = u$, there is nothing to prove, so suppose that $i < u$. By induction, it is enough to show that $\deg_{\mathbb{X}} P = \deg_{\mathbb{X}_1 \cup \cdots \cup \mathbb{X}_{u-1}} P + 1$. Let $\mathbb{Y} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_{u-1}$. We know that

$$H_{\mathbb{X}}(i) = H_{\mathbb{X}_u}(i) + H_{\mathbb{Y}}(i-1) \quad \text{for all } i.$$

Let $d = \deg_{\mathbb{Y}} P$, so that

$$H_{\mathbb{Y} \setminus P}(i) = \begin{cases} H_{\mathbb{Y}}(i) & \text{for } i < d, \\ H_{\mathbb{Y}}(i) - 1 & \text{for } i \geq d. \end{cases}$$

Now, $\sigma(\mathbb{Y} \setminus P) \leq \sigma(\mathbb{Y}) \leq \alpha_D(\mathbb{X}_u)$, so, by Theorem 3.9(4)

$$\begin{aligned} H_{\mathbb{X} \setminus P}(i) &= H_{\mathbb{X}_u}(i) + H_{\mathbb{Y} \setminus P}(i-1) \quad \text{for all } i \\ &= \begin{cases} H_{\mathbb{X}_u}(i) + H_{\mathbb{Y}}(i-1) & \text{for } i-1 < d, \\ H_{\mathbb{X}_u}(i) + H_{\mathbb{Y}}(i-1) - 1 & \text{for } i-1 \geq d, \end{cases} \\ &= \begin{cases} H_{\mathbb{X}}(i) & \text{for } i < d+1, \\ H_{\mathbb{X}}(i) - 1 & \text{for } i \geq d+1. \end{cases} \end{aligned}$$

Thus, $\deg_{\mathbb{X}} P = d + 1 = \deg_{\mathbb{Y}} P + 1$. ■

Lemma 3 *Let $D = \{d_1, \dots, d_r\}$, with $\sigma(D) > 1$. Let $\mathbb{X}$ be a weak k_D-configuration of type $\mathcal{T}$, where $\mathcal{T}$ is a weak n_D-type vector. Let $P \in \mathbb{X}$. Then $\deg_{\mathbb{X}} P \geq \alpha_D(\mathbb{X})$. In particular, $\alpha_D(\mathbb{X} \setminus P) = \alpha_D(\mathbb{X})$.*

Proof We use induction on n and u, the case $u = 1$ being the induction hypothesis on n. If $n = 1$, so that $\mathcal{T} = (e)$, then $\mathbb{X}$ is a complete intersection of type $(d_1, \dots, d_r, e)$. Then for any $P \in \mathbb{X}$, we have $\deg_{\mathbb{X}} P = \sigma(D) + e - 2 \geq e = \alpha_D(\mathbb{X})$.

If $n > 1$, then $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$ and $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$. If $P \in \mathbb{X}_u$, then by the induction hypothesis on u, $\deg_{\mathbb{X}} P \geq \deg_{\mathbb{X}_u} P \geq \alpha_D(\mathbb{X}_u) \geq u = \alpha_D(\mathbb{X})$. So suppose that $P \in \mathbb{X}_i$, where $i < u$. Then by induction on u, $\deg_{\mathbb{X}} P = \deg_{\mathbb{X}_1 \cup \dots \cup \mathbb{X}_i} P + u - i \geq i + u - i = u$. In particular, $H_{\mathbb{X} \setminus P}(i) = H_{\mathbb{X}}(i)$ for $i < u$, so $\alpha_D(\mathbb{X} \setminus P) \geq u = \alpha_D(\mathbb{X}) \geq \alpha_D(\mathbb{X} \setminus P)$. Hence, $\alpha_D(\mathbb{X}) = \alpha_D(\mathbb{X} \setminus P)$, as required. ■

Remark 4.1 If $\sigma(D) = 1$, so that $\mathbb{X}$ is a k-configuration, then Lemma 3 need not hold. Indeed, any k-configuration of type $(1, e_1, \dots, e_r)$ provides a counterexample.

Lemma 4 *Let $D = \{d_1, \dots, d_r\}$, with $\sigma(D) > 1$. Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a weak k_D-configuration of type $\mathcal{T}$, where $\mathcal{T}$ is a weak n_D-type vector. Let $P \in \mathbb{X}_u$. Then* $\deg_{\mathbb{X}} P = \deg_{\mathbb{X}_u} P$.

Proof Let $\mathbb{Y} = \bigcup_{i=1}^{u-1} \mathbb{X}_i$. Note that $\sigma(\mathbb{Y}) \leq \alpha_D(\mathbb{X}_u) = \alpha_D(\mathbb{X}_u \setminus P)$, so

$$
\begin{aligned}
H_{\mathbb{X} \setminus P}(i) &= H_{\mathbb{X}_u \setminus P}(i) + H_{\mathbb{Y}}(i-1) \quad \text{for all } i \\
&= \begin{cases} H_{\mathbb{X}_u}(i) + H_{\mathbb{Y}}(i-1) & \text{for } i < \deg_{\mathbb{X}_u} P, \\ H_{\mathbb{X}_u}(i) - 1 + H_{\mathbb{Y}}(i-1) & \text{for } i \geq \deg_{\mathbb{X}_u} P, \end{cases} \\
&= \begin{cases} H_{\mathbb{X}}(i) & \text{for } i < \deg_{\mathbb{X}_u} P, \\ H_{\mathbb{X}}(i) - 1 & \text{for } i \geq \deg_{\mathbb{X}_u} P. \end{cases}
\end{aligned}
$$

Thus, $\deg_{\mathbb{X}} P = \deg_{\mathbb{X}_u} P$, as required. ■

From Lemmas 2 and 4, we obtain the following result.

Corollary 1 *Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a weak k_D-configuration, where $\sigma(D) > 1$. Let $P \in \mathbb{X}_i$. Then* $\deg_{\mathbb{X}} P = \deg_{\mathbb{X}_i} P + u - i$.

Just as was done for k-configurations [7, Theorem 5.11], we can also determine an explicit formula for the degree of each point of a weak k_D-configuration. Each point P of a given weak k_D-configuration lies in a complete intersection which corresponds to a 1_D-type vector of the corresponding weak n_D-type vector $\mathcal{T}$. We will denote α_D of this weak 1_D-type vector by $\alpha_D(P)$. Similarly, we will denote σ_D of this weak 1_D-type vector by $\sigma_D(P)$. Regarding a weak n_D-type vector as an ordinary n-type vector, we have the invariants $t_k(\mathcal{T})$ as defined in Definition 2.13 and we define $\mathcal{T}_P$ as before.

Theorem 4.2 *Let $\sigma(D) > 1$. Let $\mathbb{X}$ be a weak k_D-configuration of type $\mathcal{T}$, and $P \in \mathbb{X}$. Then* $\deg_{\mathbb{X}}(P) = \sigma_D(P) + t_1(\mathcal{T}_P) - 2$.

Proof We use induction on n, where $\mathcal{T}$ is an n_D-type vector. If $n = 1$, then $\mathcal{T} = (e)$ and $\mathbb{X}$ is a $\mathrm{CI}(d_1, \dots, d_r, e)$. Then for any $P \in \mathbb{X}$, $\deg_{\mathbb{X}}(P) = d_1 + \dots + d_r + e - r - 1 = \sigma(D) - 1 + \alpha_D(P) - 1 = \sigma_D(P) - 1$. But $t_1(\mathcal{T}_P) = 1$, so the result holds in this case.

If $n > 1$, let $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$ be a weak n_D-type vector, and let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a weak k_D-configuration of type $\mathcal{T}$. Let $P \in \mathbb{X}$. Then $P \in \mathbb{X}_i$ for some i. By induction on n, $\deg_{\mathbb{X}_i} P = \sigma_D(P) + t_1((\mathcal{T}_i)_P) - 2$ and by definition, $t_1((\mathcal{T}_i)_P) + u - i = t_1(\mathcal{T}_P)$, so $\deg_{\mathbb{X}}(P) = \deg_{\mathbb{X}_i}(P) + u - i = \sigma_D(P) + t_1(\mathcal{T}_P) - 2$. ■

Remark 4.3 Since $\sigma_D(P) + t_1(\mathcal{T}_P) - 2 = \alpha_D(P) + t_1(\mathcal{T}_P) - 2 + \sigma(D) - 1$, we see that the values that occur as the degree of some point for a weak k_D-configuration of type $\mathcal{T}$ can be obtained from the values that occur as the degree of some point for a k-configuration of type $\mathcal{T}$ simply by adding $\sigma(D) - 1$.

4.2 Maximal Subsets Lying on a Hypersurface

Let H be the Hilbert function of a finite set of points which is contained in the complete intersection $\mathbb{W} = V(F_1, \dots, F_r)$. Consider all sets $\mathbb{X}$ of points contained in $\mathbb{W}$ which have $H_{\mathbb{X}} = H$. Let F define a hypersurface in $\mathbb{P}^n$ of degree d chosen so that $(F_1, \dots, F_r, F)$ is a radical ideal of height $r + 1$. Consider all subsets $\mathbb{Y}$ of each such $\mathbb{X}$ which lie in $V(F)$. We refer to the set of all Hilbert functions of such subsets $\mathbb{Y}$ as $\mathrm{Sub}_{D,d} H$. We can then partially order $\mathrm{Sub}_{D,d} H$ as follows. Define $H_1 \le H_2$ if $H_1(i) \le H_2(i)$ for every i. We will show that, given D, if H is the Hilbert function of a k_D-configuration and $d \le \alpha_D(H)$, then $\mathrm{Sub}_{D,d} H$ has a unique maximal element. We need some preliminary results.

Definition 4.4 Let $\mathbb{W} = V(F_1, \dots, F_r)$ be a reduced $\mathrm{CI}(d_1, \dots, d_r)$. Let $\mathbb{V}$ be a hypersurface in $\mathbb{P}^n$ of degree d chosen general enough so that $\mathbb{V} \cap \mathbb{W}$ is a reduced $\mathrm{CI}(d_1, \dots, d_r, d)$. For a finite set of points $\mathbb{X}$ in $\mathbb{V} \cap \mathbb{W}$, we put

$$\alpha_{D,\mathbb{V}}(\mathbb{X}) := \min\{i | H_{\mathbb{X}}(i) < H_{\mathbb{V}\cap\mathbb{W}}(i)\}.$$

Note that if $d = 1$, then $\alpha_{D,\mathbb{V}} = \alpha_{D,n-1}$.

Theorem 4.5 *Let $\mathbb{V}$ and $\mathbb{W}$ be as above, and let $D = \{d_1, \dots, d_r\}$. Let H be the Hilbert function of a k_D-configuration of type $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$, an n_D-type vector. Suppose that* $\deg \mathbb{V} = u = \alpha_D(\mathcal{T})$. *Then* $\alpha_{D,\mathbb{V}}(H) = \alpha_{D,n-1}(\mathcal{T}_1) + u - 1$.

Proof Let $\mathbb{X}$ be a k_D-configuration of type $\mathcal{T}$, so that $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$, where each $\mathbb{X}_i$, contained in the hyperplane $\mathbb{H}_i$, is a k_D-configuration of type $\mathcal{T}_i$. Let H_i be the Hilbert function of $\mathbb{X}_i$. We know that $H(i) = H_u(i) + H_{u-1}(i-1) + \cdots + H_1(i-u+1)$ for all i. If $i \ge \alpha_{D,n-1}(\mathcal{T}_1) + u - 1$, then $H_j(i-u+j) \le H_{\mathbb{W}\cap\mathbb{H}_j}(i-u+j)$ for $1 \le j \le u$, so

$$H(i) < H_{\mathbb{W}\cap\mathbb{H}_u}(i) + H_{\mathbb{W}\cap\mathbb{H}_{u-1}}(i-1) + \cdots + H_{\mathbb{W}\cap\mathbb{H}_1}(i-u+1) = H_{\mathbb{W}\cap\mathbb{V}}(i).$$

If $i < \alpha_{D,n-1}(\mathcal{T}_1) + u - 1$, then $H_1(i-u+1) = H_{\mathbb{W}\cap\mathbb{H}_1}(i-u+1)$. Furthermore, $i - j < \alpha_{D,n-1}(\mathcal{T}_1) + u - j - 1 \le \alpha_{D,n-1}(\mathcal{T}_{u-j})$, so we have $H_{u-j}(i-j) = H_{\mathbb{W}\cap\mathbb{H}_{u-j}}(i-j)$. Thus, $H(i) = H_u(i) + \cdots + H_1(i-u+1) = H_{\mathbb{W}\cap\mathbb{H}_u}(i) + \cdots + H_{\mathbb{W}\cap\mathbb{H}_1}(i-u+1) = H_{\mathbb{W}\cap\mathbb{V}}(i)$. ■

Corollary 2 *Let $D = \{d_1, \dots, d_r\}$ and let $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$ be an n_D-type vector. If $H_1 \leftrightarrow (\mathcal{T}_1, \dots, \mathcal{T}_{u-d})$ and $H_1' \leftrightarrow (\mathcal{T}_{u-d+1}, \dots, \mathcal{T}_u)$ and $\mathbb{V}$ is a hypersurface of degree d in $\mathbb{P}^{n+r}$ such that $\mathbb{V} \cap \mathbb{W}$ is a $\mathrm{CI}(D, d)$ in $\mathbb{P}^{n+r}$, then $\sigma_D(H_1) + d \leq \alpha_{D,\mathbb{V}}(H_1')$.*

Proof Since $\sigma_D(H_1) < \alpha_{D,n-1}(\mathcal{T}_{u-d+1})$, we have $\sigma_D(H_1) + d \leq \alpha_{D,n-1}(\mathcal{T}_{u-d+1}) + d - 1 = \alpha_{D,\mathbb{V}}(H_1')$. ∎

We are now ready to show that, given D, if H is the Hilbert function of a k_D-configuration and $d \leq \alpha_D(H)$, then $\mathrm{Sub}_{D,d} H$ has a unique maximal element.

Theorem 4.6 *Let H be the Hilbert function of a k_D-configuration of type $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$. Let $d \leq u$ and let H' be the Hilbert function of a k_D-configuration of type $(\mathcal{T}_{u-d+1}, \dots, \mathcal{T}_u)$. Then H' is the maximal element of $\mathrm{Sub}_{D,d} H$.*

Proof Let $\mathbb{Z}$ be any set of points in $\mathbb{P}^n$ with Hilbert function H which is contained in $\mathbb{W} = V(F_1, \dots, F_r)$ and let F be a form of degree d defining a hypersurface in $\mathbb{P}^n$ for which $(F_1, \dots, F_r, F)$ is a radical ideal of height $r + 1$. We will show that $\Delta H_{\mathbb{Z} \cap V(F, F_1, \dots, F_r)}(j) \leq \Delta H'(j)$ for all $j \geq 0$.

Now, $H'(j)$ is generic in $V(F_1, \dots, F_r, F)$, which is a $\mathrm{CI}(d_1, \dots, d_r, d)$ in $\mathbb{P}^n$ for $0 \leq j < \alpha_{D,\mathbb{V}}(H')$, so we obviously have $\Delta H_{\mathbb{Z} \cap V(F_1, \dots, F_r, F)}(j) \leq \Delta H'(j)$ for $0 \leq j < \alpha_{D,\mathbb{V}}(H')$.

Since $\Delta H_{\mathbb{Z} \cap V(F_1, \dots, F_r, F)}(j) \leq \Delta H_{\mathbb{Z}}(j) = \Delta H(j)$ for all j, it is enough to show that $\Delta H'(j) = \Delta H(j)$ for all $j \geq \alpha_{D,\mathbb{V}}(H')$. Let $\tilde{\mathcal{T}} = (\mathcal{T}_1, \dots, \mathcal{T}_{u-d})$. and let $\tilde{H} \leftrightarrow \tilde{\mathcal{T}}$. Then $H(j) = H'(j) + \tilde{H}(j - d)$ for all j, from the correspondence between Hilbert functions of k_D-configurations and n_D-type vectors. Also, $\sigma(\tilde{H}) + d \leq \alpha_{D,\mathbb{V}}(H')$, from Corollary 2. Let s be the eventually constant value of $\tilde{H}$, so that $\tilde{H}(t) = s$ for all $t \geq \sigma(\tilde{H}) - 1$. Then for all $j \geq \alpha_{D,\mathbb{V}}(H') - 1$, we have that $H(j) = H'(j) + s$. Thus, $\Delta H(j) = \Delta H'(j)$ for $j \geq \alpha_{D,\mathbb{V}}(H')$, as required. ∎

Given D, not every Hilbert function is the Hilbert function of some k_D-configuration. Indeed, when $\sigma(D) > 1$, there is an obvious restriction on which sequences can be the Hilbert function of even a weak k_D-configuration.

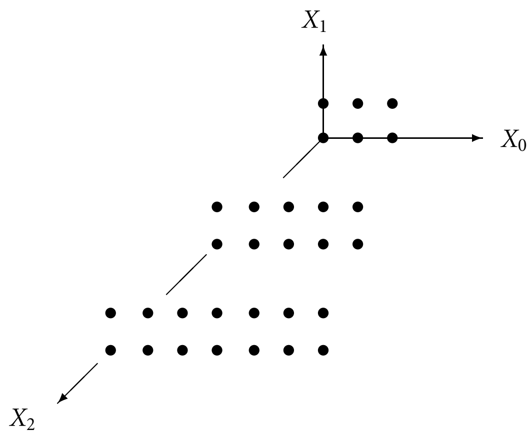
Fact 1 *With D as above, if $\mathbb{X}$ is a weak k_D-configuration of cardinality s, then*

$$(d_1 d_2 \cdots d_r) \mid s.$$

In fact, if $\mathbb{X}$ is a weak k_D-configuration of type $\mathcal{T}$, then $\frac{s}{d_1 d_2 \cdots d_r}$ is the sum of the 1-type vectors in $\mathcal{T}$.

Thus, Theorem 4.6 only applies to very special Hilbert functions, but for those Hilbert functions to which it does apply, it provides a generalization of Theorem 2.19.

Example 4.7 Let $D = \{2\}$, $\mathcal{T} = (3, 5, 7)$. Let $\mathbb{X}$ be the union of the two k-configurations shown below in the projective planes defined by $X_1 = 0$ and $X_1 = X_3$. $\mathbb{X}$ is drawn in the affine portion ($X_3 = 1$) of projective 3-space.



$\mathbb{X}$ is a k_D-configuration with respect to $V(F)$, where $F = X_1(X_1 - X_3)$. Then if H is a hyperplane for which (H, F) is a radical ideal of height 2, then $|H \cap \mathbb{X}| \leq 14$. However, if H is either X_1 or $X_1 - X_3$, then $|H \cap \mathbb{X}| = 3+5+7 = 15$, so the hypothesis in the definition of $\mathrm{Sub}_{D,d} H$ that (H, F) has height 2 is essential.

If $\mathbb{X}$ is any set of points having a subset $\mathbb{Y}$ with the extremal Hilbert function, then we can determine the Hilbert function of $\mathbb{X} \setminus \mathbb{Y}$, thus generalizing (for special Hilbert functions) Theorem 2.20.

Theorem 4.8 *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^n$ contained in the complete intersection $V(F_1, \ldots, F_r)$ of type $(d_1, \ldots, d_r)$, and let $D = \{d_1, \ldots, d_r\}$. Let F be a form of degree d such that $V(F_1, \ldots, F_r, F)$ is a radical ideal of height $r+1$. Let $H = H_{\mathbb{X}}$ be the Hilbert function of a k-configuration with respect to $V(F_1, \ldots, F_r)$ of type $\mathcal{T}$. Let $U \subset \mathbb{X} \cap V(F)$ be such that the Hilbert function H_U of U satisfies $H_U \leftrightarrow \mathcal{T}' = (\mathcal{T}_{u-d+1}, \ldots, \mathcal{T}_u)$. Let $\tilde{\mathbb{X}} = \mathbb{X} - U$. Then $H_{\tilde{\mathbb{X}}} \leftrightarrow \tilde{\mathcal{T}} := (\mathcal{T}_1, \ldots, \mathcal{T}_{u-d})$.*

Proof We have the following exact sequence:

$$0 \to [I_{\mathbb{X}} : F](-d) \xrightarrow{\times F} I_{\mathbb{X}} \to (I_{\mathbb{X}} + F)/F \to 0.$$

Note that there cannot be more points of $\mathbb{X}$ on $V(F)$ than those of U, since H_U is the maximal element of $\mathrm{Sub}_{D,d} H$. Then since $\tilde{\mathbb{X}}$ is precisely the set of points of $\mathbb{X}$ which do not lie on $V(F)$, we see that $I_{\tilde{\mathbb{X}}} = [I_{\mathbb{X}} : F]$, so we have the following exact sequence:

$$0 \to I_{\tilde{\mathbb{X}}}(-d) \xrightarrow{\times F} I_{\mathbb{X}} \to (I_{\mathbb{X}} + F)/F \to 0.$$

Thus, $H_{\mathbb{X}}(t) = H_{\tilde{\mathbb{X}}}(t-d) + H_{R/(I_{\mathbb{X}}+F)}(t)$. From the correspondence between Hilbert functions of k_D-configurations and n_D-type vectors, we know that $H_{\mathbb{X}}(t) = H_{\tilde{\mathcal{T}}}(t -$

$d) + H_{\mathcal{T}'}(t)$, so it is enough to show that $H_{R/(I_{\mathbb{X}}+F)} = H_{\mathcal{T}'} := H_U$. Certainly, $I_{\mathbb{X}} + F \subseteq I_U$, so we only need to show that $H_{R/(I_{\mathbb{X}}+F)}(t) \le H_U(t)$ for all t. Now,

$$\begin{aligned} H_U(t) = H_{\mathcal{T}'}(t) &= H_{\mathcal{T}_u}(t) + H_{\mathcal{T}_{u-1}}(t-1) + \cdots + H_{\mathcal{T}_{u-d+1}}(t-d+1) \\ &= H_{R/(F_1,\dots,F_r)}(t) + H_{R/(F_1,\dots,F_r)}(t-1) + \cdots \\ &\qquad + H_{R/(F_1,\dots,F_r)}(t-d+1) \quad \text{for } t-d+1 < \alpha_D(\mathcal{T}_{u-d+1}) \\ &= H_{R/I(V)}(t) \quad \text{for } t < \alpha_D(\mathcal{T}_{u-d+1}) + d - 1. \end{aligned}$$

But $H_{\mathcal{T}'}(t) \le H_{I_{\mathbb{X}}+F}(t) \le H_V(t)$ for all t since $(F_1, \dots, F_r, F) \subseteq (I_{\mathbb{X}} + F) \subseteq I_U$, so $H_{\mathcal{T}'}(t) = H_{I_{\mathbb{X}}+F}(t)$ for $t < \alpha_D(\mathcal{T}_{u-d+1}) + d - 1$.

Now, $\sigma_D(\tilde{\mathcal{T}}) = \sigma_D(\mathcal{T}_{u-d}) < \alpha_D(\mathcal{T}_{u-d+1})$. So, $\Delta H_{\tilde{\mathcal{T}}}(t) = 0$ for $t \ge \alpha_D(\mathcal{T}_{u-d+1}) - 1$. But, $\Delta H_{\mathbb{X}} = \Delta H_{\tilde{\mathbb{X}}}(t-d) + \Delta H_{R/I_{\mathbb{X}}+F}(t) = \Delta H_{\tilde{\mathcal{T}}}(t-d) + \Delta H_U(t)$.

Since $\Delta H_{\tilde{\mathcal{T}}}(t-d) = 0$ for $t - d \ge \alpha_D(\mathcal{T}_{u-d+1}) - 1$ and $\Delta H_{\tilde{\mathbb{X}}}(t-d) \ge 0$ for all t, we see that $\Delta H_{R/(I_{\mathbb{X}}+F)}(t) \le \Delta H_U(t)$ for $t \ge \alpha_D(\mathcal{T}_{u-d+1}) + d - 1$. Thus, $H_U(t) = H_{R/I_{\mathbb{X}}+F}(t)$ for all t, and hence $H_{\tilde{\mathbb{X}}} = H_{\tilde{\mathcal{T}}}$, as claimed. ■

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