Instability, Chaos and Predictability
in Celestial Mechanics and
Stellar Dynamics

ISBN 1-56072-054-9

THE LIBRATION POINTS AND HILL SURFACES IN THE RESTRICTED PROBLEM OF THREE VARIABLE-MASS BODIES

A.A. Bekov
Institute of Astrophysics, Kazakh SSR, Academy of Sciences, 480068 Alma-Ata, Russia

ABSTRACT

The paper deals with the study of the arising and disappearence of collinear (Eulerian) L_1, L_2, L_3, triangular (Lagrangian) L_4, L_5, coplanar L_6, L_7, ring L_0 and infinitely distant $L_{\pm\infty}$ solutions in a restricted problem of three variable-mass bodies for different time dependencies of main bodies masses and for some additional conditions imposed on the systems parameters. In this case it is assumed that the motion of variable-mass main bodies is determined by the Gylden-Mestschersky problem. The Hill surfaces in the restricted three-body problem where main bodies masses variate isotropically according to the Mestschersky law are studied. Certain possibilities of applying the results of investigations to nonstationary double stellar systems are discussed.

LIBRATION POINTS

The importance of the libration points in the analysis of the motion in the classical restricted three-body problem is well known. The investigation of the libration points in the restricted three-body problem with variable masses was marked in papers by Gelfgat (1973), Horedt (1984), Sing and Ishwar (1984) under the various assumptions relatively the mass variation of the main bodies and the passive gravitating material point.

https://doi.org/10.1017/S0252921100066173 Published online by Cambridge University Press

Let us consider the restricted problem of three variable mass bodies with isotropic mass variation of main bodies according to the unified Mestschersky law (Mestschersky, 1902). In this case it is assumed that the motion of variable mass main bodies is determined by the Gylden-Mestschersky problem (Gylden, 1884, Mestschersky, 1902).

The equations of motion of the passive gravitating material point in the revolving barycentric coordinate system O_{xyz}, the plane xy of that coincides with the plane of the motion of main bodies, and the x-axis always pass through these points, have following form

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x - \dot{\omega} y - \mu_1 \frac{x - x_1}{r_1^3} - \mu_2 \frac{x - x_2}{r_2^3} ,$$

$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \dot{\omega} x - \mu_1 \frac{y}{r_1^3} - \mu_2 \frac{y}{r_2^3} , \qquad (1)$$

$$\ddot{z} = - \mu_1 \frac{z}{r_1^3} - \mu_2 \frac{z}{r_2^3} .$$

Here r_1, r_2 - the distances of gravitating point from the main bodies, ω-their angular velocity of motion, and the quantities μ_1 and μ_2 are determined by equalities.

$$\mu_i = GM_i(t), \ (i = 1,2), \ \mu(t) = \mu_1 + \mu_2 = \frac{\mu_0}{\sqrt{\alpha t^2 + 2\beta t + \gamma}} \qquad (2)$$

where G - the gravitational constant, $M_1(t)$ and $M_2(t)$-masses of main bodies, and their ratio M_1/M_2 is constant.

The equations (1) by the transformation (Gelfgat, 1973)

$$\vec{r}(x,y,z) = \frac{\mu_0}{\mu} \vec{\rho}(\xi,\eta,\zeta), \ d\tau = \left(\frac{\mu}{\mu_0}\right)^2 dt, \ \omega = \left(\frac{\mu}{\mu_0}\right)^2 \omega_0 \qquad (3)$$

are reduced to autonomous form

$$\xi'' - 2\omega_0\eta' = \partial U/\partial\xi, \quad \eta'' + 2\omega_0\xi' = \partial U/\partial\eta, \quad \zeta'' = \partial U/\partial\zeta \qquad (4)$$

where

$$U = \frac{x\omega_o^2}{2}(\xi^2 + \eta^2 + \zeta^2) - \frac{\omega_o^2\zeta^2}{2} + \frac{\mu_{o1}}{\rho_1} + \frac{\mu_{o2}}{\rho_2},$$

$$\rho_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2, \quad (i = 1,2)$$

$$\xi_1 = -\frac{\mu_{o2}}{\mu_o}\rho_{12}, \quad \xi_2 = \frac{\mu_{o1}}{\mu_o}\rho_{12} \tag{5}$$

Here ρ_{12}, x-constants, are defined by the main bodies motion (the particular solution of the Gylden-Mestschersky problem)

$$r_{12}\mu = xC_o^2, \quad (x > 0) \tag{6}$$

where C_o - the constant of the area integral r_{12} - the distance between the main bodies.

Let us choose the units of measurement so, that in Nechvil coordinates (ξ, η, ζ, τ) the distance between the main bodies ρ_{12} and common mass μ_o are equal to unit, i.e. $\rho_{12} = 1$, $\mu_o = 1$, then $x\omega_o^2 = 1$.

Besides introduce the mass parameter

$$\frac{\mu_{20}}{\mu_o} = \nu, \quad (0 < \nu \leq \frac{1}{2}), \quad \frac{\mu_{01}}{\mu_o} = 1 - \nu \tag{7}$$

We show that the equations (4) are satisfied by some constant quantities of coordinates ξ, η, ζ. The libration points - the constant solutions of equations (4) in choosing units of measurement are satisfied by the following system

$$\xi - \frac{1-\nu}{\rho_1^3}(\xi+\nu) - \frac{\nu}{\rho_2^3}(\xi+\nu-1) = 0,$$

$$(1 - \frac{1-\nu}{\rho_1^3} - \frac{\nu}{\rho_2^3})\eta = 0,$$

$$(\frac{x-1}{x} - \frac{1-\nu}{\rho_1^3} - \frac{\nu}{\rho_2^3})\zeta = 0, \tag{8}$$

where

$\rho_1 = \sqrt{(\xi+\nu)^2 + \eta^2 + \zeta^2}$, $\rho_2 = \sqrt{(\xi+\nu-1)^2 + \eta^2 + \zeta^2}$.

The first two equations in the system (8) are complete equal to analogous equations of classical restricted three-body problem. Therefore, as in classical problem, for an arbitrary values of ν and independently of x, there exist three collinear $L_{1,2,3}$ and two triangular $L_{4,5}$ solutions (Gelfgat, 1973; Bekov, 1988; Luk'yanov, 1989a). The motion of all three bodies in the variables (x,y,z,t) is accomplished, in distinction from the classical case, by some similar spirals.

The third equation in the system (8) is different from the classical one by the presence of the term (x-1)/x, therefore, as it show in paper (Bekov, 1988), the system (8), in distinction from the classical case, is admitted in the existence of the coplanar solutions $(\xi, 0, \zeta)$, that are determined from following system (x > 1):

$$\xi - \frac{1-\nu}{\rho_1^3}(\xi+\nu) - \frac{\nu}{\rho_2^3}(\xi+\nu-1) = 0,$$

$$\frac{x-1}{x} - \frac{1-\nu}{\rho_1^3} - \frac{\nu}{\rho_2^3} = 0, \qquad (9)$$

where $\rho_1 = \sqrt{(\xi+\nu)^2 + \zeta^2}$, $\rho_2 = \sqrt{(\xi+\nu-1)^2 + \zeta^2}$

The value ζ enter in equations only under the square symbol, therefore the coplanar solutions always exists by pairs $(\xi, 0, \pm\zeta)$.

The analysis of the system (9) shows (Luk'yanov, 1989a; Bekov; Bekov, 1988) that in the domain

$$-(1-\nu)(x-1) \le \xi \le \nu(x-1) \qquad (10)$$

in which ρ_1 and ρ_2 the real distances, for the arbitrary ν and x > 1 are exist two coplanar solutions $L_{6,7}$.

For solutions $L_{6,7}$ the motion of the third body M in distinction from the solutions L_{1-5}, is accomplished by the spatial winded on the surface of the round cone, formed by the rotation of the generatrix $x = \pm kz$, $k = \left|\frac{\xi}{\zeta}\right|$, where ξ, ζ - coordinates of solutions $L_{6,7}$ around the z-axis.

In the case of the straight-line restricted three-body problem with the variable masses ($C_O = 0$, $x = \infty$) the equations (8) allows, besides the collinear $L_{1,2,3}$ solutions, the spacial solutions L_O (ξ,η,ζ), for which

$$\rho_1 = \rho_2 = 1 \qquad (11)$$

where $\rho_1 = \sqrt{(\xi+\nu)\zeta^2 + \eta^2 + \zeta^2}$, $\rho_2 = \sqrt{(\xi+\nu-1)^2 + \eta^2 + \zeta^2}$

For this solutions all three bodies always are formed the equilateral triangle in space. The dense set of solutions L_O can fill the ring with radius $\rho = \sqrt{3}/2$, this is the consequences of the axis symmetry of the problem when $C = 0$, therefore the spacial L_O solutions we can call the ring solutions. The solutions - the Lagrange ring - for the main bodies mass variation according the unified Mestschersky law was considered by Sersic (1970, 1973), but by the other way.

We note that in the straight-line problem the ring solutions L_O in particular always allows the solutions $L_{4,5}$ and $L_{6,7}$ (the solutions in given planes), therefore they always are attending to L_O solutions and we'll note their presence side by side with the L_O solutions.

The motion of M body in this case so, that the configuration of three bodies always remains similar itself and the motion of all three bodies is taken place by some straight lines.

In the classical problem besides collinear and triangular solutions are exists also the infinitely distant $L_{\pm\infty}$ solutions (Luk'yanov, 1988), for which $\xi = \zeta = 0$, and $\zeta = \pm\infty$. The system (8) admit this solutions only when $x = 1$, i.e. for the first Mestschersky law $\mu = 1/(at+b)$.

At last, there exists the collinear $L_{1,2,3}$ and the spacial ring L_O solutions (and as consequences the triangular $L_{4,5}$ and coplanar $L_{6,7}$ solutions) in the case of the classical straight-line three-body problem with constant masses (Bekov, 1990).

For arbitrary mass variation of the main bodies, when the ratio of masses is constant, and in the more general case, when we have the arbitrary time variable masses of main bodies and $\mu_1/\mu_2 \neq$ const., it may be point out (Luk'yanov, 1989b; Bekov; Bekov, 1990) on the existence of the analogous particular solutions. The results of the investigations are presented in the table.

The table of particular solutions (the existing solutions are marked by "plus" sign)

N	Conditions		Particular solutions					References
	For main bodies mass variation $\mu_1(t)$ and $\mu_2(t)$	For system parameters C_o, x, ν	$L_{1,2,3}$	$L_{4,5}$	$L_{6,7}$	L_o	$L_{\pm\infty}$	
1.	Arbitrary and independent functions of time	$C_o \neq 0$	-	+	-	-	-	[10], [12]
2.	Arbitrary functions of time when μ_1/μ_2 = const.	$C_o = 0$	-	+	+	+	-	[4]
		$C_o \neq 0$	+	+	-	-	-	[10]
		$C_o = 0$	+	+	+	+	-	[4]
3.	Unified Mestschersky law functions $\mu_1(t)=\mu_{10}u(t)$, $\mu_2(t)=\mu_{20}u(t)$, $u(t) = 1/\sqrt{\alpha t^2+2\beta t+\gamma}$	$C_o \neq 0$, $0<x<1$	+	+	-	-	-	[5]
		$C_o \neq 0$, $x > 1$	+	+	+	-	-	$L_{6,7}$ in [2]
		$C_o = 0$, $x = \infty$	+	+	+	+	-	L_0 in [13], [14]
4.	First Mestschersky law functions $\mu_1=\mu_{10}u(t)$, $\mu_2=\mu_{20}u(t)$,	$C_o \neq 0$, $x = 1$	+	+	-	-	+	[9], [5]
		$C_o = 0$	+	+	+	+	-	[4]
5.	Classical problem when μ_1 = const., μ_2 = const.	$C_o \neq 0$	+	+	-	-	+	$L_{\pm\infty}$ [8]
		$C_o = 0$	+	+	+	+	-	[4]

https://doi.org/10.1017/S0252921100066173 Published online by Cambridge University Press

HILL SURFACES

Hill surfaces in restricted three-body problem gives the possibility to find out some general properties of the relative motion of the body with small mass in the gravitational field of two main bodies with finite masses. It may be important the role of analogues of surfaces in the problem with variable masses.

Let us consider the Hill surfaces for the main bodies mass variation according to the Unified Mestschersky law(2). The equations (1) are reduced to autonomous form (4) by transformation (3). The system (4) have the first integral, which in taking units of measurement is written down in the form

$$V^2 = 2\Omega - C,$$

$$\Omega = \frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{2}\left(1 - \frac{1}{x}\right)\zeta^2 + \frac{1-\nu}{\rho_1} + \frac{\nu}{\rho_2},$$

$$\rho_1 = \sqrt{(\xi+\nu)^2 + \eta^2 + \zeta^2}, \quad \rho_2 = \sqrt{(\xi+\nu-1)^2 + \eta^2 + \zeta^2} \qquad (12)$$

where $V^2 = \xi'^2 + \eta'^2 + \zeta'^2$ - velocity of relative motion, C - Jacobi constant. The integral (12) is the analogue of the Jacobi integral of the restricted problem of three variable-mass bodies (Gelfgat, 1973; Bekov, 1987).

We note that the expression (12) differs from the Jacobi integral of the classical problem by the presence in the right hand the term $(1 - 1/x)\zeta^2$, so it make dependent on the change of the Hill surfaces. Supposing in (12) V = 0 we obtain the zero velocity surfaces

$$2\Omega = C \qquad (13)$$

which in the ξ, η, ζ space are restricted the region of possible motion of the investigating body.

Transiting of initial space-time (x,y,z,t) by the transformation (3), we came to conclusion that the knowledge of coordinates and velocities by the formulars (12) and (13) gives us besides the regions of possible motion in initial coordinates. Consequently, in order to know the regions of possible motion in variables (x,y,z,t) it is necessary to investigate the Hill surfaces (13). The analysis of equation (13) is made as in classical case, allows to obtain the qualitative character and the properties of Hill surfaces. The knowledge of libration points and the expression (13) for the zero velocity surfaces allows to make the qualitative analysis of Hill surfaces for different values of parameters ν and x. The Hill surfaces

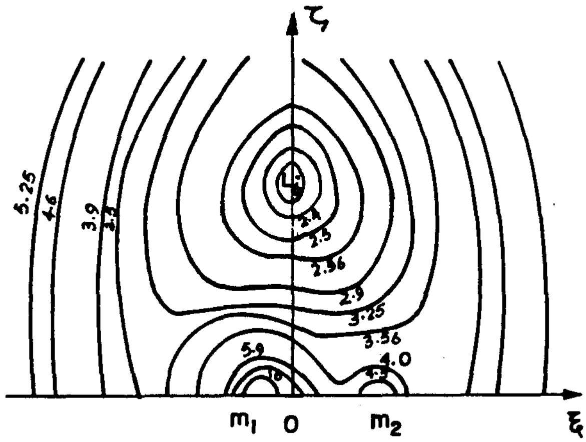

Fig. 1 Zero velocity curves on $\xi\zeta$ plane: $\nu = 0.3$, $x = 1.66$

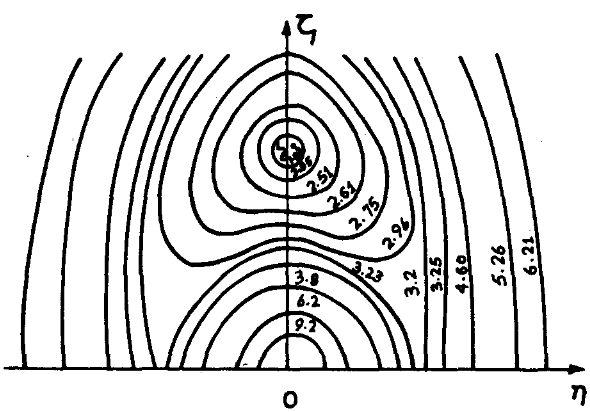

Fig. 2 Zero velocity curves on $\eta\zeta$ plane: $\nu = 0.3$, $x = 1.66$

https://doi.org/10.1017/S0252921100066173 Published online by Cambridge University Press

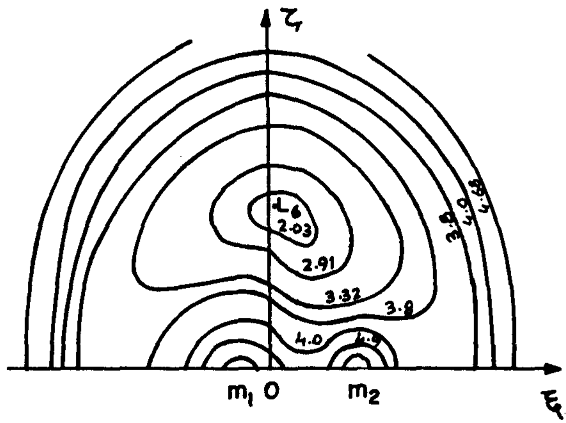

Fig. 3 Zero velocity curves on ξζ plane: ν = 0.3, x = 5.

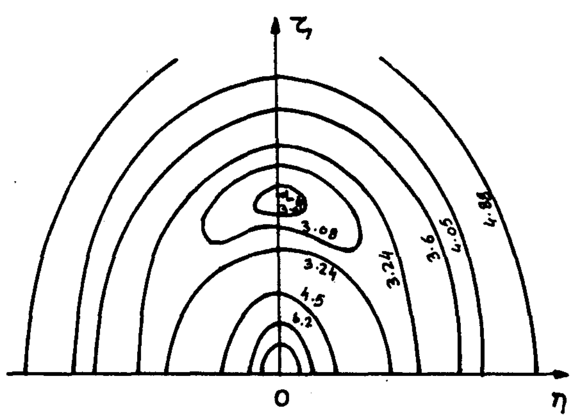

Fig. 4 Zero velocity curves on ηζ plane: ν = 0.3, x = 5.

https://doi.org/10.1017/S0252921100066173 Published online by Cambridge University Press

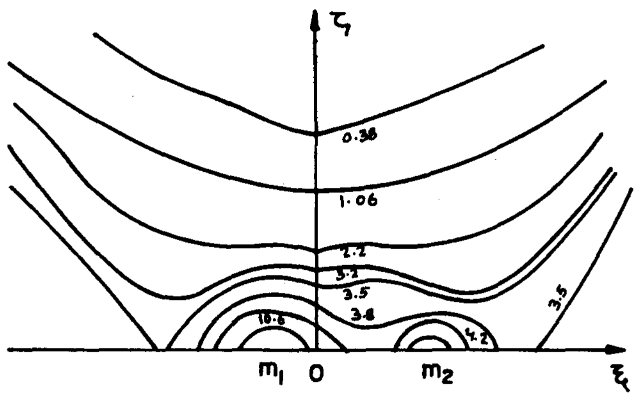

Fig. 5 Zero velocity curves on $\xi\zeta$ plane: $\nu = 0.3$, $x = 0.5$

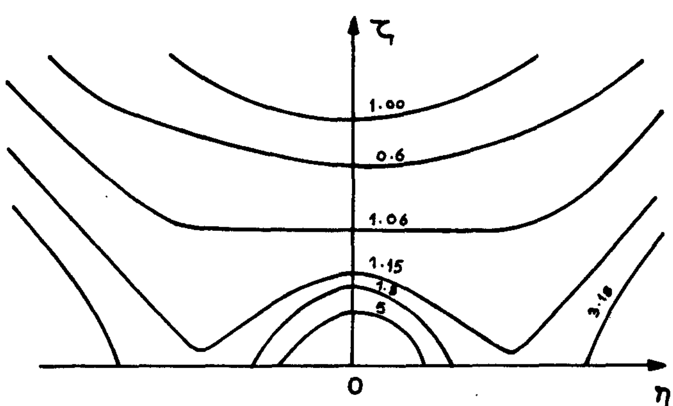

Fig. 6 Zero velocity curves on $\eta\zeta$ plane: $\nu = 0.3$, $x = 0.5$

$$\xi^2 + \eta^2 + (1 - \frac{1}{x})\zeta^2 + \frac{2(1-\nu)}{\rho_1} + \frac{2\nu}{\rho_2} = C \qquad (14)$$

are various in view of presence the parameters ν and x.

In the plane $\xi\eta$ the zero velocity curves (14) are coincide with the classical curves. Therefore, it is enough to consider the surfaces (14) for the case $\zeta \neq 0$. Figures 1-6 shows the zero velocity curves for values of parameter ν = 0.3 and x = 0.5; 1.66; 5. This curves gives the obvious presentation about the Hill surfaces and their properties and manifests new qualitative singularities of the Hill surfaces of the restricted problem of three variable-mass bodies.

CONCLUSIONS

The results of investigation of the libration points and Hill surfaces in the restricted three-body problem with variable masses are presented important, because of transiting from coordinates (ξ, η, ζ, τ) to initial space-time (x, y, z, t) we can to investigate the properties of conforming homographic solutions and analogous surfaces.

The masses of celestial bodies are change during evolution, therefore the restricted three-body problem with variable masses may be used in different astronomical problems. Especially it is concerned to double stars which masses changes rather intensively. The obtained results allows to reveal some singularities of structure and dynamics of double stellar systems, which evolution is accompanied by mass variation of themselves systems.

ACKNOWLEDGEMENTS

The author thanks Dr. L.G. Luk'yanov for useful remarks and Dr.R.K. Muchametkalieva for help with graphic calculations in this presentation.

REFERENCES

[1] Bekov, A.A. (1987). The Labours of Astrophys. Inst. of Academy of Sciences of Kazakh SSR, Alma-Ata. 47,12.
[2] Bekov, A.A. (1988), Astron. J. of Academy of Sciences of USSR, 65,202.
[3] Bekov, A.A. (in preparation).
[4] Bekov, A.A. (in press).
[5] Gelfgat, B.E.(1973), Current problems of celest.Mech. and Astrodynamics, Moscow, p.7.
[6] Gylden, H. (1884), Astron. Nachr. 109,1.

[7] Horedt, G.P. (1984), Celest. Mech. 33, 367.
[8] Luk'yanov, L.G. (1988), Astron. J. of Academy of Sciences of USSR, 65, 1308.
[9] Luk'yanov, L.G. (1989a), Astron. J. of Academy of Sciences of USSR, 66, 180.
[10] Luk'yanov, L.G. (1989b), Astron. J. of Academy of Sciences of USSR, 66, 385.
[11] Mestschersky, I.W. (1902), Astron. Nachr. 159, 229.
[12] Orlov, A.A. (1939), Astron. J. of Academy of Sciences of USSR, 16, 52.
[13] Sersic, J.L. (1970), Periodic orbits, stability and resonces (edited by G.E.O. Giacaglia), Dordrecht, p.314.
[14] Sersic, J.L. (1973), Bull. Astron. Inst. Czechoslovakia, 24, 150.
[15] Singh, J., Ishwar, B., (1984), Celest. Mech. 32, 297.