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SHILNIKOV TYPE SOLUTIONS UNDER STRONG NON-AUTONOMOUS PERTURBATION

ELÍAS TUMA

We study the behaviour of solutions in a neighbourhood of the origin for a certain type of non-autonomous system of partial differential equations whose linear approximation is non autonomous.

1. INTRODUCTION

To study either the bifurcation which arises from a homoclinic orbit Γ, when a system of differential equations is perturbed, or the behaviour of solutions close to Γ, it is necessary to know the Poincare map defined in a transversal section of Γ,with some precision. The Poincare map is defined as a combination of two dynamics, one of them in a neighbourhood of the origin and the other in a tubular neighbourhood of the orbit Γ.

Far away from the origin we may appeal to the continuity of the solution with respect to the initial data. So we consider the solutions in a neighbourhood of the origin [**1**, **5**].

This paper focuses on the derivation of exponential expansions (Deng [**3**], Blázquez-Tuma [**2**]) for solutions of systems of the type employed in the Shilnikov theorem [**5**], with a non linear, non autonomous perturbation and with a non-autonomous linear part.

Let us consider the equation

$$\dot{z} + Az = f(t, z) \tag{1.1}$$

where A is a sectorial operator in a Banach space X and f is both locally Holder in t and $f \in C^k(X^\alpha, X), k > 2, 0 \leqslant \alpha < 1$, in z. The equation (1.1) has a local solution.

We assume that the origin is a hyperbolic equilibrium point, that is, $f(t,0) = 0$, $\forall t \in \mathbb{R}$ and the linearisation about the origin is:

$$\dot{z} + Az = Bz + C(t)z + g(t, z) \tag{1.2}$$

where the non linear part $g(t,z) = zg_1(t,z)$ with $\|g_1(t,z)\| = 0(\|z\|^a)$, some $a > 0$, or

$$\|g(t,z_1) - g(t,z_2)\| < k(p)\|z_1 - z_2\|_\alpha, \quad \forall t \in \mathbb{R}$$

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with $\|z_i\|_\alpha < p$, $i = 1,2$ and $\lim_{S\to+\infty} k(s) = 0$ Let us denote by $L = A - B$ and $L(t) = A - B - C(t)$ and we assume

H1. $\sigma(L) = \sigma_1 \bigcup \sigma_2$ where

(i) $\sigma_1 = \sigma(L) \cap \{\lambda/Re(\lambda) > 0\}$, and there exists a simple, real eigenvalue $\beta \in \sigma_1$ such that $Re\lambda > \beta > 0, \quad \forall\lambda \in \sigma_1 - \{\beta\}$.

(ii) $\sigma_2 = \sigma(L) \cap \{\lambda/Re\lambda < 0\}$, and there exist two complex conjugate eigenvalues $\rho \pm i\omega \in \sigma_2$ such that $0 > \rho > Re(\lambda)$; $\forall\lambda \in \sigma_2, Re\lambda \neq \rho$;

(iii) $\rho + \beta > 0$

Under some conditions on $C(t)$, we prove that the solution $z(t)$ of the non autonomous system (1.2) has an exponential expansion in a neighbourhood of the origin. That is, for small $\|z_0\|_\alpha = \|z(0)\|_\alpha$, there exist $0 < \varphi < q$, such that

$$z(t, t_0, z_0) = K(z_0, t_0)e^{-\varphi(t-t_0)} + \varepsilon(t, t_0)$$

with $\|\varepsilon(t, t_0)\|_\alpha < C\|z_0\|_\alpha e^{-q(t-t)}$ and $K(z_0, t_0) \in \ker(L - \beta I)$.

The Banach space X^α can be written locally as $X^\alpha = X_1^\alpha \oplus X_2^\alpha$ where $X_i^\alpha, i = 1,2$ are invariant sub manifolds associated to the spectral sets σ_i with projections $E_i, i = 1,2$. If $L_i = L/X_i^\alpha$, we have that $\sigma(L_i) = \sigma_i$ and the bounds ([4]).

$$\begin{aligned} &\|e^{-L_1 t}E_1 z\| \leqslant Me^{-\beta t}\|E_1 z\|_\alpha \leqslant Me^{-\beta t}t^{-\alpha}\|E_1 z\| ; \qquad t > 0 \\ &\|e^{-L_2 t}E_2 z\| \leqslant Me^{-\rho t}\|E_2 z\|_\alpha \leqslant Me^{-\rho t}\|E_2 z\| ; \qquad t < 0 \end{aligned} \tag{1.3}$$

Let $x = E_1 z$, and $y = E_2 z$. Then equation (1.1) can be written as

$$\begin{aligned} \dot{x} + L_1(t)x = E_1 g(x, y, t) = g_1(x, y, t) \\ \dot{y} + L_2(t)y = E_2 g(x, y, t) = g_2(x, y, t). \end{aligned} \tag{1.4}$$

Let us assume either one of the following conditions on $C(t)$

H2.1 $\int_{\mathbb{R}} |C(t)|^2 dt < P^2 < \infty$.

H2.2 $\mathbf{C(t)}$ is bounded that is $\|\mathbf{C(t)}\| < \mathbf{k}$ some $\mathbf{k} > 0$.

We know that the linear systems $\dot{x} + L_1(t)x = 0$, and $\dot{y} + L_2(t)y = 0$, have unique solutions $x(t) = x(t; t_0, x_0); y(t) = y(t; t_0, y_0)$ such that $x(t_0) = x_0 \quad , y(t_0) = y_0$. These solution generate a family of evolution operator $\{T_1(t, s)/t > s\}$ and $\{T_2(t, s)/t < s\}$ such that

$$x(t; t_0, x_0) = T_1(t, t_0)x_0; y(t; t_0, y_0) = T_2(t, t_0)y_0.$$

Using Gronwall's inequality [4, Lemma 7.11], we obtain the following.

Lemma 1. *Under the hypothesis* H1, *we have:*

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(i) *If $C(t)$ satisfies* H2.1 *then for $0 < \alpha < 1/2$ there exist a constant K such that*

$$\|T_1(t,s)\| < Ke^{-\beta_1(t-s)}\,; \qquad t > s$$
$$\|T_2(t,s)\| < Ke^{-\rho(t-s)}\,; \qquad t < s$$

with $\beta_1 = \beta - PM/\sqrt{(}1-2\alpha) > 0$.

(ii) *If $C(t)$ satisfies* H2.2 *then there exists a constant K such that*

$$\|T_1(t,s)\| \leqslant Ke^{-(\beta-(kM\Gamma(1-\alpha))^{1/1-\alpha})(t-s)}\,; \qquad t \geqslant s$$
$$\|T_2(t,s)\| \leqslant Ke^{\delta(t-s)}\,; \qquad t < s$$

with $\delta = \beta - kM > 0$

Proof: Let us prove the bounds for $T_1(t,s)$:

(i)

$$x(t) = e^{-L(t-s)}x(s) + \int_s^t e^{-L(t-r)}C(r)x(r)\,dr \|x(t)\|_\alpha$$
$$\leqslant Me^{-\beta(t-s)}\|x(s)\|_\alpha + \int_s^t Me^{-\beta(t-r)}(t-r)^{-\alpha}\|C(r)\|\|x(r)\|_\alpha\,dr \Rightarrow \|x(t)\|_\alpha e^{\beta(t-s)}$$
$$\leqslant M\|x(s)\|_\alpha + \int_s^t Me^{\beta(r-s)}(t-r)^{-\alpha}\|C(r)\|\|x(r)\|_\alpha\,dr$$

using Gronwall's inequality [4]

$$\|x(t)\|_\alpha e^{\beta(t-s)} \leqslant M\|x(s)\|_\alpha e^{M\int_s^t (t-r)^{-\alpha}|C(r)|\,dr}$$

Since for $0 < \alpha < 1/2$

$$\int_s^t (t-r)^{-\alpha}\|C(r)\|\,dr \leqslant \left(\int_s^t (t-r)^{-2\alpha}\,dr\right)^{1/2}\left(\int_s^t \|C(r)\|^2\,dr\right)^{1/2}$$
$$\leqslant \left[P/\sqrt{(}1-2\alpha)\right](t-s)^{(}1/2-\alpha)$$

the result follows.

(ii)

$$\|x(t)\|_\alpha \leqslant M\overline{e}^{\beta(t-s)}\|x(s)\|_\alpha + \int_s^t Me^{-\beta(t-r)}(t-r)^{-\alpha}\|C(r)\|\|x(r)\|_\alpha\,dr$$

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then

$$\|x(t)\|_\alpha e^{\beta(t-s)} \leqslant M\|x(s)\|_\alpha + \int_s^t Me^{\beta(r-s)}(t-r)^{-\alpha}\|C(r)\|\|x(r)\|_\alpha\, dr$$

Using Gronwall's inequality [4]

$$\|x(t)\|_\alpha e^{\beta(t-s)} \leqslant K\|x(s)\|_\alpha E_{1-\alpha}(\Theta(t-s))$$

where

$$\Theta = (kM\Gamma(1-\alpha))^{1/1-\alpha};\, E_{1-\alpha}(\Theta(t-s)) \approx 1/(1-\alpha)e^{\Theta(t-s)}$$

Hence we have

$$\|T_1(t,s)x(s)\|_\alpha \leqslant K\|x(s)\|_\alpha e^{-(\beta-(kM\Gamma(1-\alpha)^{1/1-\alpha})(t-s)}.$$

□

Inmediately, from the lemma, we have.

Theorem 1. *There exist local stable (W^s) and unstable (W^u) manifolds of* (1.4).

Proof: Let

$$S = \left\{z_0/\|E_1z_0\| < p/K, \|z(t,t_0,z_0)\|_\alpha\langle p,t\rangle t_0\right\}$$

If $z_0 \in S$ then

$$z(t) = x(t) + y(t) \in X_1^\alpha \oplus X_2^\alpha,$$

where

$$y(t) = T_2(t,t_0)E_2z_0 + \int_{t_0}^t T_2(t,s)E_2g(s,z(s))\, ds.$$

Hence

$$T_2(0,t)y(t) = T_2(0,t_0)E_0z_0 + \int_{t_0}^t T_2(0,s)E_2g(s,z(s))\, ds.$$

But

$$\begin{aligned}\|T_2(0,t)y(t)\|_\alpha &\leqslant Ke^{-\delta t}\|y(t)\|_\alpha \to 0, \;\; as\;\; t\to\infty \;\;\Rightarrow T_2(0,t_0)E_2z_0\\ &= -\int_{t_0}^\infty T_2(0,s)E_2g(s,z(s))\, ds \;\;\Rightarrow E_2z_0\\ &= -\int_{t_0}^\infty T_2(0,s)E_2g(s,z(s))\, ds, \quad t\geqslant t_0 \;\;\Rightarrow z(t)\\ &= T_1(t,t_0)a + \int_{t_0}^t T_1(t,s)E_1g(s,z(s))\, ds - \int_t^\infty T_2(t,s)E_2g(s,z(s))\, ds \trianglelefteq R(z)\end{aligned}$$

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say.

Similary if $a \in X_1$ with $\|a\| < p/2K$ then we shall prove that there exists a unique solution, $z(t, t_0, a)$, with $E_1 z_0 = E_1\big(z(t_0, t_0, a)\big) = a$, and $\|z\|_\alpha < p$, $\forall t > t_0$. In fact

$$\|z\|_\alpha \leqslant K e^{-\beta_1(t-t_0)}\|a\| + \int_{t_0}^{t} K e^{-\beta_1(t-s)}\Big\|E_1 g\big(s, z(s)\big)\Big\|\, ds + \int_{t}^{\infty} K e^{\delta(t-s)}\Big\|E_2 g\big(s, z(s)\big)\Big\|\, ds$$

$$\leqslant p/2 + Kk(p)\left(\|E_1\|\int_0^\infty e^{-\beta_1 u}\, du + \|E_2\|\int_0^\infty e^{-\delta u}\, du\right) < p$$

so $R(z)$ is a contraction map, in the space of continuous functions with sup $\|z\|_\alpha < p$ and satisfying $E_1 z(t_0) = a$. Hence there exist a unique fixed point $z(t; t_0, a)$. Furthermore, from the integral representation it follows that the application $t \to z(t, t_0, a)$ is Holder continuous. Therefore if z is a solution of the equation (1.2) with initial conditions.

$$h(a) \equiv z(t_0, t_0, a) = a - \int_{t_0}^{\infty} T_2(t_0, s) E_2 g\big(s, z(s)\big)\, ds$$

then $E_1 h(a) \equiv a$ Morover

$$S = \big\{h(a)/a \in X_1^\alpha, \|a\|_\alpha \leqslant p/2K\big\}$$

and $\big\|h(a) - a\big\|_\alpha = 0\big(\|a\|_\alpha\big)$. Similarly

$$S = W^u(0) = \big\{h(a)/a \in X_2^\alpha, \|a\|_\alpha \leqslant p/2K\big\}. \qquad \square$$

REMARK. The stable and unstable manifolds are given locally by

$$W^s_{\text{loc}} : y = h(x, t); \quad W^u_{\text{loc}} : x = k(y, t)$$

Letting

$$x \to x - k(y, t), \quad y \to y - h(x, t)$$

then

$$W^s_{\text{loc}} : y = 0; \quad W^u_{loc} : x = 0$$

and thus equation (1.4) becomes

$$\begin{aligned} \dot{x} + L_1(t)x &= f_1(x, y, t)x \\ \dot{y} + L_2(t)y &= f_2(x, y, t)y \end{aligned} \tag{1.5}$$

with $f_i(0, 0, t) = 0 \quad \forall t > 0$ The integral form of (1.5) is given by

$$x(t) = T_1(t, t_0)x_0 + \int_{t_0}^{t} T_1(t, s) f_1\big(s, x(s), y(s)\big)x(s)\, ds$$

$$y(t) = T_2(t, t_1)y_1 + \int_{t_1}^{t} T_2(t, s) f_2\big(s, x(s), y(s)\big)y(s)\, ds \tag{1.6}$$

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for any t_0 and t_1.

Lemma 2. *For $\|x_0\|_\alpha$, $\|y_1\|_\alpha$ sufficiently small there exist a unique solution of* (1.5) *in a neighbourhood of the origin for* $0 < t_0 < t_1$

Proof: Let

$$H = \left\{(x,y)/\left\|x(t)\right\|_\alpha, \left\|y(s)\right\|_\alpha < K_1 < \infty\right\},$$

H is a complete metric space with the norm:

$$d\big((x_1,y_1),(x_2,y_2)\big) = \|x_2 - x_1\|_\alpha + \|y_2 - y_1\|_\alpha.$$

Let $T(x,y) = \big(T_1(x,y), T_2(x,y)\big)$ where T_1 and T_2 are given a the right hand side of (1.6). Then $T : H \to H$ is a contraction map:

$$\left\|T_1(x,y)\right\|_\alpha \leqslant Ke^{-\beta(t-t_0)}\|x_0\|_\alpha + \int_{t_0}^{t} Ke^{-\beta(t-s)}k(p)\left\|x(s)\right\|_\alpha ds \quad \Rightarrow \left\|T_1(x,y)\right\|_\alpha e^{\beta(t-t_0)}$$

$$\leqslant K\|x_0\|_\alpha + K\int_{t_0}^{t} e^{\beta(s-t_0)}k(p)\left\|x(s)\right\|_\alpha ds$$

$$\leqslant K\left\|x(t_0)\right\|_\alpha\left(1 + \int_{t_0}^{t} k(p)ds\right) \leqslant p$$

Similarly

$$\left\|T_2(x,y)\right\|_\alpha \leqslant Ke^{\delta(t-t_1)}\|y_1\|_\alpha + \int_{t}^{t_l} Ke^{\delta(t-s)}k(p)\left\|y(s)\right\|_\alpha ds \leqslant p$$

Furthermore

$$\left\|T_1(x,y) - T_1(x_1,y_1)\right\|_\alpha \leqslant \int_{t_0}^{t} Ke^{-\beta(t-s)}k(p)\left\|x(s) - x_1(s)\right\|_\alpha ds \leqslant q\|x - x_1\|_\alpha$$

and

$$\left\|T_2(x,y) - T_2(x_1,y_1)\right\|_\alpha \leqslant \int_{t}^{t_l} Ke^{\delta(t-s)}k(p)\left\|x(s) - x_1(s)\right\|_\alpha ds \leqslant q\|y - y_1\|_\alpha$$

with $q < 1$, for small $k(p)$.

Then the result follows by the fixed point theorem. ∎

Lemma 3. *The solution of* (1.1) *under the hypothesis* H1–H2.1 *satisfies*

$$\left\|z(t)\right\|_\alpha \leqslant K_3e^{-\beta_1(t-s)}\left\|z(s)\right\|_\alpha$$

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PROOF: Let

$$H = \left\{ z(t) / \left\|z(t)\right\|_\alpha \leqslant K_3 e^{-\beta_1(t-s)} \left\|z(s)\right\|_\alpha \right\}$$

and define on H

$$F(z) = T(t,s)z(s) + \int_s^t T(t,r) g_1(r,z) z(r)\, dr$$

Then

$$\left\|F(z)\right\|_\alpha \leqslant K e^{-\beta_1(t-s)} \left\|z(s)\right\|_\alpha + \int_s^t K e^{-\beta_1(t-r)} \left\|g_1(r,z)\right\| \left\|z(r)\right\|_\alpha dr \quad \Rightarrow \left\|F(z)\right\|_\alpha e^{\beta_1(t-s)}$$

$$\leqslant K \left\|z(s)\right\|_\alpha + \int_s^t K K_3 e^{\beta_1(r-s)} \left\|g_1(r,z)\right\| \left\|z(s)\right\|_\alpha e^{-\beta(r-s)}\, dr$$

$$\leqslant K \left\|z(s)\right\|_\alpha \left(1 + \int_s^t K_3 \left\|g_1(r,z)\right\| dr\right) \leqslant K_1$$

but

$$\int_s^t \left\|g_1(r,z)\right\| dr \leqslant \int_s^t K_3 \left(e^{-\beta_1(t-s)} \left\|z(s)\right\|\right)^a dr < K_2$$

Then $F(z) \in H$ and F is a contraction. In effect

$$\left\|F(u) - F(v)\right\|_\alpha \leqslant \int_s^t \left\|T(t,r)\big(g(r,u) - g(r,v)\big)\right\|_\alpha dr \leqslant \int_s^t K e^{-\beta_1(t-r)} k(p)|, dr \|u - v\|_\alpha$$

For small $k(p)$ we have

$$\left\|F(u) - F(v)\right\|_\alpha \leqslant q \|u - v\|_\alpha \quad q < 1.$$

THEOREM 2. *Under the conditions of the lemma the solution $z(t)$ has an exponential expansion (taking $\varphi = \beta_1$) if*

$$(2\beta)^{1-2\alpha} > 4P^2 M^2 \Gamma(1 - 2\alpha)$$

PROOF: Let $X = X_1^\alpha \oplus X_2^\alpha$ where $X_1^\alpha = \ker(L - \beta I)$; $X_2^\alpha = \Im(L - \beta I)$ and $L_i = L / X_i^\alpha$ and E_i be projections, $i = 1, 2$. Then $z = u + v \in X^\alpha$. Then

$$v(t) = e^{-L_2(t-t_0)} E_2(z_0) + \int_{t_0}^t e^{-L_2(t-s)} \big(C(s) + E_2 g(t,z)\big) v(s)\, ds$$

Let γ, σ such that $0 < \beta < \sigma < \gamma$. Then we have

$$\left\|v(t)\right\|_\alpha \leqslant K e^{-\sigma(t-t_0)} \left\|E_2(z_0)\right\|_\alpha.$$

Let

$$H = \left\{ v(t) / \left\| v(t) \right\|_\alpha \leqslant K_3 e^{-\sigma(t-t_0)} \left\| E_2(z_0) \right\|_\alpha \right\}$$

Then if $F(v)$ is the right hand side of the integral equation, we have

$$\begin{aligned}
\left\| F(v) \right\|_\alpha &\leqslant M e^{-\gamma(t-t_0)} \left\| E_2(z_0) \right\|_\alpha + \int_{t_0}^{t} K e^{-\gamma(t-s)} (t-s)^{-\alpha} \Big(\left\| E_2 g(s,z) \right\| \\
&\qquad + \left\| C(s) \right\| \Big) \left\| v(s) \right\|_\alpha ds \\
&\leqslant M e^{-\gamma(t-t_0)} \| E_0 z_0 \|_\alpha + M e^{-\gamma(t-t_0)} \int_{t_0}^{t} (t-s)^{-\alpha} \Big(\left\| C(s) \right\| \\
&\qquad + \left\| E_2 g(s,z) \right\| \Big) e^{-(\gamma-\sigma)(t-s)} \| E_2 z_0 \|_\alpha \, ds \\
&\leqslant M e^{-\sigma(t-t_0)} \| E_2 z_0 \|_\alpha \Big(1 + \int_{t_0}^{t} (t-s)^{-\alpha} \left\| C(s) \right\| e^{-(\gamma-\sigma)(t-s)} \, ds \\
&\qquad + \int_{t_0}^{t} (t-s)^{-\alpha} \| E_2 g \| e^{-(\gamma-\sigma)(t-s)} \, ds \Big) \\
&\leqslant M e^{-\sigma(t-t_0)} \| E_2 z_0 \|_\alpha \Big(1 + \Big[\int_{t_0}^{t} \left\| C(s) \right\|^2 ds \Big]^{1/2} \Big[\int_{t_0}^{t} (t-s)^{-2\alpha} e^{-2(\gamma-\sigma)(t-s)} \, ds \Big]^{1/2} \\
&\qquad + k(p) \int_{t_0}^{t} (t-s)^{-\alpha} e^{-(\gamma-\sigma)(t-s)} \, ds \Big) \leqslant K e^{-\sigma(t-t_0)}
\end{aligned}$$

Furthermore F is a contraction, since

$$\begin{aligned}
&\left\| F(v_1) - F(v_2) \right\|_\alpha \\
&\quad \leqslant M \int_{t_0}^{t} e^{-\gamma(t-s)} (t-s)^{-\alpha} \Big(\left\| C(s) \right\| + \| E_2 g \| \Big) \| v_1 - v_2 \|_\alpha \, ds \Big) \\
&\quad \leqslant M \Big(\int_{t_0}^{t} e^{-\gamma(t-s)} (t-s)^{-\alpha} \Big(\left\| C(s) \right\| ds + \int_{t_0}^{t} e^{-\gamma(t-s)} (t-s)^{-\alpha} k(p) \, ds \Big) \| v_1 - v_2 \|_\alpha \Big) \\
&\quad \leqslant M \Big(P \Big[\int_{t_0}^{t} e^{-2\gamma(t-s)} (t-s)^{-2\alpha} \, ds \Big]^{1/2} + k(p) \int_{t_0}^{t} e^{-\gamma(t-s)} (t-s)^{-\alpha} \, ds \Big) \| v_1 - v_2 \|_\alpha \\
&\quad \leqslant M \Big(P \Big[\frac{\Gamma(1-2\alpha)}{(2\gamma)^{I-2\alpha}} \Big]^{1/2} + L k(p) \Big) \| v_1 - v_2 \|_\alpha \\
&\quad \leqslant \Big(P \Big[\frac{\Gamma(1-2\alpha)}{(2\beta)^{I-2\alpha}} \Big]^{1/2} + L k(p) \Big) \| v_1 - v_2 \|_\alpha
\end{aligned}$$

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So let us choose p such that $MLk(p) < 1/2$, and taking

$$P^2M^2\Big[\frac{\Gamma(1-2\alpha)}{(2\beta)^{1-2\alpha}}\Big] < 1/4$$

it follows that F is a contraction as required.

On the other hand

$$\begin{aligned}\int_{t_0}^{\infty} & (t-s)^{-\alpha} e^{\beta_1(t-t_0)} e^{-\beta(t-s)} \Big(\|C(s)\| + \|E_1 g\|\Big) \|z(s)\|_\alpha \, ds \\ & \leqslant \int_{t_0}^{\infty} (t-s)^{-\alpha} e^{-\beta(t-s)} e^{\beta_1(t-t_0)} \Big(\|C(s)\| + \|E_1 g\|\Big) K e^{-\beta_1(s-t_0)} E_1 z_0 \, ds \\ & \leqslant K \int_{t_0}^{\infty} (t-s)^{-\alpha} e^{-(PM/\sqrt{(1-2\alpha)}(t-s)} \Big(\|C(s)\| + \|E_1 g\|\Big) E_1 z_0 \, ds < \infty\end{aligned}$$

then

$$\begin{aligned}K(z_0,t_0) &= \lim_{t\to\infty} z(t) e^{\beta_1(t-t_0)} \\ &= E_1 z_0 + \lim_{t\to\infty} \int_{t_0}^{t} e^{-\beta(s-t_0)} e^{\beta_1(t-t_0)} \left(C(s) + E_1 g\right) z(s)\, ds\end{aligned}$$

and $E_2 K(z_0, t_0) = 0$. ∎

Lemma 4. *The solution of* (1.1) *under the hypothesis* H1-H2 (b) *satisfies*

$$\|z(t)\|_\alpha \leqslant K_1 e^{-(\beta-(kM\Gamma(1-\alpha))^{1/1-\alpha}(t-t_0))} \|z(t)\|_\alpha$$

Proof:

$$\begin{aligned}\|z(t)\|_\alpha \leqslant{} & M e^{-\beta(t-t_0)} \|z(t_0)\|_\alpha \\ & + M \int_{s}^{t} e^{-\beta(t-s)} (t-s)^{-\alpha} \Big(\Big\|C(s) + \Big\|g\big(s,u(s)\big)\Big\|\Big\|_\alpha\Big) \|z(s)\|_\alpha \, ds\end{aligned}$$

Then, using the inequality of [**4**, Lemma 7.11], we have

$$\|z(t)\|_\alpha e^{\beta(t-t_0)} \leqslant M \|z(t)\|_\alpha E_{1-\alpha}\big(\Theta(t-t_0)\big) \leqslant M \|z(t_0)\|_\alpha e^{-(\beta-\theta)(t-t_0)}$$

where

$$\theta = \Big(M\big(k + k(p)\big)\Gamma(1-\alpha)\Big)^{1/1-\alpha}$$

and the result follows. ∎

Theorem 3. *Under the condition of the lemma the solution $z(t)$ has an exponential expansion taking*

$$\varphi = \beta - \left(kM\Gamma(1-\alpha)\right)^{1/1-\alpha} > 0$$

Proof: As in Theorem 2 let us put $X^\alpha = X_1^\alpha \oplus X_2^\alpha$, $z(t) = u(t) + v(t) \in X^\alpha$ Then

$$\left\|v(t)\right\|_\alpha \leqslant Me^{-\delta(t-t_0)}\left\|E_2 z(t_0)\right\|_\alpha + M\int_s^t e^{-\delta(t-s)}(t-s)^{-\alpha}\Big(\left\|C(s)\right\| + \|E_2 g\|\Big)\left\|v(s)\right\|_\alpha ds$$

So that

$$\left\|v(t)\right\|_\alpha e^{\delta(t-t_0)} \leqslant M\left\|E_2 z(t_0)\right\|_\alpha + M\int_{t_0}^t e^{\delta(t-t_0)}(t-s)^{-\alpha}\big(k + k(p)\big)\left\|v(s)\right\|_\alpha ds.$$

Using Gronwall's inequality, we have:

$$\left\|v(t)\right\|_\alpha \leqslant M\left\|E_2 z(t_0)\right\|_\alpha e^{[-\delta+(M(k+k(p))\Gamma(1-\alpha))^{1/1-\alpha}](t-t_0)}.$$

On the other hand

$$u(t) = T(t,t_0)E_1 z_0 + \int_{t_0}^t T(t,s)E_1 g\big(s, z(s)\big)u(s)\, ds,$$

so

$$\left\|u(t)\right\|_\alpha \leqslant K\left\|E_1 z(t_0)\right\|_\alpha e^{-[\beta-(kM\Gamma(1-\alpha))^1/1-\alpha](t-t_0)} + K\int_{t_0}^t e^{-[\beta-(kM\Gamma(1-\alpha))^{1/1-\alpha}](t-s)}(t-s)^{-\alpha}\,\|E_1 g\|\,\left\|u(s)\right\|_\alpha ds.$$

Thus

$$\left\|u(t)\right\|_\alpha e^{[\beta-(kM\Gamma(1-\alpha))^{1/1-\alpha}]}(t-t_0) \leqslant K\left\|E_1 z(t_0)\right\|_\alpha + K\int_{t_0}^t e^{[\beta-(kM\Gamma(1-\alpha))^{1/1-\alpha}](s-t_0)}(t-s)^{-\alpha} e^{(-a-1)[\beta-(M(k+k(p))\Gamma(1-\alpha))^{1/1-\alpha}](s-t_0)}\, ds.$$

Choosing $k(p)$ small enough the integral is bounded and

$$0 < \beta - (kM\Gamma(1-\alpha))^{1/1-\alpha} < \delta - \Big(M\big(k + k(p)\Gamma(1-\alpha)\big)^{1/1-\alpha}\Big).$$

Then

$$K(z_0, t_0) = \lim_{t\to\infty} z(t)e^{-(\beta-(kM\Gamma(1-\alpha))^{1/1-\alpha}(t-t_0)}.$$

□

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Department of Mathematics
Santa Maria University
PO Box 110-V
Valparaiso
Chile
e-mail: elias.tuma@usm.cl