4

Gauge field on the lattice

Gauge invariance is formulated in position space (as opposed to momentum space), which makes the lattice very well suited as a regulator for gauge theories. In this chapter we shall first review the classical QED and QCD actions, then put these theories on the lattice and define gauge-invariant path integrals. Subsequently a natural quantum-mechanical Hilbert-space interpretation will be given. Gauge-invariant couplings to external sources will be shown to correspond to Wilson loops.

4.1 QED action

The QED action for electrons is given by

$$S = \int dx \, \mathcal{L}(x), \tag{4.1}$$

$$\mathcal{L}(x) = -\tfrac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - \bar{\psi}(x) \gamma^{\mu} [\partial_{\mu} + ie A_{\mu}(x)] \, \psi(x) - m \bar{\psi}(x) \psi(x), \tag{4.2}$$

where ψ is the electron field, A_{μ} the photon field and

$$F_{\mu\nu}(x) = \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x) \tag{4.3}$$

is the electromagnetic field-strength tensor. The γ^{μ} are Dirac matrices (cf. appendix D) acting on the ψ's, e is the elementary charge ($e > 0$) and m is the electron mass. It can be useful to absorb the coupling constant e in the vector potential:

$$A_{\mu} \to \frac{1}{e} A_{\mu}. \tag{4.4}$$

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Then $\mathcal{L}$ takes the form

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}\gamma^\mu(\partial_\mu - iqA_\mu)\psi - m\bar{\psi}\psi, \qquad (4.5)$$

$$q = -1, \quad \text{for the electron,} \qquad (4.6)$$

in which the function of e as a coupling constant (characterizing the strength of the interaction) is separated from the charge q, which characterizes the behavior under gauge transformations.

The action is invariant under gauge transformations,

$$\psi'(x) = e^{i\omega(x)q}\psi(x), \quad \bar{\psi}'(x) = e^{-i\omega(x)q}\bar{\psi}(x), \qquad (4.7)$$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\omega(x), \qquad (4.8)$$

$$S(\bar{\psi}, \psi, A) = S(\bar{\psi}', \psi', A'), \qquad (4.9)$$

where $\omega(x)$ is real. The phase factors

$$\Omega(x) = e^{i\omega(x)} \qquad (4.10)$$

form a group, for each x: the gauge group $U(1)$. We may rewrite (4.7) and (4.8) entirely in terms of $\Omega(x)$,

$$\psi'(x) = \Omega(x)^q\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)\Omega^*(x)^q, \qquad (4.11)$$

$$A'_\mu(x) = A_\mu(x) + i\Omega(x)\partial_\mu\Omega^*(x). \qquad (4.12)$$

The covariant derivative

$$D_\mu \equiv \partial_\mu - iA_\mu q \qquad (4.13)$$

has the property that $D_\mu\psi(x)$ transforms just like $\psi(x)$ under gauge transformations:

$$D'_\mu\psi'(x) \equiv [\partial_\mu - iqA'_\mu(x)]\psi'(x) = \Omega(x)^q D_\mu\psi(x), \qquad (4.14)$$

such that $\bar{\psi}\gamma^\mu D_\mu\psi = \bar{\psi}'\gamma^\mu D'_\mu\psi'$.

Above we have interpreted the gauge transformations as belonging to the group $U(1)$. We may also interpret (4.10) as a unitary representation of the group (under addition) of real numbers, R:

$$\omega(x) \to e^{i\omega(x)q}, \quad \omega(x) \in R. \qquad (4.15)$$

Another unitary representation of R could be

$$\omega \to e^{i\omega T} \qquad (4.16)$$

where T is a real number. For the group $U(1)$, however, the mapping

$$\Omega = e^{i\omega} \to D(\Omega) = e^{i\omega T} \qquad (4.17)$$

is a representation only if T is an integer. If T is not an integer, $D(\Omega) = \Omega^T$ is not single valued as a function of Ω. Even if we restrict e.g. $\omega \in [-\pi, \pi]$ to make $\Omega \to e^{i\omega T}$ unique, the product rule would be violated for some Ω's. (For example, for $T = \frac{1}{2}$, $\Omega_1 \Omega_2 = \Omega_3$ with $\omega_{1,2} = 0.9\,\pi$, $\omega_3 = 1.8\,\pi - 2\pi = -0.2\,\pi$ would result in $e^{i\omega_1 T} e^{i\omega_2 T} = e^{i0.9\,\pi} \neq e^{i\omega_3 T} = e^{-i0.1\,\pi}$.)

If the gauge group were necessarily $U(1)$, charge would have to be quantized. Suppose there are fields ψ_r transforming with the representations $T = q_r =$ integer:

$$\psi_r'(x) = D^r(x)\psi_r(x), \quad D^r(x) = \Omega(x)^{q_r}. \qquad (4.18)$$

Then we have to use a corresponding covariant derivative D_μ^r,

$$D_\mu^r = \partial_\mu - iq_r A_\mu(x), \qquad (4.19)$$

such that the action density

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \sum_r \bar{\psi}_r \gamma^\mu D_\mu^r \psi_r - \sum_r m_r \bar{\psi}_r \psi_r \qquad (4.20)$$

is $U(1)$-gauge invariant. It follows that the charges eq_r are a multiple of the fundamental unit e, which is called charge quantization. If the gauge group were R, there would be no need for charge quantization. In Grand Unified Theories the gauge group of electromagnetism is a $U(1)$ group which is embedded as a subgroup in the Grand Unified gauge group. This could provide the explanation for the quantization of charge observed in nature.

4.2 QCD action

The QCD action has the form

$$S = -\int d^4x \left(\frac{1}{4g^2} G_{\mu\nu}^k G^{k\mu\nu} + \bar{\psi}\gamma^\mu D_\mu \psi + \bar{\psi}m\psi \right). \qquad (4.21)$$

The gauge group is $SU(3)$, the group of unitary 3×3 matrices with determinant equal to 1. An element of $SU(3)$ can be written as

$$\Omega = \exp(i\omega^k t_k), \tag{4.22}$$

where the t_k, $k = 1, \ldots, 8$, are a complete set of Hermitian traceless 3×3 matrices. Then $\Omega^{-1} = \Omega^\dagger$ and

$$\operatorname{Tr} t_k = 0 \Rightarrow \det \Omega = \exp(\operatorname{Tr} \ln \Omega) = 1. \tag{4.23}$$

The t_k are the generators of the group in the defining representation. A standard choice for these matrices is patterned after the $SU(2)$ spin matrices $\frac{1}{2}\sigma_k$ in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.24}$$

namely,

$$t_k = \tfrac{1}{2}\lambda_k, \tag{4.25}$$

with

$$\lambda_k = \begin{pmatrix} \sigma_k & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad k = 1, 2, 3, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \tag{4.26}$$

These λ's are the well-known Gell-Mann matrices. They have the properties

$$\operatorname{Tr}(t_k t_l) = \tfrac{1}{2}\delta_{kl}, \tag{4.27}$$

$$[t_k, t_l] = i f_{klm} t_m, \tag{4.28}$$

where the f_{klm} are the real structure constants of $SU(3)$, totally anti-symmetric in k, l and m (cf. appendix A.1).

The quark fields ψ and $\bar{\psi}$ are in the defining representation of $SU(3)$. They carry three discrete indices,

$$\psi^{\alpha a f}: \begin{array}{ll} \alpha = 1, \ldots, 4, & \text{Dirac index;} \\ a = 1, 2, 3 \quad \text{(or red, white, blue),} & \text{color index;} \\ f = 1, \ldots, n_{\mathrm{f}} \ \text{(or } u, d, s, c, b, t\text{),} & \text{flavor index.} \end{array} \qquad (4.29)$$

The gauge transformations and the covariant derivative D_μ act on a, the Dirac gamma matrices act on α and the diagonal mass matrix m acts on f,

$$m = \mathrm{diag}(m_u, m_d, m_s, \ldots). \qquad (4.30)$$

We shall now explain the covariant derivative D_μ and the gauge-field term $G^{\mu\nu}G_{\mu\nu}$. It is instructive to assume for the moment that the ψ fields transform under some arbitrary unitary irreducible representation of the color group $SU(3)$, not necessarily the defining representation. For notational convenience we shall still denote the matrices by Ω; they can be written as

$$\Omega = e^{i\omega_k T_k}, \qquad (4.31)$$

with T_k the generators in the chosen representation, which satisfy

$$[T_k, T_l] = i f_{klm} T_m, \qquad (4.32)$$

$$\mathrm{Tr}\,(T_k T_l) = \rho\,\delta_{kl}. \qquad (4.33)$$

For the defining representation $\rho = \frac{1}{2}$, for the adjoint representation $\rho = 3$ (an expression for ρ is given in (A.47) in appendix A.2).

We assume that D_μ has a form similar to (4.13). However, here it is a matrix,

$$D_{\mu ab} = \delta_{ab}\partial_\mu - iG_\mu(x)_{ab}, \quad \text{or} \quad D_\mu = \partial_\mu - iG_\mu. \qquad (4.34)$$

The matrix gauge field $G_\mu(x)$ should transform such that, under the gauge transformation

$$\psi'(x) = \Omega(x)\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)\Omega^\dagger(x), \qquad (4.35)$$

$D_\mu\psi$ transforms just like ψ,

$$D'_\mu\psi'(x) = \Omega(x)D_\mu\psi(x) = \Omega(x)D_\mu\big[\Omega^\dagger(x)\psi'(x)\big]. \qquad (4.36)$$

Treating ∂_μ as an operator gives the requirement

$$D'_\mu = \Omega D_\mu \Omega^\dagger, \qquad (4.37)$$

or more explicitly

$$\partial_\mu - iG'_\mu = \Omega(\partial_\mu - iG_\mu)\Omega^\dagger$$
$$= \Omega\partial_\mu\Omega^\dagger + \partial_\mu - i\Omega G_\mu\Omega^\dagger. \tag{4.38}$$

It follows that G_μ has to transform as

$$G'_\mu = \Omega G_\mu\Omega^\dagger + i\Omega\partial_\mu\Omega^\dagger. \tag{4.39}$$

Note that this reduces to (4.12) for an Abelian group.

What is the general form of $G_\mu(x)$? How to parameterize this matrix field? Suppose $G_\mu = 0$. Then $G'_\mu = i\Omega\partial_\mu\Omega^\dagger$, so the parameterization of G_μ must at least incorporate the general form of $i\Omega\partial_\mu\Omega^\dagger$. We shall now show that the latter can be written as a linear superposition of the generators T_m. We write

$$i\Omega\partial_\mu\Omega^\dagger = i\Omega\frac{\partial}{\partial\omega^k}\Omega^\dagger\partial_\mu\omega^k = S_k\partial_\mu\omega^k, \tag{4.40}$$

where

$$S_k(\omega) = i\Omega(\omega)\frac{\partial}{\partial\omega^k}\Omega^\dagger(\omega). \tag{4.41}$$

Let $\omega + \varepsilon$ be a small deviation of ω and consider

$$\Omega(\omega)\Omega^\dagger(\omega+\varepsilon) = \Omega(\omega)\left[\Omega^\dagger(\omega) + \varepsilon^k\frac{\partial}{\partial\omega^k}\Omega^\dagger(\omega) + O(\varepsilon^2)\right]$$
$$= 1 - i\varepsilon^k S_k(\omega) + O(\varepsilon^2). \tag{4.42}$$

The left-hand side of this equation is only a small deviation of the unit matrix, so it is possible to write it as $1 - i\varphi_m(\omega,\varepsilon)T_m$, where φ_m is of order ε, so $\varphi_m(\omega,\varepsilon) = S_{km}(\omega)\varepsilon^k$. Hence, $S_k(\omega)$ can be written as

$$S_k(\omega) = S_{km}(\omega)T_m. \tag{4.43}$$

It will be shown in appendix A.2 (eq. (A.43)) that the coefficients $S_{km}(\omega)$ are independent of the representation chosen for Ω. It follows from (4.41) and (4.43) that the Ansatz

$$G_\mu(x) = G_\mu^m(x)T_m \tag{4.44}$$

incorporates the general form of $i\Omega\partial_\mu\Omega^\dagger$. Furthermore, since the generators T_m transform under the adjoint representation of the group (cf. appendix A.2)

$$\Omega T_m\Omega^\dagger = R_{mn}^{-1}T_n, \tag{4.45}$$

the form (4.44) is preserved under the gauge transformation (4.39):

$$\Omega G_\mu^m T_m \Omega^\dagger + i\Omega\partial_\mu\Omega^\dagger = \left(G_\mu^m R_{mn}^{-1} + S_{kn}\partial_\mu\omega^k\right)T_n,$$

$$G_\mu^{'n} = R_{nm}G_\mu^m + S_{kn}\partial_\mu\omega^k, \tag{4.46}$$

where we used $R^{-1} = R^{\mathrm{T}}$. The R_{nm}, S_{nm} and ω^k are real, so we may take (4.44) as the parameterization of $G_\mu(x)$ with real fields $G_\mu^m(x)$. Note that the transformation law for $G_\mu^m(x)$ depends only on the group (its adjoint representation), not on the particular representation chosen for Ω.

The gauge-field G_μ transforms inhomogeneously under the gauge group. The field tensor $G_{\mu\nu}$ is constructed out of G_μ such that it transforms homogeneously,

$$G'_{\mu\nu} = \Omega G_{\mu\nu}\Omega^\dagger. \tag{4.47}$$

Analogously to the electrodynamic case (4.3), $G_{\mu\nu}$ can be written as

$$G_{\mu\nu} = D_\mu G_\nu - D_\nu G_\nu = \partial_\mu G_\nu - \partial_\nu G_\mu - i[G_\mu, G_\nu]. \tag{4.48}$$

Using operator notation, this can also be written as

$$G_{\mu\nu} = [D_\mu, D_\nu]. \tag{4.49}$$

Indeed, using (4.37) we have

$$[D'_\mu, D'_\nu] = D'_\mu D'_\nu - (\mu \leftrightarrow \nu) = \Omega D_\mu D_\nu \Omega^\dagger - (\mu \leftrightarrow \nu) = \Omega[D_\mu, D_\nu]\Omega^\dagger, \tag{4.50}$$

which verifies the transformation rule (4.47). The matrix field $G_{\mu\nu}(x)$ can be written in terms of $G_\mu^m(x)$, using (4.44) and (4.32):

$$G_{\mu\nu} = G_{\mu\nu}^k T_k, \tag{4.51}$$

$$G_{\mu\nu}^k = \partial_\mu G_\nu^k - \partial_\nu G_\mu^k + f_{kmn}G_\mu^m G_\nu^n. \tag{4.52}$$

According to (4.45) and (4.47), $G_{\mu\nu}^k$ transforms in the adjoint representation of the group. The combination

$$G_{\mu\nu}^k G^{k\mu\nu} = \frac{1}{\rho}\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) \tag{4.53}$$

is gauge invariant (ρ has been defined in (4.33)). Notice that the right-hand side does not depend on the representation chosen for Ω.

The action (4.21) can now be written in more detail,

$$S = -\int d^4x \left[\frac{1}{4g^2}G_{\mu\nu}^k G^{k\mu\nu} + \bar\psi\gamma^\mu\left(\partial_\mu - iT_m G_\mu^m\right)\psi + \bar\psi m\psi\right]. \tag{4.54}$$

Since $G^k_{\mu\nu}G^{k\mu\nu}$ contains terms of higher order than quadratic in G^m_μ, the non-Abelian gauge field is self-coupled. The coupling to the ψ field is completely determined by the generators T_m, i.e. by the representation Ω under which the ψ's transform. For the quark fields, Ω is the defining representation $T_m \to t_m$.

4.3 Lattice gauge field

We shall mimic the steps leading to the QCD action (4.54) with lattice derivatives,[1] except for choosing to work in Euclidean space–time. The QCD action has a straightforward generalization to $SU(n)$ gauge groups with $n \neq 3$. We shall assume the gauge group $\mathcal{G} = U(1)$ or $SU(n)$. The case of the non-compact group $\mathcal{G} = R$ will be discussed later.

Let the fermion field ψ_x be associated with the sites $x_\mu = m_\mu a$ of the lattice, analogously to the scalar field. Under local gauge transformations it transforms as

$$\psi'_x = \Omega_x \psi_x, \tag{4.55}$$

where as before Ω_x is an irreducible representation of the gauge group $\mathcal{G}$. Since the lattice derivative

$$\partial_\mu \psi_x = (\psi_{x+a\hat{\mu}} - \psi_x)/a \tag{4.56}$$

contains ψ both at x and at $x + a\hat{\mu}$, we try a covariant derivative of the form

$$D_\mu \psi_x = \frac{1}{a}(\psi_{x+a\hat{\mu}} - \psi_x) - i(\tilde{C}_{\mu x}\psi_x + C_{\mu x}\psi_{x+a\hat{\mu}}). \tag{4.57}$$

Here $C_{\mu x}$ and $\tilde{C}_{\mu x}$ are supposed to compensate for the lack of gauge covariance of the lattice derivative, analogously to the matrix gauge potential $G_\mu(x)$. The covariant derivative has to satisfy

$$D'_\mu \psi'_x = \Omega_x D_\mu \psi_x, \tag{4.58}$$

or

$$
\begin{aligned}
&\frac{1}{a}(\psi'_{x+a\hat{\mu}} - \psi'_x) - i(\tilde{C}'_{\mu x}\psi'_x + C'_{\mu x}\psi'_{x+a\hat{\mu}}) \\
&= \Omega_x\left[\frac{1}{a}(\psi_{x+a\hat{\mu}} - \psi_x) - i(\tilde{C}_{\mu x}\psi_x + C_{\mu x}\psi_{x+a\hat{\mu}})\right] \\
&= \Omega_x\left[\frac{1}{a}(\Omega^\dagger_{x+a\hat{\mu}}\psi'_{x+a\hat{\mu}} - \Omega^\dagger_x\psi'_x 0 - i(\tilde{C}_{\mu x}\Omega^\dagger_x\psi'_x + C_{\mu x}\Omega^\dagger_{x+a\hat{\mu}}\psi'_{x+a\hat{\mu}})\right].
\end{aligned}
\tag{4.59}
$$

Comparing coefficients of ψ'_x and $\psi'_{x+a\hat{\mu}}$ gives

$$\tilde{C}'_{\mu x} = \Omega_x \tilde{C}_{\mu x} \Omega^\dagger_x, \tag{4.60}$$

$$C'_{\mu x} = \Omega_x C_{\mu x} \Omega^\dagger_{x+a\hat{\mu}} + \frac{i}{a}(\Omega_x \Omega^\dagger_{x+a\hat{\mu}} - 1) \tag{4.61}$$

$$= \Omega_x C_{\mu x} \Omega^\dagger_{x+a\hat{\mu}} + \Omega_x i\partial_\mu \Omega^\dagger_x. \tag{4.62}$$

It is consistent to set

$$\tilde{C}_{\mu x} \equiv 0. \tag{4.63}$$

For $C_{\mu x}$ we then find the transformation rule (4.62), which resembles the transformation behavior of the continuum gauge potentials quite closely. By analogy with the continuum theory we try for the field strength the form

$$C_{\mu\nu x} = D_\mu C_{\nu x} - D_\nu C_{\mu x}$$
$$= \frac{1}{a}(C_{\nu,x+a\hat{\mu}} - C_{\nu x}) - iC_{\mu x}C_{\nu,x+a\hat{\mu}} - (\mu \leftrightarrow \nu). \tag{4.64}$$

We find that $C_{\mu\nu x}$ indeed transforms homogeneously,

$$C'_{\mu\nu x} = \Omega_x C_{\mu\nu x} \Omega^\dagger_{x+a\hat{\mu}+a\hat{\nu}}, \tag{4.65}$$

and consequently $\text{Tr}\,(C_{\mu\nu x}C^\dagger_{\mu\nu x})$ is gauge invariant ((4.62) implies that $C_{\mu\nu x}$ cannot be Hermitian in general).

The question is now that of how to parameterize the matrix $C_{\mu x}$ in a way consistent with the transformation rule (4.62). This can be answered by looking at the case $C_{\mu x} = 0$, as in the continuum in the previous section. Then

$$C'_{\mu x} = \Omega_x i\partial_\mu \Omega^\dagger_x = \frac{i}{a}(\Omega_x \Omega^\dagger_{x+a\hat{\mu}} - 1), \tag{4.66}$$

which suggests that we write

$$C_{\mu x} = \frac{i}{a}(U_{\mu x} - 1), \tag{4.67}$$

where $U_{\mu x}$ is a unitary matrix of the same form as Ω_x, i.e. it is a group element in the same representation of the gauge group $\mathcal{G}$. This parameterization of $C_{\mu x}$ is indeed consistent with the transformation rule (4.62), since it gives

$$U'_{\mu x} = \Omega_x U_{\mu x} \Omega^\dagger_{x+a\hat{\mu}}. \tag{4.68}$$

To connect with the gauge potentials $G^k_\mu(x)$ in the continuum we write

$$U_{\mu x} = e^{-iaG_{\mu x}}, \quad G_{\mu x} = G^k_{\mu x} T_k, \tag{4.69}$$

and identify

$$G^k_{\mu x} = G^k_\mu(x). \tag{4.70}$$

More precisely, let $G^k_\mu(x)$ be smooth gauge potentials in the continuum which we evaluate at the lattice points $x_\mu = m_\mu a$. Then $aG_\mu(x) \to 0$ as $a \to 0$, and by construction

$$C_{\mu x} \to G_\mu(x), \quad C_{\mu\nu x} \to G_{\mu\nu}(x), \tag{4.71}$$

where $G_{\mu\nu}(x)$ is the continuum form (4.48).

A possible lattice-regulated gauge-theory action is now given by

$$S = -\sum_x \left[\frac{1}{2} (\bar\psi_x \gamma_\mu D_\mu \psi_x - \bar\psi_x \gamma_\mu D^\dagger_{\mu x} \psi_x) + \bar\psi_x m \psi_x \right.$$

$$\left. + \frac{1}{4\rho g^2} \mathrm{Tr}\, (C_{\mu\nu x} C^\dagger_{\mu\nu x}) \right], \tag{4.72}$$

with ρ the representation-dependent constant defined in (4.33). Evidently, upon inserting $\psi_x = \psi(x)$, $\bar\psi_x = \bar\psi(x)$ and $G^k_{x\mu} = G^k_\mu(x)$ with smooth functions $\psi(x)$, $\bar\psi(x)$ and $G^k_\mu(x)$, this action reduces to the continuum form (4.54) in the limit $a \to 0$. The action (4.72) is not yet satisfactory in its fermion part: it describes too many fermions in the scaling region – this is the notorious phenomenon of 'fermion doubling'. We shall come back to this in a later chapter.

The transformation property (4.68) can be written in a more suggestive form by using the notation

$$U_{x,x+a\hat\mu} \equiv U_{\mu x}, \quad U_{x+a\hat\mu,x} \equiv U^\dagger_{\mu x}, \tag{4.73}$$

because then

$$U'_{x,y} = \Omega_x U_{x,y} \Omega^\dagger_y, \quad y = x + a\hat\mu. \tag{4.74}$$

This notation suggests that it is natural to think of $U_{x,x+a\hat\mu}$ as belonging to the *link* $(x, x+a\hat\mu)$ of the lattice, rather than having four $U_{1x}, \ldots, U_{4x}$ belonging to the site x, as illustrated in figure 4.1.

In fact, there is a better way of associating $U_{x,y}$ with the continuum gauge field $G_\mu(x)$: by identifying $U_{x,y}$ with the parallel transporter from y to x along the link (x, y). The parallel transporter $U(C_{xy})$ along a path C_{xy} from y to x is defined by the path-ordered product (in the continuum)

$$U(C_{xy}) = P \exp\left[-i \int_{C_{xy}} dz_\mu\, G_\mu(z) \right], \tag{4.75}$$

$$U_{\mu x}$$

$$U_{\mu x}^{+}$$

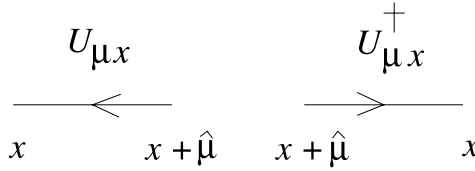$$x \qquad x + \hat{\mu} \qquad x + \hat{\mu} \qquad x$$

Fig. 4.1. Illustration of $U_{\mu x}$ and $U_{\mu x}^{\dagger}$.

where P denotes the path ordering. The path-ordered product can be defined by dividing the path into N segments $(z_n, z_n + dz_n)$, $n = 0, \ldots, N - 1$, and taking the ordered product,

$$U(C_{xy}) = \lim_{N \to \infty} \exp\left[-i \int_{z_{N-1}}^{x} dz_{\mu} \, G_{\mu}(z)\right] \cdots \exp\left[-i \int_{z_n}^{z_{n+1}} dz_{\mu} \, G_{\mu}(z)\right]$$

$$\cdots \exp\left[-i \int_{y}^{z_1} dz_{\mu} \, G_{\mu}(z)\right]$$

$$= \lim_{N \to \infty} [1 - i \, dz_{0\mu} \, G_{\mu}(z_0)] \cdots [1 - i \, dz_{N-1\,\mu} \, G_{\mu}(z_{N-1})]. \quad (4.76)$$

Under a gauge transformation $G'_{\mu}(z) = \Omega(z)G_{\mu}(z)\Omega^{\dagger}(z) + \Omega(z)i\partial_{\mu}\Omega^{\dagger}(z)$ we have

$$1 - i \, dz_{n\mu}G'_{\mu}(z_n) = \Omega(z_n)[\Omega^{\dagger}(z_n) - i \, dz_{n\mu} \, G_{\mu}(z_n)\Omega^{\dagger}(z_n)$$

$$+ dz_{n\mu} \, \partial_{\mu}\Omega^{\dagger}(z_n)] \quad (4.77)$$

$$= \Omega(z_n)[1 - i \, dz_{n\mu} \, G_{\mu}(z_n)]\Omega^{\dagger}(z_{n+1}) + O(dz^2),$$

such that all the Ω's cancel out in $U(C_{xy})$ except at the end points,

$$U'(C_{xy}) = \Omega(x)U(C_{xy})\Omega^{\dagger}(y). \quad (4.78)$$

Hence, $U(C_{xy})$ parallel transports vectors under the gauge group at y to vectors at x along the path C_{xy}. It is known that this way of associating $U_{x,y}$ with the continuum gauge field via $U(C_{xy})$ leads to smaller discretization errors in the action than does use of (4.69) and (4.70). For our lattice theory, however, the basic variables are the $U_{x,x+a\hat{\mu}} \equiv U_{\mu x}$, one for each link $(x, x + a\hat{\mu})$.

Expressing everything in terms of $U_{x,x+a\hat{\mu}}$ simplifies things and makes the transformation properties more transparent:

$$D_{\mu}\psi_x = \frac{1}{a}(U_{x,x+a\hat{\mu}}\psi_{x+a\hat{\mu}} - \psi_x), \quad (4.79)$$

$$C_{\mu\nu x} = \frac{i}{a^2}(U_{x,x+a\hat{\mu}}U_{x+a\hat{\mu},x+a\hat{\mu}+a\hat{\nu}} - U_{x,x+a\hat{\nu}}U_{x+a\hat{\nu},x+a\hat{\mu}+a\hat{\nu}}),$$
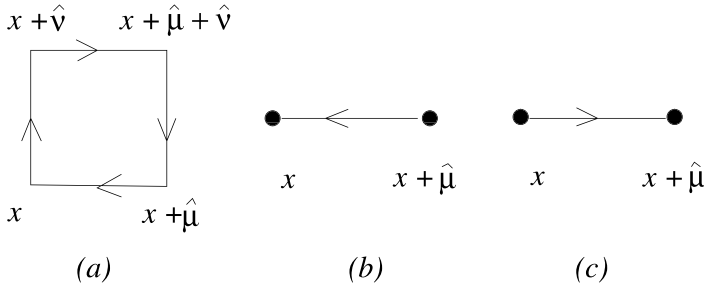
Fig. 4.2. Illustration of the terms in the action $\operatorname{Tr} U_{\mu\nu x}$ (a), $\bar{\psi}_x U_{\mu x} \psi_{x+a\hat{\mu}}$ (b) and $\bar{\psi}_{x+a\hat{\mu}} U_{\mu x}^{\dagger} \psi_x$ (c).

$$\operatorname{Tr}(C_{\mu\nu x} C_{\mu\nu x}^{\dagger}) = \frac{1}{a^4} \operatorname{Tr}(2 - U_{\mu\nu x} - U_{\mu\nu x}^{\dagger}),$$

$$U_{\mu\nu x} = U_{\nu\mu x}^{\dagger} = U_{x,x+a\hat{\mu}} U_{x+a\hat{\mu},x+a\hat{\mu}+a\hat{\nu}} U_{x+a\hat{\mu}+a\hat{\nu},x+a\hat{\nu}} U_{x+a\hat{\nu},x}.$$

We see in (4.79) how the covariant derivative involves parallel transport. The action can be written as

$$S = -\sum_{x\mu\nu} \frac{1}{2g^2 \rho a^4} \operatorname{Tr}(1 - U_{\mu\nu x})$$

$$- \sum_{x\mu} \frac{1}{2a} (\bar{\psi}_x \gamma_\mu U_{\mu x} \psi_{x+a\hat{\mu}} - \bar{\psi}_{x+a\hat{\mu}} \gamma_\mu U_{\mu x}^{\dagger} \psi_x), \qquad (4.80)$$

which is illustrated in figure 4.2. The arrows representing $U_{\mu x}$ and $U_{\mu x}^{\dagger}$ are chosen such that they flow from ψ to $\bar{\psi}$, which conforms to a convention for the Feynman rules in the weak-coupling expansion.

We continue with the theory without fermions. The elementary square of a hypercubic lattice is called a *plaquette*. It may be denoted by p ($p = (x, \mu, \nu; \mu < \nu)$ and the product of the U's around p is denoted by U_p. The gauge-field part of the action can then be written as

$$S(U) = \frac{1}{g^2 \rho} \sum_p \operatorname{Re} \operatorname{Tr} U_p + \text{constant}, \qquad (4.81)$$

in lattice units ($a = 1$). This action depends on the representation of the gauge group chosen for the U's, which in our derivation was dictated by the representation carried by ψ and $\bar{\psi}$. This is in contrast to the classical action which is independent of the group representation, as we saw in the previous section (below (4.46)).

To make the representation dependence explicit, we will from now on assume U to be in the defining representation of the gauge group.

Supposing that (4.81) refers to representation $D^r(U)$, we replace $\rho \to \rho_r$ and $\text{Tr}\, U_p \to \chi_r(U_p)$, with χ_r the character (trace) in the representation r. A more general lattice action may involve a sum over representations r,

$$S(U) = \sum_p \sum_r \beta_r \frac{\text{Re}\,\chi_r(U_p)}{\chi_r(1)}, \qquad (4.82)$$

$$\chi_r(U) \equiv \text{Tr}\,[D^r(U)], \qquad (4.83)$$

which reduces to the classical gauge-field action in the classical continuum limit, with

$$\frac{1}{g^2} = \sum_r \frac{\beta_r \rho_r}{d_r}, \quad d_r = \chi_r(1), \qquad (4.84)$$

where d_r is the dimension of representation r. For example, in an action containing both the fundamental irrep f and the adjoint irrep a of the gauge group $SU(n)$, we have $d_f = n$, $\rho_f = 1/2$, $\rho_a = n$, $d_a = n^2 - 1$ (cf. appendices A.1 and A.2), and

$$1/g^2 = \beta_f/2n + \beta_a n/(n^2 - 1). \qquad (4.85)$$

The simplest lattice formulation of QCD has a plaquette action with only the fundamental representation. It is usually called the Wilson action [39].

4.4 Gauge-invariant lattice path integral

We continue with a pure gauge theory (i.e. containing only gauge fields). The dynamical variables $U_{\mu x}$ are in the fundamental representation of the gauge group $\mathcal{G}$ and the system is described by the gauge-invariant action $S(U)$. If the gauge group is compact we can define a lattice path integral by

$$Z = \int DU \exp[S(U)], \quad DU = \prod_{x\mu} dU_{\mu x}. \qquad (4.86)$$

Here dU for a given link is a volume element in group space. For a compact group the total volume of group space is finite and therefore Z is well defined for a finite lattice. We want Z to be gauge invariant, so we want the integration measure DU to satisfy

$$DU = DU^\Omega, \quad U^\Omega_{\mu x} = \Omega_x U_{\mu x} \Omega^\dagger_{x+a\hat\mu}. \qquad (4.87)$$

On a given link with link variable U, gauge transformations $U' = \Omega_1 U \Omega_2^\dagger$ are combinations of left and right translations in group space:

$$U' = \Omega U, \quad \text{left,} \tag{4.88}$$

$$U' = U \Omega, \quad \text{right.} \tag{4.89}$$

A measure that is invariant under such translations in group space is well known: the Hurewicz or Haar measure. It can be written in a form familiar from general relativity,

$$dU = \nu \sqrt{\det g} \prod_k d\alpha^k, \tag{4.90}$$

where the α^k are coordinates on group space, $U = U(\alpha)$, and g_{kl} is a metric on this space, of the form

$$g_{kl} = \frac{1}{\rho} \text{Tr} \left(\frac{\partial U}{\partial \alpha^k} \frac{\partial U^\dagger}{\partial \alpha^l} \right), \quad \rho = \frac{1}{2}. \tag{4.91}$$

The normalization constant ν will be chosen such that

$$\int dU = 1. \tag{4.92}$$

The metric (4.91) is covariant under coordinate transformations $\alpha^k = f^k(\alpha')$,

$$g_{kl} = g_{mn} \frac{\partial \alpha'^m}{\partial \alpha^k} \frac{\partial \alpha'^n}{\partial \alpha^l}. \tag{4.93}$$

The Jacobian factors of coordinate transformations cancel out in (4.90), such that $dU' = dU$. Since left and right translations are special cases of coordinate transformations, e.g. $U = \Omega^\dagger U'$ corresponds to $U(\alpha) = \Omega^\dagger U(\alpha')$, the measure is again invariant, $dU' = dU$, or

$$d(\Omega U) = d(U \Omega) = dU. \tag{4.94}$$

The above may be illustrated by the exponential parameterization. For the one-dimensional group $U(1)$ we have

$$U = \exp(i\alpha), \quad g_{kl} = 1, \quad \int dU = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} = 1. \tag{4.95}$$

For the $(n^2 - 1)$-dimensional group $SU(n)$ we have

$$U = \exp(i\alpha^k t_k), \quad g_{kl} = S_{km} S_{lm}, \tag{4.96}$$

where we used (4.41), (4.43) and $\partial U^\dagger / \partial \alpha^k = -U^\dagger \, \partial U / \partial \alpha^k \, U^\dagger$, which follows from differentiating $UU^\dagger = 1$. An explicit form for S_{km} is given in (A.43) in appendix A.2.

This completes the definition of the partition function Z. We shall introduce gauge-invariant observables later. One such object we know already: the plaquette field $\mathrm{Tr}\,U_{\mu\nu x}$, or more simply $\mathrm{Tr}\,U_p$. It is a composite field in QCD, which will later be seen to describe 'bound states of glue' – glueballs. Expectation values are defined as usual, for example

$$\langle \mathrm{Tr}\,U_p\,\mathrm{Tr}\,U_{p'}\rangle = Z^{-1}\int DU\,\exp[S(U)]\,\mathrm{Tr}\,U_p\,\mathrm{Tr}\,U_{p'}. \tag{4.97}$$

We stress at this point that gauge fixing (which is familiar in the formal continuum approach) is not necessary with the non-perturbative lattice regulator, for a compact gauge group. The need for gauge fixing shows up again when we attempt to make a weak-coupling expansion.

4.5 Compact and non-compact Abelian gauge theory

Let us write the formulas obtained so far more explicitly for $U(1)$:

$$U_{\mu x} = \exp(-iaA_{\mu x}), \tag{4.98}$$

$$U_{\mu\nu x} = \exp[-ia(A_{\mu x} + A_{\nu x+a\hat{\mu}} - A_{\mu x+a\hat{\nu}} - A_{\nu x})] \tag{4.99}$$

$$= \exp(-ia^2 F_{\mu\nu x}), \tag{4.100}$$

$$F_{\mu\nu x} = \partial_\mu A_{\nu x} - \partial_\nu A_{\mu x}, \tag{4.101}$$

$$S = -\frac{1}{4g^2 a^4}\sum_{x\mu\nu}[2 - 2\cos(a^2 F_{\mu\nu x})], \tag{4.102}$$

$$\int DU = \prod_{x\mu}\int_{-\pi}^{\pi}\frac{d(aA_{\mu x})}{2\pi}. \tag{4.103}$$

Gauge transformations $\Omega = \exp(i\omega_x)$ are linear for the gauge potentials,

$$U'_{\mu x} = \Omega_x U_{\mu x}\Omega^{\dagger}_{x+a\hat{\mu}}, \tag{4.104}$$

$$aA'_{\mu x} = aA_{\mu x} + \omega_{x+a\hat{\mu}} - \omega_x, \mod(2\pi), \tag{4.105}$$

except for the $\mod(2\pi)$.

We used the fundamental representation in S. The more general form (4.82) is a sum over irreps $r = $ integer, with $D^r(U) = \exp(-iraA) = \chi_r(U)$, $d_r = 1$, and $\rho_r = r^2$, which takes the form of a Fourier series:

$$S = \frac{1}{2a^4}\sum_{x\mu\nu}\sum_r \beta_r \cos(ra^2 F_{\mu\nu x}) + \text{constant}, \tag{4.106}$$

$$\frac{1}{g^2} = \sum_{r=-\infty}^{\infty}\beta_r r^2. \tag{4.107}$$

We could for example choose the β_r such that

$$S = -\frac{1}{4g^2} \sum_{x\mu\nu} \left[F_{\mu\nu x}^2 \ \mathrm{mod}(2\pi/a^2) \right]. \tag{4.108}$$

The above is called the compact $U(1)$ gauge theory. It is clear that there is also a non-compact version of the Abelian gauge theory, with gauge transformations $\omega_x \in R$ acting on $A_{\mu x}$ as in (4.105), but without the mod(2π), $aA'_{\mu x} = aA_{\mu x} + \omega_{x+a\hat\mu} - \omega_x$, with $F_{\mu\nu x}$ as in (4.101), and the simple action

$$S(A) = -\frac{1}{4g^2} \sum_{x\mu\nu} F_{\mu\nu x}^2. \tag{4.109}$$

In this case the gauge-invariant measure is given by

$$\int DA = \prod_{x\mu} \int_{-\infty}^{\infty} d(aA_{\mu x}). \tag{4.110}$$

However, the path integral

$$Z = \int DA \, \exp[S(A)] \tag{4.111}$$

is ill defined because it is divergent. The reason is that $\int DA$ contains also an integration over all gauge transformations, which are unrestrained by the gauge-invariant weight $\exp S(A)$. As a consequence Z is proportional to the volume of the gauge group $\int D\Omega$. For the non-compact group $\mathcal{G} = R$ this is $\prod_x \int_{-\infty}^{\infty} d\omega_x$, which is infinite. On the other hand, this divergence formally cancels out in expectation values of gauge-invariant observables and e.g. Monte Carlo computations based on (4.111) still make sense.

To define (4.111) for the non-compact formulation, gauge fixing is needed. A suitable partition function is now given by

$$Z = \int DA \, \exp\left[S(A) - \frac{1}{2g^2\xi} \sum_{x\mu} (\partial'_\mu A_{\mu x})^2 \right], \tag{4.112}$$

where $\partial'_\mu = -\partial_\mu^\dagger$ is the backward derivative, $\partial'_\mu A_{\mu x} = (A_{\mu x} - A_{\mu x - a\hat\mu})/a$. See problem 5(i) for more details.

4.6 Hilbert space and transfer operator

We shall show here that the path integral

$$Z = \int \left(\prod_{x\mu} dU_{\mu x} \right) e^{S(U)} \tag{4.113}$$

can be expressed as the trace of a positive Hermitian transfer operator $\hat{T}$ in Hilbert space,

$$Z = \operatorname{Tr} \hat{T}^N, \tag{4.114}$$

where N is the number of time slices, thus providing the quantum-mechanical interpretation.

This Hilbert space is set up in the coordinate representation. The coordinates are $U_{m\mathbf{x}}$, $m = 1, 2, 3$, corresponding to the spatial link variables. A state $|\psi\rangle$ has a wavefunction $\psi(U) = \langle U|\psi\rangle$ depending on the $U_{m\mathbf{x}}$. The basis states $|U\rangle$ are eigenstates of operators $(\hat{U}_{m\mathbf{x}})_{ab}$, where a and b denote the matrix elements ($a, b = 1, \ldots, n$ for $SU(n)$):

$$(\hat{U}_{m\mathbf{x}})_{ab}|U\rangle = (U_{m\mathbf{x}})_{ab}|U\rangle. \tag{4.115}$$

In a parameterization $U = U(\alpha)$ with real parameters α^k one may think of the usual coordinate representation for Hermitian operators $\hat{\alpha}^k$: $\hat{\alpha}^k|\alpha\rangle = \alpha^k|\alpha\rangle$, $\hat{U}_{ab} = U(\hat{\alpha})_{ab}$, and $|U\rangle \equiv |\alpha\rangle$. The Hermitian conjugate matrix $U^\dagger$ corresponds to the operator $\hat{U}_{ba}^\dagger = U_{ba}^*(\hat{\alpha})$. We continue with the notation $|U\rangle$ for the basis states. The basis is orthonormal and complete

$$\langle U'|U\rangle = \prod_{\mathbf{x},m} \delta(U'_{m\mathbf{x}}, U_{m\mathbf{x}}), \tag{4.116}$$

$$1 = \int \left(\prod_{\mathbf{x},m} dU_{m\mathbf{x}} \right) |U\rangle\langle U|, \tag{4.117}$$

such that

$$\langle \psi_1|\psi_2\rangle = \int \left(\prod_{\mathbf{x},m} dU_{m\mathbf{x}} \right) \psi_1^*(U)\psi_2(U). \tag{4.118}$$

The delta function $\delta(U', U)$ corresponds to the measure dU such that $\int dU \, \delta(U, U') = 1$, which can of course be made explicit in a parameterization $U(\alpha)$.

After this specification of Hilbert space the trace in $Z = \operatorname{Tr} \hat{T}^N$ can be written more explicitly as

$$\operatorname{Tr} \hat{T}^N = \prod_{n=0}^{N-1} \left[\int \left(\prod_{\mathbf{x},m} dU_{m\mathbf{x}n} \right) \langle U_{n+1}|\hat{T}|U_n \rangle \right], \tag{4.119}$$

where n indicates the U variables in time slice n. Notice that the timelike link variables $U_{4\mathbf{x}n}$ are not indicated explicitly in (4.119); these are hidden in $\hat{T}$.

We now have to work the path integral (4.113) into the form (4.119). It is useful for later to allow for different lattice spacings in time and space, a_t and a. Using a notation in which the x_μ are in lattice units (i.e. $\mathbf{x}$ and $x_4 = n$ become integers), but keeping the a's, the pure-gauge part of the action (4.80), $\sum_{x\mu\nu} \operatorname{Tr} U_{\mu\nu x}/2g^2\rho a^4$, takes the form $a^3 a_t \sum_{\mathbf{x},n} [\sum_j 2 \operatorname{Re} \operatorname{Tr} U_{4j\mathbf{x}n}/g^2 a_t^2 + \sum_{i<j} 2 \operatorname{Re} \operatorname{Tr} U_{ij\mathbf{x}n}/g^2 a^2]$ ($\rho = 1/2$ in the fundamental representation), or

$$S = \frac{2}{g^2} \left(\frac{a}{a_t} \sum_{p_t} \operatorname{Re} \operatorname{Tr} U_{p_t} + \frac{a_t}{a} \sum_{p_s} \operatorname{Re} \operatorname{Tr} U_{p_s} \right), \tag{4.120}$$

where p_s and p_t are spacelike and timelike plaquettes, $U_{p_s} = U_{ij\mathbf{x}n}$, $U_{p_t} = U_{4j\mathbf{x}n}$. All the lattice-distance dependence is in the ratio a_t/a. This dependence is really a coupling-constant dependence, one coupling $g^2 a/a_t$ for the timelike plaquettes and another $g^2 a_t/a$ for the spacelike plaquettes. Inspection of (4.113) with action (4.120) shows that $\hat{T}$ can be identified in the form

$$\hat{T} = e^{-\frac{1}{2} a_t \hat{W}} \, \hat{T}'_K \, e^{-\frac{1}{2} a_t \hat{W}}, \tag{4.121}$$

$$\hat{W} = \frac{-2}{g^2 a} \sum_{p_s} \operatorname{Re} \operatorname{Tr} \hat{U}_{p_s}, \tag{4.122}$$

with the operator $\hat{T}'_K$ given by the matrix elements

$$\langle U'|\hat{T}'_K|U \rangle = \prod_{\mathbf{x},m} \int dU_{4\mathbf{x}} \exp \left[\frac{2a}{g^2 a_t} \operatorname{Re} \operatorname{Tr} \left(U_{m\mathbf{x}} U_{4\mathbf{x}+\hat{m}} U'^\dagger_{m\mathbf{x}} U^\dagger_{4\mathbf{x}} \right) \right]. \tag{4.123}$$

The way the $U_{4\mathbf{x}}$ enter in (4.123) can be viewed as a gauge transformation on $U_{m\mathbf{x}}$,

$$U^\Omega_{m\mathbf{x}} = \Omega_x U_{m\mathbf{x}} \Omega^\dagger_{x+\hat{m}}, \tag{4.124}$$

with $\Omega_x = U^\dagger_{4\mathbf{x}}$. Equivalently we can view this as a gauge transformation on $U'_{m\mathbf{x}}$. There is an integral over all such gauge transformations.

We can write this in operator notation as follows. Define the gauge-transformation operator $\hat{D}(\Omega)$ by

$$\hat{D}(\Omega)|U\rangle = |U^{\Omega^\dagger}\rangle. \qquad (4.125)$$

Then

$$\langle U|\hat{D}(\Omega)|\psi\rangle = \langle U^\Omega|\psi\rangle = \psi(U^\Omega). \qquad (4.126)$$

This operator is a unitary representation of the gauge group of time-independent gauge transformations in Hilbert space. Define furthermore $\hat{P}_0$ by

$$\hat{P}_0 = \int \left(\prod_{\mathbf{x}} d\Omega_{\mathbf{x}}\right) \hat{D}(\Omega). \qquad (4.127)$$

It follows that $\hat{T}'_K$ can be written as

$$\hat{T}'_K = \hat{T}_K \hat{P}_0 = \hat{P}_0 \hat{T}_K, \qquad (4.128)$$

with $\hat{T}_K$ given by

$$\langle U'|\hat{T}_K|U\rangle = \prod_{\mathbf{x},m} \exp\left[\frac{2a}{g^2 a_t} \operatorname{Re} \operatorname{Tr}\left(U_{m\mathbf{x}} U'^\dagger_{m\mathbf{x}}\right)\right]. \qquad (4.129)$$

The operator $\hat{P}_0$ is the projector onto the gauge-invariant subspace of Hilbert space. This follows from the fact that $\hat{D}(\Omega)$ is a representation of the gauge group,

$$\hat{D}(\Omega_1)\hat{D}(\Omega) = \hat{D}(\Omega_1\Omega), \qquad (4.130)$$

$$\hat{D}(\Omega_1)\hat{P}_0 = \int \left(\prod_{\mathbf{x}} d\Omega_{\mathbf{x}}\right) \hat{D}(\Omega_1\Omega) = \int \left(\prod_{\mathbf{x}} d(\Omega_{1\mathbf{x}}\Omega_{\mathbf{x}})\right) \hat{D}(\Omega_1\Omega)$$

$$= \hat{P}_0 \qquad (4.131)$$

$$= \hat{P}_0 \hat{D}(\Omega_1), \qquad (4.132)$$

$$\hat{P}_0^2 = \hat{P}_0. \qquad (4.133)$$

We used the invariance of the integration measure in group space and the normalization $\int d\Omega_x = 1$. It follows that a state $|\psi\rangle$ of the form $|\psi\rangle = \hat{P}_0|\phi\rangle$ is gauge invariant, $\hat{D}(\Omega)|\psi\rangle = |\psi\rangle$. It also follows easily by taking matrix elements that $\hat{P}_0$ commutes with $\hat{W}$.

We shall show in the next section that $\hat{T}_K$ is a positive operator. It can therefore be written in the from

$$\hat{T}_K = e^{-a_t \hat{K}}, \qquad (4.134)$$

with $\hat{K}$ a Hermitian operator.

Summarizing, the path integral leads naturally to a quantum-mechanical Hilbert space and a transfer operator

$$\hat{T} = \hat{P}_0 \, e^{-\frac{1}{2}a_t \hat{W}} \, e^{-a_t \hat{K}} \, e^{-\frac{1}{2}a_t \hat{W}} = e^{-\frac{1}{2}a_t \hat{W}} \, e^{-a_t \hat{K}} \, e^{-\frac{1}{2}a_t \hat{W}} \hat{P}_0 \,, \quad (4.135)$$

which is positive and defines therefore a Hermitian Hamiltonian $\hat{H}$,

$$\hat{T} = \hat{P}_0 \, e^{-a_t \hat{H}} = e^{-a_t \hat{H}} \, \hat{P}_0. \quad (4.136)$$

We recognize a kinetic part ($\hat{K}$) and potential part ($\hat{W}$), analogously to the example of the scalar field. The form (4.129) for the matrix elements of $\hat{T}_K$ shows a plaquette in the *temporal gauge* $U_{4x} = 1$. The path integral has automatically provided the supplementary condition that has to be imposed in this 'gauge': physical states must be gauge invariant (i.e. invariant under time-independent gauge transformations),

$$|\text{phys}\rangle = \hat{P}_0|\text{phys}\rangle, \quad \hat{D}(\Omega)|\text{phys}\rangle = |\text{phys}\rangle. \quad (4.137)$$

In the continuum this corresponds to the 'Gauss law' condition (cf. appendix B).

4.7 The kinetic-energy operator

As we can see from its definition (4.129), the kinetic-energy transfer operator $\hat{T}_K$ is a product of uncoupled link operators. So let us concentrate on a single link $(\mathbf{x}, \mathbf{x}+\hat{m})$ and simple states $|\psi\rangle$ for which $\psi(U)$ depends only on $U_{m\mathbf{x}}$. To simplify the notation we write $U = U_{m\mathbf{x}}$. Then the single-link kinetic transfer operator is given by

$$\langle U'|\hat{T}_{K1}|U\rangle = \exp\left[\kappa \operatorname{Re} \operatorname{Tr}\left(UU'^{\dagger}\right)\right], \quad (4.138)$$

$$\kappa = \frac{2a}{g^2 a_t}, \quad (4.139)$$

where the subscript 1 reminds us of the fact that we are dealing with a single link.

Realizing that $\operatorname{Re} \operatorname{Tr}\left(UU'^{\dagger}\right)$ may be taken as the distance between the points U and U' in group space, we note that (4.138) is analogous to the expression (2.15) in quantum mechanics, which also involved a translation in the coordinates. So we may expect to gain understanding here too by introducing translation operators. Left and right translations $\hat{L}(V)$ and $\hat{R}(V)$ can be defined by

$$\hat{L}(V)|U\rangle = |V^{\dagger}U\rangle, \quad (4.140)$$

$$\hat{R}(V)|U\rangle = |UV\rangle. \quad (4.141)$$

By comparing matrix elements we see that $\hat{T}_{K1}$ can be written as

$$\hat{T}_{K1} = \int dV \, \exp[\kappa \operatorname{Re} \operatorname{Tr} V] \, \hat{L}(V), \qquad (4.142)$$

$$= \int dV \, \exp[\kappa \operatorname{Re} \operatorname{Tr} V] \, \hat{R}(V). \qquad (4.143)$$

The eigenstates of the translation operators can be found among the eigenstates of the Laplacian on group space, as summarized in appendix A.3 (eq. (A.76)). The eigenfunctions are the finite-dimensional unitary irreducible representations (irreps) $D^r_{mn}(U)$ of the group,

$$U \to D^r_{mn}(U) = \langle U | rmn \rangle. \qquad (4.144)$$

Here r labels the irreps and m and n label the matrix elements. These unitary matrices form a complete orthogonal set of basis functions,

$$\int dU \, D^{r'}_{m'n'}(U)^* \, D^r_{mn}(U) = \delta_{r'r}\delta_{m'm}\delta_{n'n}\frac{1}{d_r}, \qquad (4.145)$$

$$\sum_{rmn} d_r D^r_{mn}(U) \, D^r_{mn}(U')^* = \delta(U, U'), \qquad (4.146)$$

where d_r is the dimension of the representation ($D^r(U)$ is a $d_r \times d_r$ matrix). A function $\psi(U)$ can be expanded as

$$\psi(U) = \langle U | \psi \rangle = \sum_{rmn} \psi_{rmn} d_r D^r_{mn}(U), \qquad (4.147)$$

with the inversion

$$\psi_{rmn} = \langle rmn | \psi \rangle = \int dU \, D^r_{mn}(U)^* \psi(U). \qquad (4.148)$$

The action of $\hat{T}_{K1}$ on $|\psi\rangle$ now follows from

$$\langle U | \hat{T}_{K1} | \psi \rangle = \int dV \, \exp(\kappa \operatorname{Re} \operatorname{Tr} V) \, \langle U | \hat{L}(V) | \psi \rangle \qquad (4.149)$$

$$= \sum_{rmn} \psi_{rmn} d_r \int dV \, \exp(\kappa \operatorname{Re} \operatorname{Tr} V) \, D^r(VU)_{mn}.$$

Using the group-representation property $D^r(VU) = D^r(V)D^r(U)$, the integral in the above expression, i.e. the complex conjugate of

$$c_{rmn} \equiv \int dV \, D^r_{mn}(V)^* \, \exp(\kappa \operatorname{Re} \operatorname{Tr} V), \qquad (4.150)$$

is the coefficient for the expansion of the exponential in irreps,

$$\exp(\kappa \operatorname{Re} \operatorname{Tr} V) = \sum_{rmn} d_r c_{rmn} D^r_{mn}(V), \qquad (4.151)$$

as follows from the orthogonality of the irreps. A change of variables $V \to V^\dagger$ in (4.150) shows that $c_{rmn} = c^*_{rnm}$. Making a transformation of variables $V \to WVW^\dagger$, with arbitrary group element W, gives the relation

$$c_{rmn} = D^r_{mm'}(W)c_{rm'n'}D^r_{n'n}(W^\dagger). \tag{4.152}$$

Using Schur's lemma it follows that c_{rmn} can be written in the form

$$c_{rmn} = c_r\,\delta_{mn}, \tag{4.153}$$

with real c_r. Returning to (4.149) we get

$$\langle U|\hat{T}_{K1}|\psi\rangle = \sum_{rmn} \psi_{rmn}d_r c_r\, D^r(U)_{mn}. \tag{4.154}$$

Every irrep r is just multiplied by the number c_r. The irrep states $|rmn\rangle$ are eigenstates of $\hat{T}_{K1}$ with eigenvalue c_r,

$$\hat{T}_{K1}|rmn\rangle = c_r\,|rmn\rangle. \tag{4.155}$$

The relation (4.153) holds generally for expansion coefficients of functions on the group which are invariant under $V \to WVW^\dagger$, i.e. class functions. These have a *character expansion*,

$$\exp(\kappa\,\mathrm{Re}\,\mathrm{Tr}\,V) = \sum_r d_r c_r \chi_r(V), \tag{4.156}$$

with

$$\chi_r(V) = \mathrm{Tr}\,D^r(V) = D^r_{mm}(V) \tag{4.157}$$

the character in the representation r. The characters are orthonormal,

$$\int dV\,\chi_r(V)^*\chi_s(V) = \delta_{rs}, \tag{4.158}$$

as follows from (4.145). Writing

$$d_r c_r = \int dV\,\exp\left[\frac{\kappa}{2}\chi_f(V) + \frac{\kappa}{2}\chi_f(V)^*\right]\chi_r(V)^*, \tag{4.159}$$

where f is the fundamental (defining) representation, we can show that the c_r are positive. Expansion of the right-hand side of (4.159) in powers of κ leads to

$$c_r = \sum_{n=0}^{\infty} \frac{(\kappa/2)^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \int dV\,\chi_f(V)^{*k}\chi_f(V)^{n-k}\chi_r(V)^*. \tag{4.160}$$

Reducing the tensor product representation $D^{r_1} \cdots D^{r_k}$ to irreducible components, we see that

$$\int dV \, \chi_{r_1}(V) \cdots \chi_{r_k}(V) = n(r_1, \ldots, r_k) \qquad (4.161)$$

is the number of times the singlet irrep occurs. Since κ is positive the c_r are positive.

It follows that the eigenvalues of $\hat{T}_{K1}$ are positive, i.e. $\hat{T}_{K1}$ is a positive operator. The full kinetic transfer operator $\hat{T}_K$, being the product of single-link operators $\hat{T}_{K1}$, is also positive.

4.8 Hamiltonian for continuous time

In the Hamiltonian approach to lattice gauge-theory time is kept continuous while space is replaced by a lattice. Taking the formal limit $a_t \to 0$ we get the appropriate Hamiltonian from

$$\hat{T} = \hat{P}_0 \, \exp\Big[-a_t \hat{H} + O(a_t^2)\Big]. \qquad (4.162)$$

Some work is required for $\hat{T}_K$ as $a_t \to 0$ since it depends explicitly on a/a_t through $\kappa = 2a/g^2 a_t$. Consider again the form (4.142) for one link,

$$\hat{T}_{K1} = \int dV \, \exp[\kappa \operatorname{Re} \operatorname{Tr} V] \, \hat{L}(V). \qquad (4.163)$$

Since $\kappa \to \infty$ as $a_t \to 0$ we can evaluate this expression with the saddle-point method. The highest saddle point is at $V = 1$. It is convenient to use the exponential parameterization,

$$V = \exp(i\alpha^k t_k), \qquad (4.164)$$

$$\operatorname{Re} \operatorname{Tr} V = d_f - \tfrac{1}{4}\alpha^k \alpha^k + O(\alpha^4), \qquad (4.165)$$

$$dV = \prod_k d\alpha_k \, [1 + O(\alpha^2)], \qquad (4.166)$$

$$\hat{L}(V) = 1 + i\alpha^k \hat{X}_k(L) - \tfrac{1}{2}\alpha^k \alpha^l \hat{X}_k(L)\hat{X}_l(L) + O(\alpha^3), \quad (4.167)$$

where we have written the left translator $\hat{L}$ in terms of its generators $\hat{X}_k(L)$ (cf. appendix A.3). Gaussian integration over α^k gives

$$\hat{T}_{K1} = \text{constant} \times \left(1 - \frac{1}{\kappa}\hat{X}^2 + \cdots\right), \qquad (4.168)$$

$$\hat{X}^2 = \hat{X}_k(L)\hat{X}_k(L) = \hat{X}_k(R)\hat{X}_k(R), \qquad (4.169)$$

$$\text{constant} = \int \prod_k d\alpha_k \, \exp[\kappa(d_f - \alpha^2/4)]. \qquad (4.170)$$

Here the constant could have been avoided by changing the measure in the path integral by an overall constant.

The Hamiltonian can be written as

$$\hat{H} = \hat{K} + \hat{W} \tag{4.171}$$
$$= \frac{1}{a}\left[\frac{g^2}{2}\sum_{l_s}\hat{X}_{l_s}^2 + \frac{2}{g^2}\sum_{p_s}\mathrm{Re\,Tr}\left(1 - \hat{U}_{p_s}\right)\right] + \mathrm{constant},$$

where the l_s denote the spatial links and p_s the spatial plaquettes. In the coordinate representation $\hat{U} \to U$ and $\hat{X}^2$ becomes the covariant Laplacian on group space. The above Hamiltonian is known as the Kogut–Susskind Hamiltonian [40].

It is good to keep in mind that, with continuous time and a lattice in space, the symmetry between time and space is broken. It is necessary to renormalize the velocity of light, which amounts to introducing different couplings g_K^2 and g_W^2 for the kinetic and potential terms in the Hamiltonian (4.171).

The formal continuum limit $a \to 0$, $U_{\mu x} = \exp(-iaG_{\mu x}) \to 1 - iaG_\mu(x) + \cdots$ leads to the formal continuum Hamiltonian in the temporal gauge:

$$H = \int d^3x \left(\frac{g^2}{2}\Pi_k^p\Pi_k^p + \frac{1}{4g^2}G_{lm}^p G_{lm}^p\right) = \int d^3x \left(\frac{1}{2}E^2 + \frac{1}{2}B^2\right), \tag{4.172}$$

where

$$\Pi_k^p = -i\frac{\delta}{\delta G_k^p}, \quad p = 1,\ldots,n^2 - 1, \quad k = 1,2,3, \tag{4.173}$$
$$G_{lm}^p = \partial_l G_m^p - \partial_m G_l^p + f_{pqr}G_l^q G_m^r, \tag{4.174}$$

and the conventional 'electric' and 'magnetic' fields are given by

$$E_k^p = -g\Pi_k^p, \quad B_k^p = \frac{1}{g}\epsilon_{klm}G_{lm}^p. \tag{4.175}$$

In the continuum the canonical quantization in the temporal gauge is often lacking in text books, because it is less suited for weak-coupling perturbation theory. A brief exposition is given in appendix B.

4.9 Wilson loop and Polyakov line

In the classical Maxwell theory an external current J^μ enters in the weight factor in the real-time path integral as

$$e^{iS} \rightarrow e^{iS + i \int d^4x \, J^\mu A_\mu}. \tag{4.176}$$

For a line current along a path $z^\mu(\tau)$,

$$J^\mu(x) = \int d\tau \, \frac{dz^\mu(\tau)}{d\tau} \, \delta^4(x - z(\tau)), \tag{4.177}$$

the phase $\exp(i \int J^\mu A_\mu)$ takes the form

$$\exp\left[i \int dz^\mu A_\mu(z)\right], \tag{4.178}$$

where the integral is along the path specified by $z(\tau)$. The current is 'conserved' (i.e. $\partial_\mu J^\mu = 0$) for a closed path or a never-ending path. In classical electrodynamics one thinks of $z^\mu(\tau)$ as the trajectory of a point charge. Then $dz^\mu/d\tau$ is timelike. For a positive static point charge at the origin the phase is

$$\exp\left[i \int dz^0 \, A_0(\mathbf{0}, z^0)\right]. \tag{4.179}$$

We may however choose the external current as we like and use also spacelike $dz/d\tau$. For a line current running along the coordinate 3-axis the phase is

$$\exp\left[i \int dz^3 \, A_3(0, 0, z^3, 0)\right]. \tag{4.180}$$

The Euclidean form is obtained from the Minkowski form by the substitution $J^0 = -iJ_4$, $dz^0 = -i \, dz_4$, $A_0 = iA_4$. The phase remains a phase,

$$\exp\left[i \int \sum_{\mu=1}^{4} dz_\mu A_\mu(z)\right]. \tag{4.181}$$

The source affects the places where the time components enter in the action and we have to take a second look at the derivation of the transfer operator. Consider therefore first in the compact $U(1)$ theory the path integral

$$Z(J) = \int DU \, \exp\left[S(U) + \sum_{x\mu} J_{\mu x} A_{\mu x}\right], \tag{4.182}$$
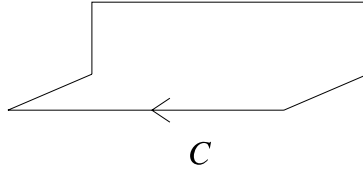
C

Fig. 4.3. A contour C specifying a line current or Wilson loop.

$$S(U) = \frac{1}{4g^2} \sum_{x\mu\nu} U_{\mu\nu x}, \tag{4.183}$$

$$U_{\mu x} = \exp(-iA_{\mu x}), \tag{4.184}$$

$$U_{\mu\nu x} = \exp[i(A_{\mu x+\hat\nu} - A_{\mu x} - A_{\nu x+\hat\mu} + A_{\nu x})]. \tag{4.185}$$

We have written the source term in conventional Euclidean form as a real-looking addition to $S(U)$, but the current J_μ is purely imaginary. For a line current of unit strength over a closed contour C as illustrated in figure 4.3 we have

$$\begin{aligned} J_{\mu x} &= -i \quad \text{for links } (x, x + \hat\mu) \in C \\ &= +i \quad \text{for links } (x + \hat\mu, x) \in C \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{4.186}$$

This current is 'conserved',

$$\partial'_\mu J_{\mu x} = \sum_\mu (J_{\mu x} - J_{\mu x-\hat\mu}) = 0, \tag{4.187}$$

and the integrand of the path integral is gauge invariant. The phase factor associated with the current can be written in another way,

$$\exp\left(\sum_{x\mu} J_{\mu x} A_{\mu x}\right) = \prod_{l \in C} U_l \equiv U(C), \tag{4.188}$$

where l denotes a directed link, $U_l = U_{\mu x}$ for $l = (x, x + \hat\mu)$. Such a product $U(C)$ of U's around a loop C is called a Wilson loop [39]. It is gauge invariant. The simplest Wilson loop is the plaquette $U_{\mu\nu x}$.

The Wilson-loop form of the interaction with an external line source generalizes easily to non-Abelian gauge theories. For a source in irrep r we have

$$\operatorname{Tr} D^r(U(C)) = \chi_r(U(C)), \tag{4.189}$$

with $D^r(U(C))$ the ordered product of the link matrices $D^r(U_l)$ along the loop C. Denoting the links l by the pair of neighbors (x, y), we have

for example in the fundamental representation

$$U(C) = \text{Tr}\,(U_{x_1 x_2} U_{x_2 x_3} \cdots U_{x_n x_1}). \qquad (4.190)$$

The gauge invariance is obvious: the gauge transformations cancel out pairwise in the product along the closed loop.

Consider now the derivation of the transfer operator. For the parts of C where it runs in spacelike directions it represents an operator in Hilbert space through $U_l \to \hat{U}_l$ as before. What about the timelike links? Suppose that between two time slices there are only two such links, say the links $(y, y + \hat{4})$ and $(z + \hat{4}, z)$. Then, for these time slices, (4.123) is modified to

$$\langle U'|\hat{T}'_K|U\rangle = \prod_{\mathbf{x},m} \int dU_{4\mathbf{x}} \, \exp[\cdots] \, D^r_{mn}(U_{4\mathbf{y}}) D^r_{pq}(U^\dagger_{4\mathbf{z}}), \qquad (4.191)$$

where $\exp[\cdots]$ is the same as in (4.123) and the indices m, n, p, q hook up to the other D^r's of the Wilson loop. We see that the operator $\hat{P}_0$ defined in (4.127) is replaced by

$$\hat{P}_0 \to \int \prod_{\mathbf{x}} dV_{\mathbf{x}} \, \hat{D}(V) D^r_{mn}(V_{\mathbf{y}}) D^r_{pq}(V^\dagger_{\mathbf{z}}), \qquad (4.192)$$

where we used the notation $\Omega^\dagger_{\mathbf{x}} = V_{\mathbf{x}} = U_{4\mathbf{x}}$, $d\Omega = dV$. The gauge-transformation operator $\hat{D}(V)$ is the product of operators $\hat{D}(V_{\mathbf{x}})$ at sites $\mathbf{x}$. With the notation

$$\hat{P}^{r\mathbf{x}}_{mn} = \int dV_x \, D^r_{mn}(V_{\mathbf{x}})\hat{D}(V_{\mathbf{x}}), \qquad (4.193)$$

the right-hand side of (4.192) can be written as

$$\hat{P}'_0 \, \hat{P}^{r\mathbf{y}}_{mn} \, \hat{P}^{\bar{r}\mathbf{z}}_{pq}, \qquad (4.194)$$

with

$$\hat{P}'_0 = \prod_{\mathbf{x}\neq\mathbf{y},\mathbf{z}} \hat{P}^{0\mathbf{x}} \qquad (4.195)$$

the projector onto the gauge-invariant subspace except at $\mathbf{y}$ and $\mathbf{z}$. The irrep $\bar{r}$ is the Hermitian conjugate of the irrep r. The operator $\hat{P}^{r\mathbf{x}}_{mn}$ projects onto the subspace transforming at $\mathbf{x}$ in the irrep r in the following way. Let $|skl\rangle$ be an irrep state for some link $(\mathbf{u}, \mathbf{v})$,

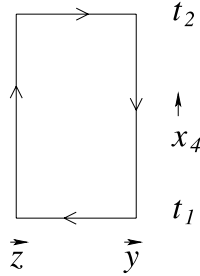$$\langle U|skl\rangle = D^s_{kl}(U), \quad U = U_{\mathbf{u},\mathbf{v}}; \qquad (4.196)$$

Fig. 4.4. A rectangular timelike Wilson loop.

then, for $\mathbf{x} = \mathbf{u}$,

$$\sum_n \langle U|\hat{P}^{r\mathbf{u}}_{mn}|snl\rangle = \int dV\, D^r_{mn}(V)D^s_{nl}(VU) = \delta_{r\bar{s}}D^s_{ml}(U),$$

$$\sum_n \hat{P}^{\bar{s}\mathbf{u}}_{mn}|snl\rangle = |sml\rangle; \tag{4.197}$$

similarly, for $\mathbf{x} = \mathbf{v}$,

$$\sum_m \langle U|\hat{P}^{r\mathbf{v}}_{mn}|skm\rangle = \int dV\, D^r_{mn}(V)D^s_{km}(UV^\dagger) = \delta_{rs}D^s_{kn}(U),$$

$$\sum_m \hat{P}^{s\mathbf{v}}_{mn}|skm\rangle = |skn\rangle. \tag{4.198}$$

The $\hat{P}^r$ are Hermitian projectors in the following sense:

$$(\hat{P}^{r\mathbf{x}}_{mn})^\dagger = \hat{P}^{r\mathbf{x}}_{nm}, \tag{4.199}$$

$$\sum_n \hat{P}^{r\mathbf{x}}_{mn}\hat{P}^{r\mathbf{x}}_{nq} = \hat{P}^{r\mathbf{x}}_{mq}. \tag{4.200}$$

Consider next a Wilson loop of the form shown in figure 4.4. In the $U(1)$ case this corresponds to two charges that are static at times between t_1 and t_2, a charge $+1$ at $\mathbf{z}$ and a charge -1 at $\mathbf{y}$:

$$J_4(\mathbf{x}, x_4) = -i[\delta_{\mathbf{x},\mathbf{y}} - \delta_{\mathbf{x},\mathbf{z}}], \quad t_1 < x_4 < t_2. \tag{4.201}$$

In the $SU(n)$ case the interpretation is evidently that we have a source in irrep $\bar{r}$ at $\mathbf{y}$ and a source in irrep r at $\mathbf{z}$. If r is the defining representation of the gauge group $SU(3)$ we say that we have a static quark at $\mathbf{z}$ and an antiquark at $\mathbf{y}$. The path-integral average of this Wilson loop

$$W(C) = \frac{1}{Z}\int DU\, \exp[S(U)]\, \chi_r(U(C)), \tag{4.202}$$

$$Z = \int DU \, \exp[S(U)] = \operatorname{Tr} \hat{T}^N, \qquad (4.203)$$

can be expressed as

$$W(C) = \frac{1}{Z} \operatorname{Tr} \left[\hat{T}^{N-t} D_{kl}^r(\hat{U}^\dagger)\,(\hat{T}')^t \, \hat{P}_{lm}^{\bar{r}\mathbf{z}} \, D_{mn}^r(\hat{U}) \, \hat{P}_{nk}^{r\mathbf{y}} \right]. \quad (4.204)$$

Here C is the rectangular loop shown in figure 4.4, $t = t_2 - t_1$, $\hat{U}$ is the operator corresponding to the product of U's at time t_1 and similarly for $\hat{U}^\dagger$ at t_2, and $\hat{T}'$ is the transfer operator with $\hat{P}_0'$ (cf. (4.195)). In the zero-temperature limit $N \to \infty$, the trace in (4.204) is replaced by the expectation value in the ground state $|0\rangle$, $\hat{T}|0\rangle = \exp(-E_0)\,|0\rangle$. Inserting intermediate states $|n\rangle$, which are eigenstates of $\hat{T}' \, \hat{P}_{lm}^{\bar{r}\mathbf{z}} \, \hat{P}_{nk}^{r\mathbf{y}}$ with eigenvalues $\exp(-E_n')$, gives the representation

$$W(C) = \sum_n R_n \, e^{-(E_n' - E_0)t}, \quad N = \infty, \qquad (4.205)$$

where R_n and E_n' depend on $\mathbf{y}$ and $\mathbf{z}$. For large times t the lowest energy level E_0' will dominate. This is the energy of the ground state $|0'\, rmn\rangle$ in that sector of Hilbert space which corresponds to the static sources at $\mathbf{y}$ and $\mathbf{z}$. By definition, the difference $E_0' - E_0$ is the *potential V*:

$$W(C) \overset{t\to\infty}{\to} R_0 \, e^{-Vt}, \quad V = V^r(\mathbf{y}, \mathbf{z}), \quad R_0 = R_0(\mathbf{y}, \mathbf{z}), \quad (4.206)$$
$$R_0 = \sum_{mn} \langle 0|D_{mn}^r(\hat{U}^\dagger)|0'\, rmn\rangle \langle 0'\, rmn|D_{mn}^r(\hat{U})|0\rangle.$$

Hence, we have found a formula for the static potential (e.g. for a quark–antiquark pair) in terms of the expectation value of a Wilson loop.

Another interesting quantity is the Polyakov line [41], which is a string of U's closed by periodic boundary conditions in the Euclidean time direction. (In case of closure by periodic boundary conditions in the spatial direction, this is often called a Wilson line.) For example, the situation illustrated in figure 4.5 corresponds to a single static quark, a source which is always switched on. The expectation value $W(L)$ of the Polyakov line operator at $\mathbf{x}$, e.g. in the defining representation

$$\operatorname{Tr} U(L) = \operatorname{Tr}(U_{4\mathbf{x},0} U_{4\mathbf{x},1} \cdots U_{4\mathbf{x},N-1}), \qquad (4.207)$$

can be written as

$$W(L) = \langle \operatorname{Tr} U(L) \rangle = \frac{1}{Z} \operatorname{Tr} \left[(\hat{T}')^N \sum_m \hat{P}_{mm}^{r\mathbf{x}} \right]. \qquad (4.208)$$

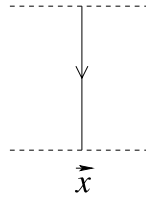Fig. 4.5. A Polyakov line.

It is the free energy of a static quark at inverse temperature N. For temperature going to zero it behaves as

$$W(L) \propto e^{-\epsilon N}, \quad N \to \infty, \tag{4.209}$$

with ϵ the self-energy of a static quark.

4.10 Problems

(i) *The case $SU(2)$*

(a) Work out the metric $g_{kl} = 2\operatorname{Tr}\left[(\partial U/\partial \alpha_k)\,(\partial U^\dagger/\partial \alpha_l)\right]$ using the exponential parameterization $U = \exp(i\alpha^k \tau_k/2)$.

(b) Determine the normalization constant ν in $dU = \nu\sqrt{\det g}\,d\alpha_1\,d\alpha_2\,d\alpha_3$ such that $\int dU = 1$.

(c) Find the characters $\chi_j(U) = \operatorname{Tr} D^j(U)$ as a function of α^k $(j = \frac{1}{2}, 1, \frac{3}{2}, \ldots)$.

(d) Check the orthogonality relation $\int dU\,\chi_j(U)\chi_{j'}^*(U) = \delta_{jj'}$.

(e) Verify for a one-link state that $_1\langle U|\hat{X}_k(L)|\psi\rangle_1 = X_k(L)\psi_1(U)$.

(f) Verify that $X^2(L) = X^2(R)$.

(ii) *Two-dimensional $SU(n)$ gauge-field theory*

Consider two-dimensional $SU(n)$ gauge theory with action

$$S = \sum_p L(U_p), \tag{4.210}$$

$$L(U) = \frac{1}{g^2}\sum_r \kappa_r \operatorname{Re}\chi_r(U), \tag{4.211}$$

$$\sum_r \kappa_r \rho_r = 1, \tag{4.212}$$

and periodic boundary conditions in space. In ordinary units g has the dimension of mass (in two dimensions), such that in lattice

units $g \to 0$ in the continuum limit. The transfer operator is given by

$$\hat{T} = \hat{T}_K \hat{P}_0, \quad \hat{T}_K = \prod_l \hat{T}_{Kl}, \quad (4.213)$$

where l, $l = 0, \ldots, N - 1$ labels the spatial links $(x, x + 1)$, $x = 0, \ldots, N - 1$. Since there is only one space direction, the link variables in the spatial direction may be denoted by U_x. Consider the wavefunction

$$\psi_{\{r,m,n\}}(U) = \prod_x D^{r_x}_{m_x n_x}(U_x), \quad (4.214)$$

which is just a product of irreps r_x at each x.

(a) Show that

$$P_0 \psi_{\{r,m,n\}}(U) \equiv \langle U | \hat{P}_0 | \psi \rangle \quad (4.215)$$
$$= d_{r_0}^{-N} \text{Tr} \left[D^{r_0}(U_0) D^{r_1}(U_1) \cdots D^{r_{N-1}}(U_{N-1}) \right]$$
$$\times \delta_{r_0 r_1} \cdots \delta_{r_{N-1} r_0} \, \delta_{n_0 m_1} \delta_{n_1 m_2} \cdots \delta_{n_{N-1} m_0}.$$

Hence, the gauge-invariant component is non-zero only if all irreps are equal, say r, and it is a Wilson line in the spatial direction.

(b) Show that the energy spectrum of the system is given by

$$E_r = -N \left[\ln a_0 + \ln \left(\frac{\langle \chi_r \rangle_1}{d_r} \right) \right], \quad (4.216)$$

where

$$\langle \chi_r \rangle_1 = \frac{\int dU \, e^{L(U)} \chi_r(U)}{\int dU \, e^{L(U)}}. \quad (4.217)$$

(c) Show for $g \to 0$, using a saddle-point expansion about $U = 1$, that

$$\frac{\langle \chi_r \rangle_1}{d_r} \to 1 - \tfrac{1}{2} C_2^r g^2 + O(g^4), \quad (4.218)$$

where C_2^r is the value of the quadratic Casimir operator in the representation r. This result holds independently of the detailed choice of κ_r's, as long as they satisfy the constraint (4.212),

(d) Restoring the lattice spacing a, $L = Na$, deduce from the result above that, in the continuum limit, the energy spectrum takes the universal form

$$E_r - E_0 = \tfrac{1}{2} g^2 C_2^r L. \quad (4.219)$$

(iii) *Glueball masses and string tension*

Simple glueball operators may be defined in terms of the plaquette field $\text{Tr}\, U_p$, where $p = (\mathbf{x}, m, n)$ denotes a spacelike plaquette. When this operator acts on the ground state (vacuum state) it creates a state with the quantum numbers of the plaquette. Similarly, a string state may be created by the operator $U_{\mathbf{x},\mathbf{y}} = \prod_{l \in C} U_l$, where the links l belong to an open contour from $\mathbf{x}$ to $\mathbf{y}$. The string state defined this way is not gauge invariant at $\mathbf{x}$ and $\mathbf{y}$; it has to be interpreted as a state with external sources at these points.

Using the transfer-operator formalism, derive to leading order a strong-coupling formula for the glueball mass corresponding to the plaquette, and for the string mass corresponding to $U_{\mathbf{x},\mathbf{y}}$. Use lattice units $a = a_t = 1$. Note that the potential-energy factors $\exp(-a_t \hat{W}/2)$ in the transfer operator may be neglected to leading order in $1/g^2$.