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DETECTION OF SINGULAR POINTS

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ABSTRACT. A theorem is proved which gives a new test for singular points on the circle of convergence of the function $f(z)$ defined by a power series $\sum a_n z^n$ with finite non-zero radius of convergence.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence 1. The circle of convergence, $\{z : |z| = 1\}$, then contains at least one singular point for f. A classical problem in complex analysis is to determine conditions on the sequence of coefficients which ensure that a particular point on the circle is a singular point. There is no loss of generality in supposing that the point to be considered is $z = 1$.

A standard method for devising these tests is to consider an analytic univalent function $z = g(t)$ whose inverse $t = h(z)$ has the properties: (i) $h(\{z : |z| < 1\}) = R$, (ii) $h(1) = \alpha$, $\alpha > 0$, (iii) $\{t : |t| \leq \alpha, t \neq \alpha\} \subset R$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (g(t))^n = \sum_{n=0}^{\infty} b_n t^n = F(t),$$

where the b_n are determined in terms of the a_n. It follows that $z = 1$ is a singularity for f if and only if $t = \alpha$ is a singular point for F. Moreover, $t = \alpha$ is a singular point for F if and only if the radius of convergence of $\sum_{n=0}^{\infty} b_n t^n$ is α, since all points of $\{t : |t| \leq \alpha, t \neq \alpha\}$ are regular because of the properties of the mapping $t = h(z)$. Hence, $z = 1$ is a singular point for f if and only if $\limsup \sqrt[n]{|b_n|} = 1/\alpha$.

Some theorems which are proved using this method may be found in [1], [2], and [3, p. 216].

The following theorem gives a new test for singular points. The proof exploits the general method but the calculation is not quite routine because, here, the function $z = g(t)$ is not univalent. This means that the inverse $t = h(z)$ is defined on a Riemann surface so that care must be taken in the analysis to ensure that $h(\{z : |z| < 1\})$ has the required properties. An interesting feature of the theorem is that, for each fixed n, the coefficient b_n of the rearranged series involves roughly only $n/2$ of the coefficients of the original series.

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THEOREM. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence* 1. *Let* $0<k<\frac{3}{4}$. *Then* $z=1$ *is a singular point for f if and only if*

$$\limsup \left|\sum_{s=0}^{[n/2]} a_{n-s}\binom{n-s}{s}k^s\right|^{1/n}=\frac{1+\sqrt{1+4k}}{2}.$$

Proof. Let $z=g(t)=t+kt^2$. It is easy to see that the Riemann surface for the inverse function consists of two copies of the z-plane slit along the negative real axis from $-\frac{1}{4}k=g(-\frac{1}{2}k)$ to infinity. The function $t=h(z)=(-1+\sqrt{1+4kz})/2k$ maps one of these sheets onto $\{t:\operatorname{Re} t>-\frac{1}{2}k\}$, with the origin mapping to the origin.

Now let $\alpha=(-1+\sqrt{1+4k})/2k=h(1)$. Moreover, $|t|\leq\alpha$ implies $|z|=|t+kt^2|\leq\alpha+k\alpha^2=1$. Thus $g(\{t:|t|\leq\alpha\}\subset\{z:|z|\leq 1\}$. Also $g(\alpha)=1$ and $t=\alpha$ is the only point of $\{t:|t|\leq\alpha\}$ which maps to a point of $\{z:|z|\leq 1\}$. For if $g(\alpha e^{i\theta})=z_0=x_0+iy_0$ then $x_0=\alpha\cos\theta+k\alpha^2\cos 2\theta$ and $y_0=\alpha\sin\theta+k\alpha^2\sin 2\theta$. So if $|z_0|=1$ then it follows that $\alpha^2+k^2\alpha^4+2k\alpha^3\cos\theta=1$ and hence $\cos\theta=(1-\alpha^2-k^2\alpha^4)/2k\alpha^3$. This reduces to $\cos\theta=1$ so that $\theta=0$.

Write $f(z)=\sum_{n=0}^{\infty} a_n z^n=\sum_{n=0}^{\infty} a_n(g(t))^n=\sum_{n=0}^{\infty} b_n t^n=F(t)$. Now $t=h(z)$ maps some subset, say D, of $\{z:|z|\leq 1\}$ onto $\{t:|t|\leq\alpha\}$. Moreover $D\setminus\{1\}\subset\{|z|<1\}$ and $h(1)=\alpha$. If $k<\frac{3}{4}$ then $\alpha<\frac{1}{2}k$ and D contains no point of the slit from $-\frac{1}{4}k$ to infinity. It follows that $z=1$ is a singular point for f if and only if $t=\alpha$ is a singular point for F, that is if and only if $\overline{\lim}\sqrt[n]{|b_n|}=1/\alpha$. But

$$F(t)=\sum_{s=0}^{\infty} a_s(t+kt^2)^s=\sum_{s=0}^{\infty} a_s t^s\sum_{n=0}^{s}\binom{s}{n}k^n t^n.$$

A rearrangement gives $F(t)=\sum_{n=0}^{\infty}\left(\sum_{s=0}^{[n/2]} a_{n-s}\binom{n-s}{s}k^s\right)t^n$, and the theorem follows.

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