Adv. Appl. Prob. **39,** 492–509 (2007)
Printed in Northern Ireland

FIRST-CROSSING AND BALLOT-TYPE RESULTS FOR SOME NONSTATIONARY SEQUENCES

CLAUDE LEFÈVRE,* *Université Libre de Bruxelles*

Abstract

In this paper we consider the problem of first-crossing from above for a partial sums process $m + S_t,\ t \geq 1$, with the diagonal line when the random variables $X_t,\ t \geq 1$, are independent but satisfying nonstationary laws. Specifically, the distributions of all the X_ts belong to a common parametric family of arithmetic distributions, and this family of laws is assumed to be stable by convolution. The key result is that the first-crossing time distribution and the associated ballot-type formula rely on an underlying polynomial structure, called the generalized Abel–Gontcharoff structure. In practice, this property advantageously provides simple and efficient recursions for the numerical evaluation of the probabilities of interest. Several applications are then presented, for constant and variable parameters.

Keywords: Parametric distributions; stability by convolution; first-crossing of partial sums; generalized Abel–Gontcharoff polynomials; discrete distributions; total progeny; busy period; ruin probability

2000 Mathematics Subject Classification: Primary 60G40; 60E05; 62P05

1. Introduction

A considerable amount of literature is devoted to the first-crossing problem of a certain stochastic process with a given boundary, lower or upper, linear or nonlinear. This is especially true for Poisson and compound Poisson processes (e.g. Stadje (1993), Picard and Lefèvre (1996), Ignatov and Kaishev (2000), (2004), Perry *et al.* (2002), and Zacks (1991), (2005)) and for Brownian motion (e.g. Perry *et al.* (2004) and the references therein).

In this framework, ballot-type results play an important role for the study of various problems in probability and statistics, in particular in queueing, risk theory, reliability, sequential analysis, and random graphs. Much on these problems can be found in the comprehensive book by Takács (1967); see also the review paper by Takács (1997).

A classical ballot theorem is concerned with a sequence $\{X_t,\ \ t = 1, 2, \dots\}$ of independent, identically distributed (i.i.d.) random variables taking on values in the set $\mathbb{N} = \{0, 1, 2, \dots\}$. Specifically, let

$$S_t = X_1 + \cdots + X_t, \qquad t \geq 1,$$

denote the associated partial sums (with $S_0 \equiv 0$). Then, for any $m = 1, \dots, n$ with $n \geq 1$,

$$\mathrm{P}(S_t < t \text{ for } t = 1, \dots, n \mid S_n = n - m) = \frac{m}{n}. \tag{1.1}$$

Received 16 October 2006; revision received 15 February 2007.
* Postal address: Département de Mathématique, Université Libre de Bruxelles, Campus de la Plaine C.P. 210, B-1050 Bruxelles, Belgium. Email address: clefevre@ulb.ac.be

We mention that (1.1) holds true when the sequence $\{X_t,\ t \geq 1\}$ is not only i.i.d. but cyclically exchangeable, i.e. when the X_ts are still equidistributed but with cyclical exchangeability as a dependence structure. A version of (1.1) also exists for a continuous time analogue of such a process.

Now, in (1.1) the left-hand side represents the conditional probability that the process S_t remains *below* the diagonal line (i.e. a line of slope one through the origin) between times 1 and n, given that at time n it is under the line at level $n - m$. Alternatively, the X_ts being i.i.d., the variables S_t and $S_n - S_{n-t}$ for $1 \leq t \leq n$ are equidistributed, so that (1.1) can be rewritten as

$$\mathrm{P}(m + S_t > t \text{ for } t = 1, \ldots, n-1 \mid m + S_n = n) = \frac{m}{n}. \tag{1.2}$$

The left-hand side in (1.2) represents the conditional probability that the process $m + S_t$ remains *above* the diagonal line between times 1 and n, given that it crosses (and meets) the line at time n; in other words, this is the probability that a crossing-time n is in fact a first-crossing time for the process. We note that the passage from (1.1) to (1.2) can be visualized graphically since it amounts to operating a rotation of 180 degrees and following the trajectories in reversed time, from time n to time 0. In the sequel, it will be more convenient for us to consider (1.2) rather than (1.1).

In this paper we consider a similar problem of first-crossing for the process $m+S_t,\ t \geq 1$, with the diagonal line, but this time for some nonstationary sequences. More precisely, the X_ts are still assumed to be independent, as above, but now they are no longer necessarily equidistributed, at least in a specific sense. The central assumption is that the laws of all the X_ts belong to the same parametric family of arithmetic distributions, possibly with different parameters, and that this family of distributions is stable by convolution. Now, we will show that under this condition, the first crossing-time distribution and the associated ballot-type formula rely on a remarkable algebraic structure of polynomial form. This result advantageously provides simple recursions, directly implementable and numerically efficient, for the computation of the probabilities of interest. To the best of our knowledge, boundary crossing for *nonstationary* processes like here has received little study to date.

The paper is organized as follows: in Section 2, we consider parametric families of arithmetic distributions that are stable by convolution, and we point out that such laws exhibit a polynomial component in the parameter. In Section 3, we give a short presentation, adapted to the present framework, of a particular family of polynomials, named generalized Abel–Gontcharoff (A.G.) polynomials. In Section 4, we examine the first-crossing problem of the process $m + S_t$, $t \geq 1$, with the diagonal line when the random variables $X_t,\ t \geq 1$, are independent and of laws belonging to a parametric family such as described before (possibly with different parameters). Using the generalized A.G. polynomials as a mathematical tool, we will first derive the distribution of the first-crossing level and then deduce an associated ballot-type formula generalizing (1.2). Finally, various applications are presented: in Section 5 for a traditional case where all the parameters are identical, and in Section 6 for several less standard situations (including risk theory).

We indicate that our approach, which is based on the recourse to polynomials of the A.G.-kind, is inspired from previous works covering the study of epidemic models. The reader is referred, for instance, to Lefèvre and Picard (1990), (2005) and Ball and O'Neill (1999) for standard A.G. polynomials; the generalized version used here was initially introduced in Picard and Lefèvre (1996).

2. Parametric distributions stable by convolution

Let $X(\theta)$ be a discrete random variable valued in $\mathbb{N}$, with a parametric distribution, $\mathcal{L}(\theta)$ say, depending on some parameter θ. It is assumed that θ takes on values in a set $\mathcal{D}$ which corresponds either to $\mathbb{R}^+ = [0, \infty)$ or to $\mathbb{N}$.

Denote $p_i(\theta) = \mathrm{P}[X(\theta) = i]$, for $i \geq 0$. To avoid degenerate cases, we put $p_i(0) = \delta_{i,0}$ (the Kronecker delta), $i \geq 0$, and $0 < p_0(\theta) < 1$ for all $\theta > 0$. When $\mathcal{D} = \mathbb{R}^+$, we add the assumption that $p_i(\theta)$ is continuous at $\theta = 0$ for every i. Now, the main property required for $\mathcal{L}(\theta)$ is its stability by convolution, meaning that, for any $\theta, \tilde{\theta} \in \mathcal{D}$,

$$p_i(\theta + \tilde{\theta}) = \sum_{j=0}^{i} p_j(\theta) p_{i-j}(\tilde{\theta}), \qquad i \geq 0. \tag{2.1}$$

Under these conditions, it is easily seen that when $\theta \in \mathbb{R}^+$, $X(\theta)$ has a compound Poisson law, and when $\theta \in \mathbb{N}$, $X(\theta)$ is the θ-fold convolution of $X(1)$. This (known) property is derived below in order to point out that for both cases, the terms $p_i(\theta)/p_0(\theta)$, $i \geq 0$, are of polynomial form in θ.

Property 2.1. *For $\theta \in \mathcal{D}$, $e_i(\theta)$ defined by*

$$e_i(\theta) \equiv p_i(\theta)/p_0(\theta), \qquad i \geq 0, \tag{2.2}$$

is a polynomial of degree i in θ. Moreover, the following properties hold:

$$e_0(\theta) = 1, \tag{2.3}$$

$$e_i(0) = \delta_{i,0}, \qquad i \geq 0, \tag{2.4}$$

$$e_i(\theta + \tilde{\theta}) = \sum_{j=0}^{i} e_j(\theta) e_{i-j}(\tilde{\theta}), \quad i \geq 0. \tag{2.5}$$

Proof. Let $f(\theta, z)$ denote the probability generating function of $X(\theta)$, with argument $z \in [0, 1]$. By (2.1), we have

$$f(\theta + \tilde{\theta}, z) = f(\theta, z) f(\tilde{\theta}, z) \quad \text{for } \theta, \tilde{\theta} \in \mathcal{D}. \tag{2.6}$$

First, take $\mathcal{D} = \mathbb{R}^+$. As, by hypothesis, $f(\theta, z)$ is continuous at $\theta = 0$ for every z, we get from (2.6) that $f(\theta, z)$ is an exponential function in θ (see, e.g. Breiman (1968, Chapter 14)). In other words, $f(\theta, z)$ is infinitely divisible with

$$f(\theta, z) = \exp[\theta\lambda(g(z) - 1)] \quad \text{for } \theta \in \mathbb{R}^+, \tag{2.7}$$

where $\lambda > 0$ and $g(z)$ is the probability generating function of some distribution $\{q_i,\ i \geq 1\}$. Thus, $\mathcal{L}(\theta)$ is a compound Poisson distribution, i.e.

$$p_i(\theta) = \mathrm{e}^{-\lambda\theta} \sum_{j=0}^{i} \frac{(\lambda\theta)^j}{j!} q_i^{*j}, \qquad i \geq 0, \tag{2.8}$$

where $\{q_i^{*j},\ i \geq 1\}$, $j \geq 1$, is the jth convolution of $\{q_i,\ i \geq 1\}$, and $q_0^{*j} = \delta_{j,0}$, $j \geq 0$.

https://doi.org/10.1239/aap/1183667620 Published online by Cambridge University Press

Now, take $\mathcal{D} = \mathbb{N}$. By (2.6), we have

$$f(\theta, z) = [f(1, z)]^{\theta} \quad \text{for } \theta \in \mathbb{N}, \tag{2.9}$$

and thus, $\mathcal{L}(\theta)$ is the θth convolution of $\mathcal{L}(1)$. To express $p_i(\theta)$, we rewrite (2.9) in the form

$$f(\theta, z) = [p_0(1)]^{\theta} \left\{ 1 + \frac{1 - p_0(1)}{p_0(1)} \sum_{i=1}^{\infty} q_i z^i \right\}^{\theta}, \tag{2.10}$$

where $\{q_i = p_i(1)/[1 - p_0(1)],\ \ i \geq 1\}$ denotes the conditional distribution of $X(1)$ given that $X(1) \geq 1$. By expanding (2.10), we obtain

$$\begin{aligned} f(\theta, z) &= [p_0(1)]^{\theta} \sum_{j=0}^{\infty} \binom{\theta}{j} \left[\frac{1 - p_0(1)}{p_0(1)} \right]^j \left(\sum_{i=1}^{\infty} q_i z^i \right)^j \\ &= p_0(\theta) \sum_{j=0}^{\infty} \binom{\theta}{j} \left[\frac{1 - p_0(1)}{p_0(1)} \right]^j \sum_{i=j}^{\infty} q_i^{*j} z^i, \end{aligned}$$

which yields

$$p_i(\theta) = p_0(\theta) \sum_{j=0}^{i} \binom{\theta}{j} \left[\frac{1 - p_0(1)}{p_0(1)} \right]^j q_i^{*j}, \qquad i \geq 0. \tag{2.11}$$

From (2.8) and (2.11), we then see that in both cases, $e_i(\theta)$ defined by (2.2) is indeed a polynomial of degree i in θ. Furthermore, (2.3), (2.4), and (2.5) are straightforward since (2.1) implies that $p_i(0) = \delta_{i,0}$ for $i \geq 0$ and $p_0(\theta + \tilde{\theta}) = p_0(\theta) p_0(\tilde{\theta})$ for $\theta, \tilde{\theta} \in \mathcal{D}$.

Note that in the case of an algebraic treatment, the right-hand sides of (2.8) and (2.11) may be directly extended to any real value of θ, positive or negative. In other words, the polynomials $e_i(\theta)$ may be extended to $\mathbb{R} = (-\infty, \infty)$ in an obvious way; their convolution property (2.5), for instance, will then remain valid on $\mathbb{R}$. Of course, whenever a probabilistic interpretation is needed, only $\theta \in \mathcal{D}$ can be taken into account.

The existence of this polynomial structure is at the basis of our methodology to extend the ballot formula. The $e_i(\theta)$s are called below fundamental polynomials. Their exact expression will be of little use in the arguments, but (2.3), (2.4), and (2.5) will play a central role.

Note that each $e_i(\theta)$ is of degree at most i, its degree being equal to i when $q_1 \neq 0$, i.e. if $p_1(1) \neq 0$. We will assume that this condition is satisfied, in order to guarantee that the family $\{e_i(\theta),\ \ i \geq 0\}$ is linearly independent. This is not a real restriction, however, because one may, without difficulty, let $p_1(1) \to 0$.

In Section 2.1 we list a few examples of classical distributions that can be chosen for $\mathcal{L}(\theta)$ (and will be used later).

2.1. Particular examples

When $\theta \in \mathbb{R}^+$,

- the Poisson law $\mathcal{P}(\theta)$: $p_i(\theta) = \mathrm{e}^{-\theta} \theta^i / i!$, $i \geq 0$;
- the negative binomial law $\mathcal{NB}(\theta, p)$:

$$p_i(\theta) = \binom{\theta + i - 1}{i} p^i (1 - p)^{\theta}, \qquad i \geq 0,$$

  for fixed $p \in (0, 1)$;

https://doi.org/10.1239/aap/1183667620 Published online by Cambridge University Press

- the generalized Poisson law $\mathcal{GP}(\theta, \beta)$:

$$p_i(\theta) = \frac{\theta(\theta+\beta i)^{i-1}\mathrm{e}^{-\theta-\beta i}}{i!}, \qquad i \geq 0,$$

  for fixed $\beta \in [0, 1]$ (e.g. Johnson *et al.* (1992, p. 396));

- the generalized negative binomial law $\mathcal{GNB}(\theta, \beta, p)$:

$$p_i(\theta) = \frac{\theta}{\theta+\beta i}\binom{\theta+\beta i}{i} p^i (1-p)^{\theta+\beta i - i}, \qquad i \geq 0,$$

  for fixed $p \in (0, 1)$ and $\beta \in [1, 1/p]$ (e.g. Johnson *et al.* (1992, p. 230));

- the generalized Pólya–Eggenberger law $\mathcal{GPE}(\theta, \beta, p, c)$, which covers the four previous laws:

$$p_i(\theta) = \frac{\theta}{\theta+\beta i}(\theta+\beta i)^{(i,c)} \frac{1}{i!}\left(\frac{p}{c}\right)^i (1-p)^{(\theta+\beta i)/c}, \qquad i \geq 0,$$

  for fixed $p \in (0, 1)$, $c > 0$, and $\beta \in [0, c(1-p)/p]$, and using the notation $a^{(i,c)} = a(a+c)\dots[a+c(i-1)]$ with $a^{(0,c)} = 1$ (e.g. Janardan and Rao (1982));

- the compound Poisson law: $X(\theta) \stackrel{\mathrm{D}}{=} \sum_{j=1}^{\mathcal{P}(\theta)} W_j$, for a given sequence of i.i.d. $\mathbb{N}$-valued random variables W_j.

When $\theta \in \mathbb{N}$,

- the binomial law $\mathcal{B}(\theta, p)$:

$$p_i(\theta) = \binom{\theta}{i} p^i (1-p)^{\theta-i}, \qquad i \geq 0,$$

  for fixed $p \in (0, 1)$;

- the θ-fold convolution of any basic Lagrangian law, shifted by θ so that its support is $\mathbb{N}$; for instance, the shifted Poisson $\mathcal{P}(\beta)$-delta(θ) type (or Borel-Tanner law) (which is equivalent to $\mathcal{GP}(\theta\beta, \beta)$), the shifted binomial $\mathcal{B}(\beta, p)$-delta(θ) type (equivalent to $\mathcal{GNB}(\theta\beta, \beta, p)$) and the shifted negative binomial $\mathcal{NB}(\beta, p)$-delta(θ) type (equivalent to $\mathcal{GNB}(\theta\beta, \beta+1, p)$) (e.g. Johnson *et al.* (1992, p. 439) and Sibuya *et al.* (1994));

- the compound binomial law: $X(\theta) \stackrel{\mathrm{D}}{=} \sum_{j=1}^{\mathcal{B}(\theta,p)} W_j$, for a given sequence of i.i.d. $\mathbb{N}$-valued random variables W_j;

- the case $\theta = 1$ allows us to deal with the traditional i.i.d. model (for an arbitrary arithmetic law $\mathcal{L} \equiv \mathcal{L}(1)$).

3. Preliminaries: generalized A.G. polynomials

The generalized Abel–Gontcharoff (A.G.) polynomials generalize the classical Abel polynomials and their extension given by Gontcharoff (1937) (referred to as standard A.G. polynomials). They have been introduced by Picard and Lefèvre (1996), under a more general form of pseudopolynomials. Hereafter, we adapt to our framework the elements of the theory which are useful for our analysis.

We start with the above fundamental polynomials $\{e_i(\theta),\ i \geq 0\}$, for $\theta \in \mathbb{R}$. Let $\mathcal{G}$ be the real vector space generated by this family of polynomials, i.e. the set of polynomials $B_n(\theta)$ of the form

$$B_n(\theta) = \sum_{i=0}^{n} b_{n,i} e_i(\theta), \qquad n \geq 0. \tag{3.1}$$

A linear shift operator Δ is defined on $\mathcal{G}$ by

$$\Delta e_0(\theta) = 0 \quad \text{and} \quad \Delta e_i(\theta) = e_{i-1}(\theta), \qquad i \geq 1. \tag{3.2}$$

Put $\Delta^j = \Delta(\Delta^{j-1}) = \Delta^{j-1}(\Delta)$, $j \geq 1$, where Δ^0 is the identity operator. Now, any polynomial in $\mathcal{G}$ admits a Taylor-type expansion with respect to $\{e_i(\theta),\ i \geq 0\}$ and built with the operator Δ. As proved below, this is a consequence of (2.3), (2.4), and (2.5) of the fundamental polynomials.

Property 3.1. *For any $B_n(\theta) \in \mathcal{G}$, and given any $a \in \mathbb{R}$,*

$$B_n(\theta) = \sum_{j=0}^{n} \Delta^j B_n(a) e_j(\theta - a), \qquad n \geq 0. \tag{3.3}$$

Proof. Using (2.5), $B_n(\theta)$ defined in (3.1) can be expanded as

$$B_n(\theta) = \sum_{i=0}^{n} b_{n,i} \left(\sum_{k=0}^{i} e_{i-k}(a) e_k(\theta - a) \right) = \sum_{k=0}^{n} e_k(\theta - a) \left(\sum_{i=k}^{n} b_{n,i} e_{i-k}(a) \right). \tag{3.4}$$

From (3.4) and using (3.2), (2.3), and (2.4), we find that, for $0 \leq j \leq n$,

$$\Delta^j B_n(a) = \sum_{i=j}^{n} b_{n,i} e_{i-j}(a). \tag{3.5}$$

Inserting (3.5) in (3.4) yields (3.3).

Now, let us introduce a family $U = \{u_i,\ i \geq 0\}$ of arbitrary real numbers. For $r = 0, 1, \ldots,$ let $E^r U = \{u_i,\ i \geq r\}$ be the family U deprived of its first r terms. To U one can attach a (unique) family of generalized A.G. polynomials $\{\bar{G}_n(\theta \mid U),\ n \geq 0\}$ of degree n in θ by imposing the following conditions.

Definition 3.1. Each generalized A.G. polynomial $\bar{G}_n(\theta \mid U)$, $n \geq 0$, is such that

$$\Delta^r \bar{G}_n(\theta \mid U) = \bar{G}_{n-r}(\theta \mid E^r U), \quad 0 \leq r \leq n, \tag{3.6}$$

$$\Delta^r \bar{G}_n(u_r \mid U) = \delta_{n,r}, \qquad 0 \leq r \leq n. \tag{3.7}$$

In fact, by (3.3) and (3.6), $\bar{G}_n(\theta \mid U)$ can be expressed as

$$\bar{G}_n(\theta \mid U) = \sum_{j=0}^{n} \bar{G}_{n-j}(0 \mid E^j U) e_j(\theta), \qquad n \geq 0, \tag{3.8}$$

where, by (3.7), the coefficients $\bar{G}_{n-j}(0 \mid E^j U)$ are provided recursively by

$$\delta_{n,r} = \sum_{j=r}^{n} \bar{G}_{n-j}(0 \mid E^j U) e_{j-r}(u_r), \qquad 0 \leq r \leq n. \tag{3.9}$$

Using (3.6), it directly follows that any polynomial in $\mathcal{G}$ admits an Abelian-type expansion with respect to $\{\bar{G}_i(\theta \mid U),\ \ i \geq 0\}$ and built with the operator Δ.

Property 3.2. *For any $B_n(\theta) \in \mathcal{G}$, and given any real family U,*

$$B_n(\theta) = \sum_{j=0}^{n} \Delta^j B_n(u_j)\bar{G}_j(\theta \mid U), \qquad n \geq 0. \tag{3.10}$$

Choosing $B_n(\theta) \equiv e_n(\theta)$, (3.10) with (3.2) yields

$$\bar{G}_n(\theta \mid U) = e_n(\theta) - \sum_{j=0}^{n-1} e_{n-j}(u_j)\bar{G}_j(\theta \mid U), \qquad n \geq 0, \tag{3.11}$$

which constitutes another possible recursion to evaluate the $\bar{G}_n(\theta \mid U)$s.

Notice that $\bar{G}_n(\theta \mid U)$ depends on U only through its first n elements $u_0, \dots, u_{n-1}$. It can also be checked that if $U + a$ is the family $\{u_i + a,\ \ i \geq 0\}$, with $a \in \mathbb{R}$,

$$\bar{G}_n(\theta + a \mid U + a) = \bar{G}_n(\theta \mid U), \qquad n \geq 0. \tag{3.12}$$

Later we will discuss a case where the family U corresponds to a family $V = \{v_i,\ \ i \geq 0\}$ for the first $k + 1$ terms ($k \geq 0$), and to a family $W = \{w_i,\ \ i \geq 0\}$ for the next terms. For this case, the following Abelian-type expansion holds.

Lemma 3.1. *If $U = \{u_i \equiv v_i,\ \ 0 \leq i \leq k$, and $u_i \equiv w_i,\ \ i \geq k + 1\}$, then $\bar{G}_n(\theta \mid U) = \bar{G}_n(\theta \mid V)$ for $n \leq k + 1$, while for $n \geq k + 2$,*

$$\bar{G}_n(\theta \mid U) = \sum_{j=k+1}^{n} \bar{G}_{n-j}(v_j \mid E^j W)\bar{G}_j(\theta \mid V) \tag{3.13}$$

$$= \bar{G}_n(\theta \mid W) - \sum_{j=0}^{k} \bar{G}_{n-j}(v_j \mid E^j W)\bar{G}_j(\theta \mid V). \tag{3.14}$$

Proof. Let us apply (3.10) to expand $\bar{G}_n(\theta \mid U)$ with respect to the $\bar{G}_n(\theta \mid V)$s. By (3.7), $\Delta^j \bar{G}_n(v_j \mid U) = 0$ if $j \leq k$ (since $v_j = u_j$), and by (3.6), $\Delta^j \bar{G}_n(v_j \mid U) = \bar{G}_{n-j}(v_j \mid E^j W)$ if $j \geq k+1$ (since $u_{j+i} = w_{j+i}$ for $i \geq 0$), hence (3.13) follows. The equality between (3.13) and (3.14) follows from a similar expansion of $\bar{G}_n(\theta \mid W)$.

Especially interesting is the situation when u_i is an affine function of i. We prove below that each $\bar{G}_n$ then reduces to a simple variant of e_n.

Lemma 3.2. *If $U = \{u_i \equiv a + bi,\ \ i \geq 0\}$, then*

$$\bar{G}_n(\theta \mid U) = \frac{\theta - u_0}{\theta - u_n} e_n(\theta - u_n), \qquad n \geq 0. \tag{3.15}$$

Proof. The right-hand side of (3.15), $K_n(\theta \mid U)$ say, is a polynomial of degree n, with $K_0(\theta \mid U) = 1$ and $K_n(u_0 \mid U) = 0$ for $n \geq 1$. By (3.6) and (3.7), we then see that to establish (3.15), it suffices to show that $\Delta K_n(\theta \mid U) = K_{n-1}(\theta \mid EU)$ for $n \geq 1$. Now, if u_i

is affine in i, we can easily check that this identity is true by virtue of the following relation between fundamental polynomials: for $\theta \neq 0$,

$$\Delta\left[\frac{ie_i(\theta)}{\theta}\right] = \frac{(i-1)e_{i-1}(\theta)}{\theta}, \qquad i \geq 1. \tag{3.16}$$

So, it remains to prove (3.16). Let $\theta \in \mathcal{D}$, and denote by $f_e(\theta, z) = \sum_{i=0}^{\infty} e_i(\theta) z^i$ the generating function of the $e_i(\theta)$s, with argument $z \in [0, 1]$. Clearly, (3.16) is equivalent to

$$\Delta\left\{\frac{1}{\theta}\frac{\mathrm{d}f_e(\theta, z)}{\mathrm{d}z}\right\} = \frac{z}{\theta}\frac{\mathrm{d}f_e(\theta, z)}{\mathrm{d}z}. \tag{3.17}$$

By (2.2), (2.7), and (2.9), we have

$$f_e(\theta, z) = \frac{f(\theta, z)}{p_0(\theta)} = \frac{[f(1, z)]^\theta}{p_0(\theta)},$$

so that

$$\frac{1}{\theta}\frac{\mathrm{d}f_e(\theta, z)}{\mathrm{d}z} = \frac{f_e(\theta, z)}{f(1, z)}\frac{\mathrm{d}f(1, z)}{\mathrm{d}z}. \tag{3.18}$$

Substituting (3.18) into (3.17) gives

$$\Delta f_e(\theta, z) = z f_e(\theta, z),$$

which is indeed satisfied by (3.2). The extension to $\theta \in \mathbb{R}$ is then immediate.

A remarkable particular case is when $e_i(\theta) = \theta^i/i!$, $i \geq 0$, i.e. if $\mathcal{L}(\theta)$ is a Poisson law $\mathcal{P}(\theta)$. Then, by (3.2) Δ is the usual differentiation operator, and the $\bar{G}_n(\theta \mid U)$s correspond to standard A.G. polynomials, denoted by $G_n(\theta \mid U)$. It can also be seen that if aU is the family $\{au_i,\ i \geq 0\}$, with $a \in \mathbb{R}$,

$$G_n(a\theta \mid aU) = a^n G_n(\theta \mid U), \qquad n \geq 0.$$

Moreover, if u_i is affine in i, then (3.15) reduces to the classical Abel polynomials, i.e.

$$G_n(\theta \mid U) = \frac{(\theta - u_0)(\theta - u_n)^{n-1}}{n!}, \qquad n \geq 0.$$

4. First-crossing and ballot-type results

Returning to the initial question, let $\{X_t,\ t \geq 1\}$ be a sequence of independent random variables taking on values in $\mathbb{N}$. It is assumed that the laws of all X_ts belong to the same parametric family of distributions, $\mathcal{L}(\theta)$, which satisfies the conditions indicated in Section 2, for $\theta \in \mathcal{D}$. Each X_t may have its own parameter value, and we write

$$X_1 \overset{\mathrm{D}}{=} \mathcal{L}(\theta_0 + \theta_1) \quad \text{and} \quad X_t \overset{\mathrm{D}}{=} \mathcal{L}(\theta_t), \qquad t \geq 2,$$

where $\theta_0 + \theta_1, \theta_2, \theta_3, \ldots \in \mathcal{D}$.

Let us consider the associated partial sums $S_t = X_1 + \cdots + X_t$ for $t \geq 1$. $\mathcal{L}(\theta)$ being stable by convolution, we have

$$S_t \overset{\mathrm{D}}{=} \mathcal{L}(\theta_0 + \theta_t^+), \qquad t \geq 1, \tag{4.1}$$

https://doi.org/10.1239/aap/1183667620 Published online by Cambridge University Press

where $\theta_t^+ \equiv \theta_1 + \cdots + \theta_t$. It is convenient to define a family U of real numbers as follows:

$$U = \{u_i \equiv -\theta_{i+1}^+, \ i \geq 0\}; \tag{4.2}$$

so, the successive parameters of the sums $\{S_t, \ t \geq 1\}$ are provided by the family $\theta_0 - U$.

Now, fix an arbitrary integer $m \geq 1$. We will begin by examining the first-crossing problem of the process $m + S_t, \ t \geq 1$, with the diagonal line. Let T be the first-crossing time:

$$T = \inf\{t \geq m \colon m + S_t \leq t\}; \tag{4.3}$$

clearly, the first-crossing corresponds to a first-meeting, i.e. $m + S_T = T$. Let $N = S_T$ denote the first-crossing level; putting $t = m + n, \ n \geq 0$, in (4.3) yields

$$N = \inf\{n \geq 0 \colon S_{m+n} = n\}. \tag{4.4}$$

To indicate dependence on m, θ_0 and U, set $T \equiv T(m, \theta_0, U)$ and $N \equiv N(m, \theta_0, U)$.

In practice, $\theta_0 = 0$ quite often (see later). A possible interpretation for $\theta_0 > 0$ is that the initial level is not equal to a constant m but corresponds to a variable $m + X_0$ where $X_0 \overset{\mathrm{D}}{=} \mathcal{L}(\theta_0)$. Indeed, it is easily shown that $T(m + X_0, 0, U) \overset{\mathrm{D}}{=} T(m, \theta_0, U)$.

Lemma 4.1 below states that the first-crossing problem can be reduced to the case $m = 1$. As previously, we put $E^r U = \{-\theta_{i+1}^+, i \geq r\}$, for $r = 0, 1, \ldots$

Lemma 4.1.

$$N(m, \theta_0, U) \overset{\mathrm{D}}{=} N(1, \theta_0, E^{m-1} U). \tag{4.5}$$

Proof. Denote $\hat{\theta}_1 = \theta_1 + \cdots + \theta_m$ and $\hat{\theta}_t = \theta_{m+t-1}$ for $t \geq 2$. Thus, the family $\hat{U}$ defined as $\hat{U} = \{-\hat{\theta}_i^{\,+}, \ i \geq 1\}$ (similarly to (4.2)) is equivalent to $E^{m-1}U$. Now, define a sequence of independent random variables $\hat{X}_t, \ t \geq 1$, where $\hat{X}_1 \overset{\mathrm{D}}{=} \mathcal{L}(\theta_0 + \hat{\theta}_1)$ and $\hat{X}_t \overset{\mathrm{D}}{=} \mathcal{L}(\hat{\theta}_t)$ for $t \geq 2$. Putting $\hat{S}_t = \hat{X}_1 + \cdots + \hat{X}_t, \ t \geq 1$, we observe that

$$\hat{S}_{1+n} \overset{\mathrm{D}}{=} \mathcal{L}(\theta_0 + \hat{\theta}_{1+n}^+) \equiv \mathcal{L}(\theta_0 + \theta_{m+n}^+) \overset{\mathrm{D}}{=} S_{m+n}, \qquad n \geq 0. \tag{4.6}$$

From (4.4) and (4.6), we can deduce (4.5).

In the following theorem we establish a key result, namely, the probability mass function of $N(1, \theta_0, U)$ is given by the family of generalized A.G. polynomials $\bar{G}_n(\theta_0|U), \ n \geq 0$, apart from a simple multiplicative factor.

Theorem 4.1.

$$\mathrm{P}[N(1, \theta_0, U) = n] = p_0(\theta_0 - u_n)\bar{G}_n(\theta_0 \mid U), \qquad n \geq 0. \tag{4.7}$$

Proof. Obviously,

$$\mathrm{P}[N(1, \theta_0, U) = 0] = \mathrm{P}(X_1 = 0) = p_0(\theta_0 + \theta_1). \tag{4.8}$$

For $n \geq 1$, looking at time $t = 1$, we obtain

$$\mathrm{P}[N(1, \theta_0, U) = n] = \sum_{j=1}^{n} \mathrm{P}(X_1 = j)\, \mathrm{P}[N(j, -\theta_1, EU) = n - j], \qquad n \geq 1. \tag{4.9}$$

Indeed, if $X_1 = j \geq 1$, then a new situation begins at time $t = 1$ that is similar to the previous one, but now the initial state is equal to j $(= 1 + j - 1)$ and the sequence of independent

https://doi.org/10.1239/aap/1183667620 Published online by Cambridge University Press

random variables is the shifted sequence $\{X_t,\ t \geq 2\}$. Note that the successive parameters of the associated sums $\{S_t,\ t \geq 2\}$ are given by $\{\theta_2 + \cdots + \theta_t,\ t \geq 2\}$, which corresponds to the family $-\theta_1 - EU$. Thus, the first-crossing level under consideration is $N(j, -\theta_1, EU)$ and it has to be equal to $n - j$ as indicated in (4.9).

Now, applying Lemma 4.1, (4.9) becomes

$$\mathrm{P}[N(1, \theta_0, U) = n] = \sum_{j=1}^{n} p_j(\theta_0 + \theta_1)\, \mathrm{P}[N(1, -\theta_1, E^j U) = n - j], \qquad n \geq 1. \tag{4.10}$$

Remember that e_j is defined by (2.2) and $u_0 = -\theta_1$ by (4.2). Therefore, (4.10) can be rewritten as

$$\mathrm{P}[N(1, \theta_0, U) = n] = p_0(\theta_0 - u_0) \sum_{j=1}^{n} e_j(\theta_0 - u_0)\, \mathrm{P}[N(1, u_0, E^j U) = n - j], \qquad n \geq 1. \tag{4.11}$$

Finally, let us introduce a function $H_n(\theta_0 \mid U)$ by

$$H_n(\theta_0 \mid U) \equiv \frac{\mathrm{P}[N(1, \theta_0, U) = n]}{p_0(\theta_0 - u_n)}, \qquad n \geq 0. \tag{4.12}$$

We notice that if $\theta_0 = u_0 (= -\theta_1)$, then $X_1 = 0$ almost surely so that $N(1, u_0, U) = 0$ almost surely; by (4.8) and (4.11), this implies that

$$H_n(u_0 \mid U) = \delta_{n,0}, \qquad n \geq 0. \tag{4.13}$$

For $\theta_0 > u_0$, let us divide both members of (4.8) by $p_0(\theta_0 - u_0)$ and both members of (4.11) by $p_0(\theta_0 - u_n) = p_0(\theta_0 - u_0)p_0(u_0 - u_n)$; using (4.12), we so obtain

$$H_n(\theta_0 \mid U) = \sum_{j=0}^{n} e_j(\theta_0 - u_0) H_{n-j}(u_0 \mid E^j U), \qquad n \geq 0. \tag{4.14}$$

Combining (3.3), (3.6), and (3.7), we then find that (4.14) with (4.13) corresponds to the Taylor-type expansion of $\bar{G}_n(\theta_0 \mid U)$, $n \geq 0$, around the point u_0. In other words, $H_n(\theta_0 \mid U) = \bar{G}_n(\theta_0 \mid U)$, $n \geq 0$, and (4.12) thus yields (4.7).

Owing to Theorem 4.1, we are in a position to derive a ballot-type formula for the nonstationary model under study. Of course, (1.2) will then follow as a particular case (see Section 5).

Corollary 4.1. *For $m = 1, \ldots, n$,*

$$\mathrm{P}(m + S_t > t \text{ for } t = 1, \ldots, n-1 \mid m + S_n = n) = \frac{\bar{G}_{n-m}(\theta_0 \mid E^{m-1} U)}{e_{n-m}(\theta_0 - u_{n-1})}. \tag{4.15}$$

Proof. By definition,

$$\mathrm{P}(m + S_t > t,\ 1 \leq t \leq n-1 \mid m + S_n = n) = \frac{\mathrm{P}[N(m, \theta_0, U) = n - m]}{\mathrm{P}(S_n = n - m)}. \tag{4.16}$$

By (4.1), (4.2), and (2.2), we have

$$\mathrm{P}(S_n = n - m) = p_{n-m}(\theta_0 + \theta_n^+) = p_0(\theta_0 - u_{n-1}) e_{n-m}(\theta_0 - u_{n-1}), \tag{4.17}$$

https://doi.org/10.1239/aap/1183667620 Published online by Cambridge University Press

while (4.5) and (4.7) yield

$$\mathrm{P}[N(m,\theta_0,U)=n-m]=p_0(\theta_0-u_{n-1})\bar{G}_{n-m}(\theta_0\mid E^{m-1}U). \tag{4.18}$$

Substituting (4.17) and (4.18) into (4.16) leads to (4.15).

Let us underline that in (4.7) and (4.15), the probabilities of interest can now be evaluated numerically without difficulty. Indeed, $\bar{G}_n(\cdot)$ and $\bar{G}_{n-m}(\cdot)$ are easily determined by recursion, either from (3.8), (3.9) or from (3.11), and $e_{n-m}(\cdot)$ is directly computed from the distribution $\mathcal{L}(\theta)$.

To close, we examine below the related problem of first-crossing from above when the initial level is 0. Let $N(0,\theta_0,U)$ be the corresponding first-crossing level, defined as $\inf\{n\geq 1\colon S_n\leq n\}$. Its probability mass function follows from Theorem 4.1. Indeed, we observe that each trajectory linking from above level 0 at time 0 to level n at time n, can be coupled with a trajectory linking from above level 1 at time -1 to level n at time n, for the partial sums process of the variables $X_1\overset{\mathrm{D}}{=}\mathcal{L}(\theta_0+\theta_1)$, $X_2=0$ almost surely, and $X_t\overset{\mathrm{D}}{=}\mathcal{L}(\theta_{t-1})$, $t\geq 3$. Therefore, $N(0,\theta_0,U)$ has the same distribution as $N(1,\theta_0,\{u_0,U\})$ where $\{u_0,U\}$ is the family $\{u_0,u_0,u_1,u_2,\dots\}$. From (4.7) we then deduce that

$$\mathrm{P}[N(0,\theta_0,U)=n]=p_0(\theta_0-u_{n-1})\bar{G}_n(\theta_0\mid\{u_0,U\}),\qquad n\geq 1. \tag{4.19}$$

Note that by (3.14) (with $k=0$ and $\{u_0,U\}$ for U), $\bar{G}_n(\cdot)$ in (4.19) can be decomposed as

$$\bar{G}_n(\theta_0\mid\{u_0,U\})=\bar{G}_n(\theta_0\mid\{0,U\})-\bar{G}_n(u_0\mid\{0,U\}),\qquad n\geq 1. \tag{4.20}$$

5. A situation with constant parameters

Let us assume that all the parameters θ_ts, $t\geq 1$, are equal to each other, i.e. $X_1\overset{\mathrm{D}}{=}\mathcal{L}(\theta_0+\theta_1)$ and $X_t\overset{\mathrm{D}}{=}\mathcal{L}(\theta_1)$ for $t\geq 2$, where $\theta_0,\theta_1\in\mathcal{D}$. If $\theta_0=0$ and $\theta_1=1$, this reduces to the classical case where all the variables X_t for $t\geq 1$, are i.i.d. (with arbitrary law $\mathcal{L}\equiv\mathcal{L}(1)$).

5.1. Simplified formulas

By (4.2), $u_i=-\theta_1(i+1)$ for $i\geq 0$, and the sequence is thus affine in i. In this case, the polynomials $\bar{G}_n(\theta\mid U)$ for $n\geq 0$, take the form of (3.15). By substitution in the ballot-type formula (4.15), we then find that

$$\mathrm{P}(m+S_t>t\text{ for }t=1,\dots,n-1\mid m+S_n=n)=\frac{\theta_0+\theta_1 m}{\theta_0+\theta_1 n},\qquad n\geq 0, \tag{5.1}$$

which becomes (1.2) when $\theta_0=0$.

From (5.1), the distribution of $N\equiv N(m,\theta_0,\theta_1)$ is given by

$$\mathrm{P}[N(m,\theta_0,\theta_1)=n]=\frac{\theta_0+\theta_1 m}{\theta_0+\theta_1(m+n)}\,\mathrm{P}(S_{m+n}=n),\qquad n\geq 0. \tag{5.2}$$

In particular, for the main special laws listed in Subsection 2.1, we obtain that, putting $\tilde{\theta}_0\equiv\theta_0+\theta_1 m$,

- if $\mathcal{L}(\theta)$ is a Poisson law $\mathcal{P}(\theta)$, then N has a generalized Poisson law $\mathcal{GP}(\tilde{\theta}_0,\theta_1)$;
- if $\mathcal{L}(\theta)$ is a negative binomial law $\mathcal{NB}(\theta,p)$, then N has a generalized negative binomial law $\mathcal{GNB}(\tilde{\theta}_0,\theta_1+1,p)$;

- if $\mathcal{L}(\theta)$ is a generalized Poisson law $\mathcal{GP}(\theta, \beta)$, then N has a generalized Poisson law $\mathcal{GP}(\tilde{\theta}_0, \theta_1 + \beta)$;
- if $\mathcal{L}(\theta)$ is a generalized negative binomial $\mathcal{GNB}(\theta, \beta, p)$, then N has a generalized negative binomial $\mathcal{GNB}(\tilde{\theta}_0, \theta_1 + \beta, p)$;
- if $\mathcal{L}(\theta)$ is a generalized Pólya–Eggenberger law $\mathcal{GPE}(\theta, \beta, p, c)$, then N has a generalized Pólya–Eggenberger law $\mathcal{GPE}(\tilde{\theta}_0, \theta_1 + \beta, p, c)$;
- if $\mathcal{L}(\theta)$ is a binomial law $\mathcal{B}(\theta, p)$, then N has a generalized negative binomial law $\mathcal{GNB}(\tilde{\theta}_0, \theta_1, p)$.

Moreover, if $m = 0$, (4.19) and (4.20) with (3.15) yield

$$\begin{aligned} &\mathrm{P}[N(0, \theta_0, \theta_1) = n] \\ &\quad = p_0(\theta_0 + \theta_1 n)\left\{\frac{\theta_0}{\theta_0 + \theta_1 n} e_n(\theta_0 + \theta_1 n) + \frac{1}{n-1} e_n[\theta_1(n-1)]\right\}, \qquad n \geq 1. \qquad (5.3) \end{aligned}$$

5.2. A standard application

Consider a Galton–Watson branching process initiated by m ancestors and with i.i.d. family sizes distributed as a random variable X of distribution $\{p_i,\ i \geq 0\} \equiv \mathcal{L}$. Let $T(m)$ be the total number of descendants including the m initial ancestors. It is well known that extinction is almost sure and $T(m)$ has a finite distribution if and only if $E(X) \leq 1$. From the probability generating function of $T(1)$ and using the Lagrange expansion formula, it is possible to show that $T(1)$ has a basic Lagrangian distribution, $T(m)$ being the m-fold convolution of $T(1)$ (e.g. Consul and Shenton (1972)).

An alternative (known) approach to determine $T(m)$ consists in introducing a new time-scale $t = 1, 2, \ldots,$ to represent the cumulative number of deaths in the course of time. Each tth case, $t \geq 1$, gives birth to a family of size X_t before its death; thus, $m + S_t$ counts the number of ancestors plus the total number of direct descendants due to the first t death cases. It is clear that $T(m)$ can then be represented as the first-crossing time defined in (4.3); more precisely, $T(m) \overset{\mathrm{D}}{=} T(m, \theta_0 = 0, \theta_1 = 1)$ using previous notation. Therefore, $T(m) = m + N(m)$ with the distribution of $N(m)$ given by (5.2).

In the same vein, consider a G/D/1 queue where initially m customers are waiting, the service time for a customer is of constant length ($= 1$) and the numbers of arrivals per time unit are i.i.d. random variables with law $\mathcal{L}$. Then, the total number of customers served during a busy period is distributed exactly as the random variable $T(m)$ above (e.g. Takàcs (1967) and Sibuya *et al.* (1994)).

For illustration, let us consider a discrete-time queue in which per time unit, only three different events can occur: the departure of one customer, no change or the arrival of α customers (α being a fixed positive integer), always under i.i.d. conditions; the case $\alpha = 1$ has been examined by Mohanty and Panny (1990). With respect to the previous G/D/1 queue, this amounts to assuming that there is one service after each time unit (if the queue is not empty), and the number of arrivals per time unit is of law $\mathcal{L}$ with probability mass function $\{p_0, p_1, p_{\alpha+1}\}$. Thus, from (5.2) we get that $T(m)$, the number of customers served during a busy period, is of probability mass function

$$\mathrm{P}[T(m) = m + n] = \frac{m}{m+n} \sum_{a+b+c=m+n,\, b+(\alpha+1)c=n} \binom{m+n}{a, b, c} p_0^a p_1^b p_{\alpha+1}^c, \qquad n \geq 0. \quad (5.4)$$

https://doi.org/10.1239/aap/1183667620 Published online by Cambridge University Press

If $p_1 = 0$, the previous queue corresponds also to the random walk associated with the game of roulette. Here, starting with an initial capital m, a player can, after each step, either lose one unit with probability $q \equiv p_0$ (if the capital is not zero) or earn α units with probability $p \equiv p_{\alpha+1}$; when $\alpha = 1$, this is the classical ruin model with one absorbing state at zero (e.g. Feller (1957)). In this game, we see that $T(m)$ corresponds to the ruin time of the player. Its possible values are $t = m + (\alpha+1)l,\ l = 0, 1, \dots,$ and from (5.4),

$$\mathrm{P}[T(m) = m + (\alpha+1)l] = \frac{m}{m + (\alpha+1)l}\binom{m + (\alpha+1)l}{l} p^l q^{m+\alpha l}, \qquad l \geq 0, \tag{5.5}$$

a result obtained by Hill and Gulati (1981). Note that by (5.5), $N(m)/(\alpha+1)$ has a generalized negative binomial law $\mathcal{GNB}(m, \alpha+1, p)$. If $m = 1$, the associated coefficient

$$\frac{1}{1 + (\alpha+1)l}\binom{1 + (\alpha+1)l}{l}$$

is equal to the generalized Catalan number

$$\frac{1}{1 + \alpha l}\binom{(\alpha+1)l}{l},$$

which reduces to the standard Catalan number when $\alpha = 1$ (see, e.g. Hilton and Pederson (1991)). So, for the game of roulette, this number can be interpreted as the conditional probability that, starting with a capital of one, a player who is ruined at time $1 + (\alpha+1)l$ was not ruined before that time.

Again if $p_1 = 0$ but now when $m = 0$, the corresponding variable $N(0)$ represents the first time where a player with an initial capital of zero becomes ruined again (if amounts are payable at the end of each step). Its possible values are $n = (\alpha+1)l,\ l = 1, 2, \dots,$ and from (5.3), we obtain

$$\mathrm{P}[N(0) = (\alpha+1)l] = \frac{1}{(\alpha+1)l - 1}\binom{(\alpha+1)l - 1}{l} p^l q^{\alpha l}, \qquad l \geq 1. \tag{5.6}$$

It is easily checked from (5.6) that $N(0)/(\alpha+1)$ has a generalized negative binomial law $\mathcal{GNB}(\alpha, \alpha+1, p)$ shifted by 1 (also named Geeta distribution by Consul (1990)), and non-normalized by p.

6. Some situations with variable parameters

6.1. A single change in the parameters

Let us assume that the parameters θ_t are all equal to θ_1 until some time $k+1$ ($k \geq 0$), and that the next parameters are all equal to another value θ_c, where $\theta_1, \theta_c \in \mathcal{D}$. Thus, $u_i = -\theta_1(i+1) \equiv v_i$ for $0 \leq i \leq k$, and $u_i = -\theta_1(k+1) - \theta_c(i-k) \equiv w_i$ for $i \geq k+1$.

In a queueing context, as above, this means that the service time is of length θ_1 for the first $k+1$ customers and of length θ_c for the subsequent customers. Such a situation can arise if the server needs some kind of setup time before becoming quite operational (then, $\theta_1 \geq \theta_c$).

Applying (4.15) or (4.18) requires computing the value of $\bar{G}_{n-m}(\theta_0 \mid E^{m-1}U)$. This is easily done as follows. If $m \geq k+2$, $E^{m-1}U$ reduces to the affine family $E^{m-1}W$, and it suffices to use (3.15). Suppose now that $m \leq k+1$. Then, $E^{m-1}U$ is the family

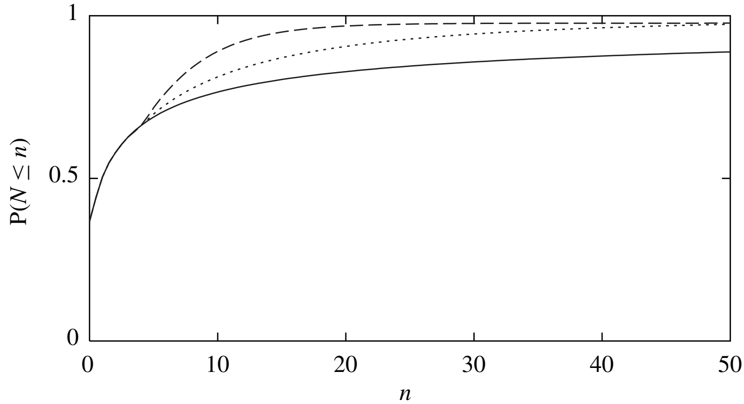

FIGURE 1: Graph of $\mathrm{P}(N \le n)$, $0 \le n \le 50$, when $m = 1$, $\mathcal{L}(\theta) = \mathcal{P}(\theta)$, $\theta_0 = 0$, $\theta_1 = 1$, $k = 3$ and either $\theta_c = 1$ (continuous line), $\theta_c = 0.8$ (dotted line) or $\theta_c = 0.5$ (dashed line).

$\{v_{m-1}, \dots, v_k, w_{k+1}, w_{k+2}, \dots\}$ which is of the form discussed in Lemma 3.1, with in addition, affine families for V and W. Thus, if $n - m \le k + 1$,

$$\bar{G}_{n-m}(\theta_0 \mid E^{m-1}U) = \bar{G}_{n-m}(\theta_0 \mid E^{m-1}V) = \frac{\theta_0 - v_{m-1}}{\theta_0 - v_{n-1}} e_{n-m}(\theta_0 - v_{n-1}).$$

If $n - m \ge k + 2$, (3.13) and (3.14) yield

$$\begin{aligned}\bar{G}_{n-m}(\theta_0 \mid E^{m-1}U) &= \sum_{j=k+1}^{n-m} a_j \\ &= \frac{\theta_0 - w_{m-1}}{\theta_0 - w_{n-1}} e_{n-m}(\theta_0 - w_{n-1}) - \sum_{j=0}^{k} a_j,\end{aligned}$$

where, for $j \ge 0$,

$$a_j \equiv \frac{v_{m-1+j} - w_{m-1+j}}{v_{m-1+j} - w_{n-1}} e_{n-m-j}(v_{m-1+j} - w_{n-1}) \frac{\theta_0 - v_{m-1}}{\theta_0 - v_{m-1+j}} e_j(\theta_0 - v_{m-1+j}).$$

Suppose that $m = 1$ and $\mathcal{L}(\theta)$ is a Poisson law $\mathcal{P}(\theta)$, and let us choose $\theta_0 = 0$, $\theta_1 = 1$ and various values for θ_c and k. If $\theta_c = \theta_1 = 1$, the parameters are constant and we know from Section 5 that N has a generalized Poisson law $\mathcal{GP}(1, 1)$. Figure 1 shows the graph of $\mathrm{P}(N \le n)$, the distribution function of N, when $k = 3$ and $\theta_c = 1$, 0.8 or 0.5. As indicated before, the effect of θ_c appears only from level $n \ge k + 2 = 5$. We observe that decreasing θ_c implies a higher distribution function for N (i.e. decreases N stochastically). This is natural: for the previous queue, decreasing the service time will stochastically decrease the number of customers served during the busy period. Figure 2 is concerned with the case where $\theta_c = 0.8$ and $k = 2$, 8 or 14. Since increasing k postpones the use of θ_c to the profit of θ_1 ($> \theta_c$), this has the effect, of course, to increase N stochastically. We underline that the numerical procedure followed is quite fast and precise; the extension to several changes in the parameters can be treated with no more difficulty.

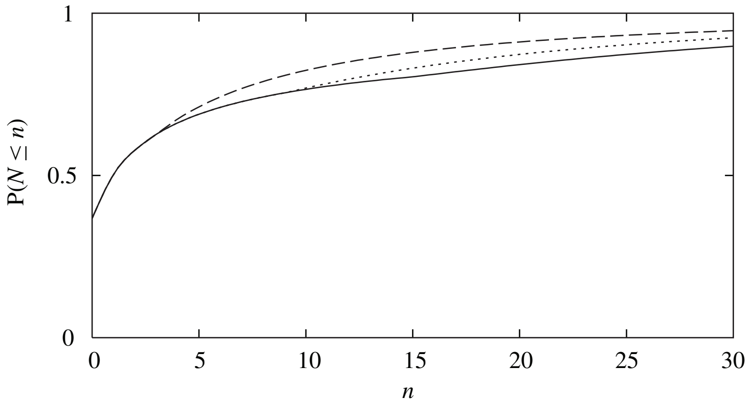

FIGURE 2: Graph of $\mathrm{P}(N \le n)$, $0 \le n \le 30$, when $m = 1$, $\mathcal{L}(\theta) = \mathcal{P}(\theta)$, $\theta_0 = 0$, $\theta_1 = 1$, $\theta_c = 0.8$ and either $k = 2$ (dashed line), $k = 8$ (dotted line) or $k = 14$ (contiuous line).

6.2. A geometric sequence of parameters

Let us assume that the θ_ts form a geometrically decreasing sequence in $\mathbb{R}^+$, denoted by $\theta_t = \lambda p q^{t-1}$ for $t \ge 1$ and where $\lambda > 0$, $0 < p = 1-q < 1$. Thus, $u_i = -\lambda p(1+q+\cdots+q^i) = -\lambda(1-q^{i+1})$, $i \ge 0$.

Such a geometric sequence of parameters has been considered by Takács (1989) for the particular case where $\mathcal{L}(\theta)$ is a Poisson law. His motivation comes from applications in queueing and graph theory, which will be discussed in more detail in a forthcoming work.

By (4.5) and (4.7), we obtain

$$\mathrm{P}[N(m,\theta_0,U) = n] = p_0[\theta_0 + \lambda(1-q^{m+n})]\bar{G}_n[\theta_0 \mid \{-\lambda(1-q^{m+i}),\ i \ge 0\}], \qquad n \ge 0. \tag{6.1}$$

It is worth pointing out that the recursive method used to compute $\bar{G}_n(\cdot)$ in (6.1) is simpler and more general than an alternative procedure proposed by Takács (1989) (in the Poisson case).

Following Takács' (1989) let us now examine the limit situation where $\lambda \to \infty$, $p \to 0$, and $\lambda p \to a$ $(0 < a < \infty)$. Then, $q^{m+i} \approx 1 - a(m+i)/\lambda$, $i \ge 0$, is an affine sequence. Therefore, from (6.1) and using (3.15) and (2.2), we obtain

$$\mathrm{P}[N(m,\theta_0,U) = n] \to \frac{\theta_0 + am}{\theta_0 + a(m+n)} p_n[\theta_0 + a(m+n)], \qquad n \ge 0,$$

a formula which is similar to (5.2).

6.3. An application in risk theory

Let us consider a discrete-time risk model for insurance. At the beginning of each period, the company receives a constant premium equal to one, and at the end of the period, the company covers the claim amounts occurred during the period. We assume that the successive claim amounts are random variables X_t, $t \ge 1$, independent and distributed as in Section 4. Let R_t denote the reserves of the company at time t, $t \ge 1$. If the initial capital is of amount u $(\in \mathbb{N})$, then $R_t = u + t - S_t$. Ruin occurs at the first time T_u when the reserves become negative or null, i.e. $T_u = \inf\{t \ge 1 : S_t \ge u + t\}$. Much attention has been paid to the ruin problem in the actuarial literature, generally for the classical model where the X_ts are i.i.d. (see, e.g.

https://doi.org/10.1239/aap/1183667620 Published online by Cambridge University Press

Asmussen (2000)). We are going to derive the exact distribution of the ruin time T_0 for the above nonstationary model; the case $u > 0$ is not treated here.

Obviously, $\mathrm{P}(T_0 = 1) = \mathrm{P}(X_1 \geq 1)$. For $t \geq 1$, we can write that

$$\mathrm{P}(T_0 \geq t+1) = \sum_{n=0}^{t-1} \mathrm{P}(S_i < i \text{ for } 1 \leq i \leq t, \text{ and } S_t = n). \tag{6.2}$$

Operating a rotation of 180 degrees, let us follow the different possible trajectories within (6.2) in reversed time (i.e. from time t to time 0). We then observe that the probability $\mathrm{P}(S_i < i,\ 1 \leq i \leq t, \text{ and } S_t = n) \equiv a_n^{(t)}$ can be re-expressed as

$$a_n^{(t)} = \mathrm{P}(t-n+X_t > 1,\ t-n+X_t+X_{t-1} > 2, \ldots, t-n+X_t+\cdots+X_2 > t-1, \\ \text{and } t-n+X_t+\cdots+X_1 = t),$$

that is, in the notation of Section 4,

$$a_n^{(t)} = \mathrm{P}[N(t-n, 0, U^{(t)}) = n], \qquad 0 \leq n \leq t-1,$$

where the family $U \equiv U^{(t)}$ depends here on t and is given by

$$U^{(t)} = \{-\theta_t, -(\theta_t+\theta_{t-1}), \ldots, -(\theta_t+\cdots+\theta_2), -(\theta_t+\cdots+\theta_1)\}.$$

By (4.5) and (4.7), this becomes

$$\begin{aligned} a_n^{(t)} &= \mathrm{P}[N(1, 0, E^{t-n-1}U^{(t)}) = n] \\ &= p_0(\theta_t+\cdots+\theta_1)\bar{G}_n(0 \mid E^{t-n-1}U^{(t)}), \qquad 0 \leq n \leq t-1, \end{aligned} \tag{6.3}$$

where

$$E^{t-n-1}U^{(t)} = \{-(\theta_t+\cdots+\theta_{n+1}), -(\theta_t+\cdots+\theta_n), \ldots, -(\theta_t+\cdots+\theta_2), -(\theta_t+\cdots+\theta_1)\}. \tag{6.4}$$

Inside (6.3), applying (3.12) with $a = \theta_t+\cdots+\theta_1 \equiv \theta_t^+$ to $\bar{G}_n(\cdot)$ yields

$$a_n^{(t)} = p_0(\theta_t^+)\bar{G}_n(\theta_t^+ \mid \{\theta_n^+, \theta_{n-1}^+, \ldots, \theta_1^+, 0\}), \qquad 0 \leq n \leq t-1. \tag{6.5}$$

Now, denoting

$$\bar{A}_n(\theta) = \bar{G}_n(\theta \mid \{\theta_n^+, \theta_{n-1}^+, \ldots, \theta_1^+, 0\}), \qquad n \geq 0, \tag{6.6}$$

we observe that $\bar{A}_0(\theta) = 1$ and $\Delta\bar{A}_n(\theta) = \bar{A}_{n-1}(\theta)$ for $n \geq 1$, that is $\bar{A}_n(\theta)$ for $n \geq 0$, is a generalized Appel polynomial in the sense given in Picard and Lefèvre (1996). In other words, for $U = \{\theta_n^+, \theta_{n-1}^+, \ldots, \theta_1^+, 0\}$, the associated generalized A.G. polynomials reduce to the generalized Appel polynomials (a known propery). By (3.8) and (3.9), $\bar{A}_n(\theta)$ can then be computed from the expansion

$$\bar{A}_n(\theta) = \sum_{j=0}^{n} \bar{A}_{n-j}(0)e_j(\theta), \qquad n \geq 0,$$

https://doi.org/10.1239/aap/1183667620 Published online by Cambridge University Press

and using the recursions (where $\theta_0^+ \equiv 0$)

$$\delta_{n,r} = \sum_{j=r}^{n} \bar{A}_{n-j}(0) e_{j-r}(\theta_{n-r}^+), \qquad 0 \le r \le n.$$

Combining (6.2), (6.5), and (6.6) we finally obtain

$$\mathrm{P}(T_0 \ge t+1) = \sum_{n=0}^{t-1} p_0(\theta_t^+) \bar{A}_n(\theta_t^+), \qquad t \ge 1,$$

a result to be compared with the formulas derived by Picard *et al.* (2003) and Ignatov and Kaishev (2004) (for a risk model with continuous claim amounts).

6.4. Randomized parameters

Let us assume that $\theta_0 = 0$ and the other parameters form a sequence of i.i.d. random variables $\{\Theta_t,\ t \ge 1\}$ valued in $\mathcal{D}$ (and distributed as Θ, say). Thus, each X_t, $t \ge 1$, has a mixed distribution $\mathcal{L}(\Theta)$.

Similarly to (4.1) and (4.2), denote $\Theta_t^+ = \Theta_1 + \cdots + \Theta_t$ for $t \ge 1$ and $U_i = -\Theta_{i+1}^+$ for $i \ge 0$, with $U = \{U_i,\ i \ge 0\}$. Now, by construction the model is of the traditional i.i.d. type. From (5.2), the probability mass function of $N \equiv N(m, \Theta)$ is thus given by

$$\mathrm{P}[N(m,\Theta) = n] = \frac{m}{m+n} \mathrm{P}(S_{m+n} = n), \qquad n \ge 0, \tag{6.7}$$

and by randomizing (4.1) and (4.2),

$$\mathrm{P}(S_{m+n} = n) = \mathrm{E}[p_0(-U_{m+n-1}) e_n(-U_{m+n-1})], \qquad n \ge 0. \tag{6.8}$$

We note that by (4.18), the law of N can also be expressed as

$$\mathrm{P}[N(m,\Theta) = n] = \mathrm{E}[p_0(-U_{m+n-1}) \bar{G}_n(0 \mid E^{m-1}U)], \qquad n \ge 0, \tag{6.9}$$

which is, a priori, more complicated than (6.8) because of the presence of $\bar{G}_n$ instead of e_n. From an algebraic point of view, it is rather surprising that (6.9) can be reduced to (6.8). This can be proved, however, using Theorem 2.2 in Picard and Lefèvre (2003).

Formula (6.7) and (6.8) are applicable to mixtures of the special laws listed in Section 2.1. For instance, if $\mathcal{L}(\Theta)$ is a mixed Poisson law $\mathcal{P}(\Theta)$, then

$$\mathrm{P}[N(m,\Theta) = n] = \frac{m}{m+n} \mathrm{E}\left[\frac{(\Theta_{m+n}^+)^n}{n!} e^{-\Theta_{m+n}^+} \right], \qquad n \ge 0, \tag{6.10}$$

and if $\mathcal{L}(\Theta)$ is a mixed negative binomial law $\mathcal{NB}(\Theta, p)$, then

$$\mathrm{P}[N(m,\Theta) = n] = \frac{m}{m+n} \mathrm{E}\left[\binom{\Theta_{m+n}^+ + n - 1}{n} (1-p)^{\Theta_{m+n}^+} \right] p^n, \qquad n \ge 0.$$

Going back to queueing, let us consider a queue model in which each service period is distributed as the random variable Θ and the number of customers arriving per service period is $\mathcal{L}(\Theta)$-distributed. Then, (6.7) and (6.8) provide the law of the number of new customers served during a busy period with initially m customers. The case where $\mathcal{L}(\Theta) = \mathcal{P}(\Theta)$ corresponds to the M/G/1 queue for which formula (6.10) is well known (e.g. Takács (1962, p. 63)).

Acknowledgements

This research was supported in part by the Banque Nationale de Belgique. I thank Ph. Picard for interesting discussions on a preliminary version, and M. Gathy for her help with the numerical part. I also wish to thank the referee for useful comments.

References

Asmussen, S. (2000). *Ruin Probabilities*. World Scientific, Singapore.

Ball, F. G. and O'Neill, P. (1999). The distribution of general final state random variables for stochastic epidemic models. *J. Appl. Prob.* **36,** 473–491.

Breiman, L. (1968). *Probability*. Addison-Wesley, Reading, MA.

Consul, P. C. (1990). Geeta distribution and its properties. *Commun. Statist. Theory Meth.* **19,** 3051–3068.

Consul, P. C. and Shenton, L. R. (1972). Use of Lagrange expansion for generating discrete generalized probability distributions. *SIAM J. Appl. Math.* **23,** 239–248.

Feller, W. (1957). *An Introduction to Probability Theory and Its Applications*. Vol. 1. John Wiley, New York.

Gontcharoff, W. (1937). *Détermination des Fonctions Entières par Interpolation.* Hermann, Paris.

Hill, J. M. and Gulati, C. M. (1981). The random walk associated with the game of roulette. *J. Appl. Prob.* **18,** 931–936.

Hilton, P. and Pederson, J. (1991). Catalan numbers, their generalization and their uses. *Math. Intelligencer* **13,** 64–75.

Ignatov, Z. G. and Kaishev, V. K. (2000). Two-sided bounds for the finite time probability of ruin. *Scand. Actuarial J.* **1,** 46–62.

Ignatov, Z. G. and Kaishev, V. K. (2004). A finite-time ruin probability formula for continuous claim severities. *J. Appl. Prob.* **41,** 570–578.

Janardan, K. G. and Rao, B. R. (1982). Characterization of generalized Markov–Polya and generalized Polya–Eggenberger distributions. *Commun. Statist. Theory Meth.* **11,** 2113–2124.

Johnson, N. L., Kotz, S. and Kemp, A. W. (1992). *Univariate Discrete Distributions*. John Wiley, New York.

Lefèvre, C. and Picard, P. (1990). A non-standard family of polynomials and the final size distribution of Reed–Frost epidemic processes. *Adv. Appl. Prob.* **22,** 25–48.

Lefèvre, C. and Picard, P. (2005). Nonstationarity and randomization in the Reed–Frost epidemic model. *J. Appl. Prob.* **42,** 950–963.

Mohanty, S. G. and Panny, W. (1990). A discrete-time analogue of the M/M/1 queue and the transient solution: a geometric approach. *Sankhyā A* **52,** 364–370.

Perry, D., Stadje, W. and Zacks, S. (2002). First-exit times for compound Poisson processes for some types of positive and negative jumps. *Commun. Statist. Stoch. Models* **18,** 139–157.

Perry, D., Stadje, W. and Zacks, S. (2004). The first rendezvous time of Brownian motion and compound Poisson-type processes. *J. Appl. Prob.* **41,** 1059–1070.

Picard, P. and Lefèvre, C. (1996). First crossing of basic counting processes with lower non-linear boundaries: a unified approach through pseudopolynomials. I. *Adv. Appl. Prob.* **28,** 853–876.

Picard, P. and Lefèvre, C. (2003). On the first meeting or crossing of two independent trajectories for some counting processes. *Stoch. Process. Appl.* **104,** 217–242.

Picard, P., Lefèvre, C. and Coulibaly, I. (2003). Problèmes de ruine en théorie du risque à temps discret avec horizon fini. *J. Appl. Prob.* **40,** 527–542.

Sibuya, M., Miyawaki, N. and Sumita, U. (1994). Aspects of Lagrangian probability distributions. In *Studies in Applied Probability* (J. Appl. Prob. Spec. Vol. **31A**), eds J. Galambos and J. Gani, Applied Probability Trust, Sheffield, pp. 185–197.

Stadje, W. (1993). Distribution of first-exit times for empirical counting and Poisson processes with moving boundaries. *Commun. Statist. Stoch. Models* **9,** 91–103.

Takács, L. (1962). *Introduction to the Theory of Queues*. Oxford University Press.

Takács, L. (1967). *Combinatorial Methods in the Theory of Stochastic Processes*. John Wiley, New York.

Takács, L. (1989). Ballots, queues and random graphs. *J. Appl. Prob.* **26,** 103–112.

Takács, L. (1997). On the ballot theorems. In *Advances in Combinatorial Methods and Applications to Probability and Statistics*, ed. N. Balakrishnan. Birkhäuser, Boston, pp. 97–114.

Zacks, S. (1991). Distributions of stopping times for Poisson processes with linear boundaries. *Commun. Statist. Stoch. Models* **7,** 233–242.

Zacks, S. (2005). Some recent results on the distributions of stopping times of compound Poisson processes with linear boundaries. *J. Statist. Planning Infer.* **130,** 95–109.