Can. J. Math., Vol. XXVI, No. 5, 1974, pp. 1169–1172

CAPACITY ESTIMATES FOR PLANAR CANTOR-LIKE SETS

CARL DAVID MINDA

Upper and lower bounds for the capacity of planar Cantor-like sets are presented. Chebichev polynomials are the principal tool employed in the derivation of these estimates. A necessary and sufficient condition for certain planar Cantor-like sets to have positive capacity is obtained. Related one-sided capacitary estimates for more general Cantor-like sets can be found in [**3**, pp. 106-109]. Techniques analogous to those used in this paper yield similar results for linear Cantor-like sets which are well-known [**2**, pp. 150-161]. The use of Chebichev polynomials to obtain these results provides an alternate, possibly more elementary, approach to these linear problems.

First of all, we recall some of the basic facts about Chebichev polynomials and their connection with capacity. Suppose E is a compact set in the complex plane $\mathbf{C}$. Let $P_k(E)$ denote the set of all complex polynomials of degree k having leading coefficient 1 and all zeros in E. It is known that there is a polynomial $T_k(z) = T_k(z; E) \in P_k(E)$ such that

$$\max_{z \in E} |T_k(z)| = \min_{p \in P_k(E)} \max_{z \in E} |p(z)| = M_k(E)$$

and

$$\lim_{k \to \infty} M_k(E)^{1/k} = c(E),$$

where $c(E)$ denotes the capacity of E [**1**, pp. 294-297]. $T_k(z; E)$ is called a Chebichev polynomial of degree k for E. Given $a \in \mathbf{C}$ let $a + E = \{a + z : z \in E\}$ and $aE = \{az : z \in E\}$. From the above it follows immediately that $c(a + E) = c(E)$ and $c(aE) = |a|\, c\,(E)$. The former inequality asserts that capacity is translation invariant; one consequence of the latter is that capacity is rotationally invariant.

Next, we describe the construction of planar Cantor-like sets. Let S be a closed square in the complex plane having sides of length l and $p > 1$ a real number. Define an operation $[p]$ on S as follows: Take four disjoint closed squares in S each containing one vertex of S and having sides of length $l/2p$. Thus, the operation $[p]$ selects four congruent, smaller subsquares from S with sides parallel to those of S and each containing one vertex of S. Now, consider the particular square $S = \{z = x + iy : 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1\}$ and let $(p_n)_{n=1}^{\infty}$ be a sequence of real numbers with $p_n > 1$ for all n. By applying the operation $[p_1]$ to S we obtain four disjoint closed squares S_{11}, S_{12}, S_{13}, S_{14} each

Received March 28, 1973.

having sides of length $(2p_1)^{-1}$ by removing from S all points $z = x + iy$ with either $1/2p_1 < x < 1 - 1/2p_1$ or $1/2p_1 < y < 1 - 1/2p_1$. Define

$$E_1 = S[p_1] = \bigcup_{j=1}^{4} S_{1j}.$$

Then perform the operation $[p_2]$ on each of the closed squares $S_{11}, S_{12}, S_{13}, S_{14}$ to obtain sixteen closed squares S_{2j} $(1 \leqq j \leqq 4^2)$ each having sides of length $(2^2 p_1 p_2)^{-1}$. Set

$$E_2 = S[p_1 p_2] = \bigcup_{j=1}^{4^2} S_{2j}.$$

Proceed inductively; in general,

$$E_n = S[p_1 p_2 \ldots p_n] = \bigcup_{j=1}^{4^n} S_{nj},$$

where the S_{nj} are pairwise disjoint closed squares with sides of length $(2^n p_1 p_2 \ldots p_n)^{-1}$. Set $E = \bigcap_{n=1}^{\infty} E_n$; then E is an uncountable, perfect, totally disconnected subset of $\mathbf{C}$. E is called a planar Cantor-like set. Note that the area of E_n is $(p_1 p_2 \ldots p_n)^{-2}$ so that the two-dimensional Lebesgue measure of E vanishes if and only if $\prod_{n=1}^{\infty} p_n = \infty$.

We shall show that $c(E)$ satisfies the double inequality

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n}\right)^{3\cdot 4^{-n}} \prod_{n=1}^{\infty} \left(\frac{1}{2p_n}\right)^{4^{-n}} \leqq c(E) \leqq 2^{1/2} \prod_{n=1}^{\infty} \left(\frac{1}{2p_n}\right)^{4^{-n}}.$$

These new estimates for $c(E)$ constitute our main result. They will be derived from the following lemma which may be of independent interest.

Let F be a compact subset of $\mathbf{C}$. Suppose that $\theta = 2\pi/m$ for some positive integer m and $R(z) = e^{i\theta}z$ is the rotation of $\mathbf{C}$ about the origin through the angle θ. Set $F_j = R^j(F) = e^{ij\theta}F$ $(1 \leqq j \leqq m)$ and assume that the sets $(F_j)_{j=1}^m$ are pairwise disjoint. Note that $F_m = F$. Write $E = \bigcup_{j=1}^m F_j$, $d = \operatorname{diam}(E)$ and $\delta = \min\{\operatorname{dist}(F, F_j) : 1 \leqq j \leqq m - 1\}$.

LEMMA. $\delta^{(m-1)/m} c(F)^{1/m} \leqq c(E) \leqq d^{(m-1)/m} c(F)^{1/m}$.

Proof. Initially, we establish the upper bound for $c(E)$. Let $T_k(z; F_j)$ be a Chebichev polynomial of degree k for F_j $(1 \leqq j \leqq m)$ and set

$$p(z) = \prod_{j=1}^{m} T_k(z; F_j).$$

Then $p \in P_{km}(E)$. Also, for $z \in F_j$, $|T_k(z; F_j)| \leqq M_k(F_j) = M_k(F)$ and $|T_k(z, F_i)| \leqq d^k$ for $i \neq j$. Therefore,

$$M_{km}(E) \leqq \max_{z \in E} |p(z)| \leqq M_k(F) d^{(m-1)k}, \; M_{km}(E)^{1/km} \leqq [M_k(F)^{1/k}]^{1/m} d^{(m-1)/m}.$$

Upon letting $k \to \infty$ we obtain

$$c(E) \leqq c(F)^{1/m} d^{(m-1)/m}.$$

Now, we derive the lower bound for $c(E)$. Let $T_k(z) = T_k(z; E)$ be a Chebichev polynomial of degree k for E and set $p(z) = \prod_{j=1}^m T_k(R^j(z))$. Given $z \in E$, $M_k(E) \geqq |T_k(R^j(z))|$ $(1 \leqq j \leqq m)$ so that $M_k(E)^m \geqq |p(z)|$. $p(z)$ is a polynomial of degree km with k roots in each F_j $(1 \leqq j \leqq m)$. Let z_i $(1 \leqq i \leqq k)$ be the k roots, repeated according to multiplicity, which lie in $F_m = F$ and set $q(z) = \prod_{i=1}^k (z - z_i)$. Then $q \in P_k(F)$. Note that $p = qr$, where r is a polynomial of degree $(m-1)k$ all of whose zeros lie in $F_1 \cup F_2 \cup \ldots \cup F_{m-1}$ and the leading coefficient of r has modulus one. Take $z_0 \in F$ with $|q(z_0)| = \max_{z \in F} |q(z)| \geqq M_k(F)$; then $|r(z_0)| \geqq \delta^{(m-1)k}$ and $|p(z_0)| \geqq M_k(F)\delta^{(m-1)k}$. Thus,

$$M_k(E)^m \geqq \max_{z \in E} |p(z)| \geqq |p(z_0)| \geqq M_k(F)\delta^{(m-1)k},$$

$$M_k(E)^{1/k} \geqq [M_k(F)^{1/k}]^{1/m}\delta^{(m-1)/m},$$

which gives

$$c(E) \geqq c(F)^{1/m}\delta^{(m-1)/m}.$$

Note that the lemma remains valid if R is replaced by a rotation of $\mathbf{C}$ about some point $a \in \mathbf{C}$. This is true because capacity is translation invariant.

THEOREM. *Let $(p_n)_{n=1}^\infty$ be a sequence of real numbers such that $p_n > 1$ for all n and let E be the Cantor-like set determined by $(p_n)_{n=1}^\infty$. Then*

$$\prod_{n=1}^\infty \left(1 - \frac{1}{p_n}\right)^{3\cdot 4^{-n}} \prod_{n=1}^\infty \left(\frac{1}{2p_n}\right)^{4^{-n}} \leqq c(E) \leqq 2^{1/2} \prod_{n=1}^\infty \left(\frac{1}{2p_n}\right)^{4^{-n}}.$$

Proof. For a fixed positive integer n let $E_{n,m} = S[p_m p_{m+1} \ldots p_n]$ for $1 \leqq m \leqq n$, $E_{n,n+1} = S$ and

$$F_{n,m} = \{z = x + iy : z \in E_{n,m}, 0 \leqq x \leqq \tfrac{1}{2}, 0 \leqq y \leqq \tfrac{1}{2}\}.$$

Observe that if the set $F_{n,m}$ is rotated about the point $\frac{1}{2}(1 + i)$ through angles of $\pi/2$, π, $3\pi/2$ and 2π, then we obtain four disjoint sets whose union is $E_{n,m}$. Therefore, the preceding lemma gives

$$(1 - 1/p_m)^{3/4} c(F_{n,m})^{1/4} \leqq c(E_{n,m}) \leqq 2^{3/8}\, c(F_{n,m})^{1/4}.$$

Clearly, $2p_m F_{n,m} = E_{n,m+1}$, so that $2p_m\, c(F_{n,m}) = c(E_{n,m+1})$. Hence, for $1 \leqq m \leqq n$

$$\left(1 - \frac{1}{p_m}\right)^{3/4}\left(\frac{1}{2p_m}\right)^{1/4} c(E_{n,m+1})^{1/4} \leqq c(E_{n,m}) \leqq 2^{3/8}\left(\frac{1}{2p_m}\right)^{1/4} c(E_{n,m+1})^{1/4}.$$

If we apply this inequality for $m = 1, 2, \ldots, n$ and then combine the resulting inequalities in light of the facts that $E_{n,1} = E_n$ and $E_{n,n+1} = S$, we obtain

$$c(S)^{4^{-n}} \prod_{j=1}^n \left(1 - \frac{1}{p_j}\right)^{3\cdot 4^{-j}} \prod_{j=1}^n \left(\frac{1}{2p_j}\right)^{4^{-j}} \leqq c(E_n) \leqq c(S)^{4^{-n}} \cdot$$

$$2^{(3/2)Q} \prod_{j=1}^n \left(\frac{1}{2p_j}\right)^{4^{-j}},$$

https://doi.org/10.4153/CJM-1974-109-1 Published online by Cambridge University Press

where $Q = \sum_{j=1}^{n} 4^{-j}$. As $n \to \infty$, $c(E_n) \to c(E)$ and $\sum_{j=1}^{\infty} 4^{-j} = 1/3$ so that the statement of the theorem is verified.

COROLLARY 1. *If $p_n \geqq p > 1$ for all n, then*

$$\frac{p-1}{p} \prod_{n=1}^{\infty} \left(\frac{1}{2p_n}\right)^{4^{-n}} \leqq c(E) \leqq 2^{1/2} \prod_{n=1}^{\infty} \left(\frac{1}{2p_n}\right)^{4^{-n}}.$$

Proof. This follows immediately from the theorem upon making use of the inequality $1 - 1/p_n \geqq (p-1)/p$ for all n.

COROLLARY 2. *If $p_n \geqq p > 1$ for all n, then $c(E) > 0$ if and only if*

$$\sum_{n=1}^{\infty} 4^{-n} \log p_n < \infty.$$

Proof. This is a simple consequence of the preceding corollary together with the fact that the infinite product $\prod_{n=1}^{\infty} (1/2p_n)^{4^{-n}}$ converges to a finite nonzero number if and only if the infinite series $\sum_{n=1}^{\infty} 4^{-n} \log(1/2p_n)$ converges to a finite real number.

COROLLARY 3. *If $p_n = p > 1$ for all n, then*

$$\frac{p-1}{p} \left(\frac{1}{2p}\right)^{1/3} \leqslant c(E) \leqslant 2^{1/2} \left(\frac{1}{2p}\right)^{1/3}.$$

Proof. Just put $p_n = p$ for all n in the conclusion of Corollary 1.

Note that the third corollary shows that there exist sets of planar measure zero which have positive capacity. The second corollary implies that an uncountable set may have capacity zero; for example, take $p_n = 4^{4^n}$.

REFERENCES

1. G. M. Goluzin, *Geometric theory of functions of a complex variable*, Translations of Mathematical Monographs, vol. *26* (Amer. Math. Soc., Providence, 1969).
2. R. Nevanlinna, *Analytic functions, Die Grundlehren der math. Wissenschaften*, Band *162* (Springer-Verlag, New York, 1970).
3. M. Tsuji, *Potential theory in modern function theory* (Maruzen, Tokyo, 1959).

University of Cincinnati,
Cincinnati, Ohio