Bull. Aust. Math. Soc. **100** (2019), 13–26
doi:10.1017/S0004972718001259

MAXIMUM CUTS IN GRAPHS WITHOUT WHEELS

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(Received 24 September 2018; accepted 16 October 2018; first published online 26 December 2018)

Abstract

For a graph G, let $f(G)$ denote the maximum number of edges in a bipartite subgraph of G. Given a fixed graph H and a positive integer m, let $f(m, H)$ denote the minimum possible cardinality of $f(G)$, as G ranges over all graphs on m edges that contain no copy of H. Alon *et al.* ['Maximum cuts and judicious partitions in graphs without short cycles', *J. Combin. Theory Ser.* B **88** (2003), 329–346] conjectured that, for any fixed graph H, there exists an $\epsilon(H) > 0$ such that $f(m, H) \geq m/2 + \Omega(m^{3/4+\epsilon})$. We show that, for any wheel graph W_{2k} of $2k$ spokes, there exists $c(k) > 0$ such that $f(m, W_{2k}) \geq m/2 + c(k)m^{(2k-1)/(3k-1)} \log m$. In particular, we confirm the conjecture asymptotically for W_4 and give general lower bounds for W_{2k+1}.

2010 *Mathematics subject classification*: primary 05C35; secondary 05C70.

Keywords and phrases: maximum cut, H-free, wheel graph.

1. Introduction

For a graph G, let $f(G)$ denote the maximum number of edges in a cut (or, equivalently, a bipartite subgraph) of G. For a positive integer m, let $f(m)$ denote the minimum value of $f(G)$, as G ranges over all graphs with m edges.

The max cut problem asks for the value of $f(m)$ and has been widely studied. It is quite easy to show that $f(m) \geq m/2$ by considering a random partition or a suitable greedy algorithm of a graph with m edges. This elementary result can be improved by providing a more accurate estimate for the error term after the main term $m/2$. Edwards [7] proved that, for every m,

$$f(m) \geq \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}, \tag{1.1}$$

which is tight for complete graphs on an odd number of vertices. More information on $f(m)$, including a determination of its precise value for some values of m, can be found in [1, 3, 5, 6, 17].

The research of the first author is supported by Youth Foundation of Fujian Province (Grant No. JAT170398) and Foundation of Fujian University of Technology (Grant No. GY-Z15086); the research of the second author is supported by the Postdoctoral Science Foundation of China (Grant No. 2018M632528) and the Fundamental Research Funds for the Central Universities (Grant No. WK0010460005); the research of the third author is supported by NSFC (Grant No. 11601001).

For a given graph H, a graph G is called H-free if G contains no copy of H. Let $f(m, H)$ denote the minimum possible cardinality of $f(G)$, as G ranges over all H-free graphs on m edges. In [2], Alon *et al.* proposed the following conjecture.

CONJECTURE 1.1 (Alon *et al.* [2]). For any fixed graph H, there exists a positive constant $\epsilon = \epsilon(H)$ such that

$$f(m, H) \geq \frac{m}{2} + \Omega(m^{3/4+\epsilon}).$$

This conjecture remains open and it obviously suffices to prove it for complete graphs H. Zeng and Hou [20] proved that, for any fixed integer $k \geq 2$ and all $m > 1$, there is a positive constant $c(k)$ such that

$$f(m, K_{k+1}) \geq \frac{m}{2} + c(k)m^{k/(2k-1)}\left(\frac{\log^2 m}{\log\log m}\right)^{(k-1)/(2k-1)}.$$

However, the problem of estimating the error term more precisely is not easy, even for relatively simple graphs H. The case $H = K_3$ has attracted most of the attention so far. After a series of papers by various researchers [8, 16, 18], Alon [1] proved that $f(m, K_3) = m/2 + \Theta(m^{4/5})$ for all m. Alon *et al.* [4] found the following extension.

THEOREM 1.2 (Alon *et al.* [4]). *Let H be a graph obtained by connecting a single vertex to all vertices of a fixed nontrivial forest. Then there is a $c = c(H) > 0$ such that*

$$f(m, H) \geq \frac{m}{2} + cm^{4/5}$$

for all m. This is tight, up to the value of c, for each such H.

Since forests are acyclic, it is natural to study the function $f(m, H)$ for any H obtained by connecting a single vertex to all vertices of a fixed graph with cycles. We consider Conjecture 1.1 for wheel graphs, the first interesting case. Throughout, graphs are finite, undirected and have no loops or parallel edges. All logarithms are with the natural base e, unless otherwise indicated. Our main result is the following theorem.

THEOREM 1.3. *Let W_r denote the wheel graph obtained by connecting a single vertex to all vertices of a cycle of length r.*

(i) *For $r = 4$ and all m, there is a constant $c > 0$ such that*

$$f(m, W_4) \geq \frac{m}{2} + cm^{3/4}.$$

(ii) *For every odd integer $r > 3$ and all m, there is a constant $c(r) > 0$ such that*

$$f(m, W_{r+1}) \geq \frac{m}{2} + c(r)m^{2r/(3r+1)}\log m.$$

(iii) *For every odd integer $r > 3$ and all m, there is a constant $c'(r) > 0$ such that*

$$f(m, W_r) \geq \frac{m}{2} + c'(r)m^{2r/(3r+1)}(\log m)^{2/(3r+1)}.$$

REMARK 1.4. The ideas in (ii) can be further used to improve the bound in (iii) by logarithmic factors.

2. W_4-free graphs

In this section, we consider the maximum cuts of W_4-free graphs. We use the arguments from [1, 2, 4] with some additional ideas. We need a lemma proved in [1] (see also [2, 8, 14]).

LEMMA 2.1 [1]. *If G is a graph with m edges and chromatic number at most χ, then*

$$f(G) \geq \frac{\chi+1}{2\chi} m.$$

A graph is *q-degenerate* if every one of its subgraphs contains a vertex of degree at most q. We require the following well-known fact (see [1, 2, 4] for a proof).

LEMMA 2.2 [1]. *Let G be a q-degenerate graph on n vertices. Then there is a labelling $v_1, \ldots, v_n$ of the vertices of G so that, for each i with $1 \leq i \leq n$, the vertex v_i has at most q neighbours v_j with $j < i$.*

Next, we employ the following three lemmas, which establish the lower bounds of $f(G)$ for graphs G in terms of different parameters.

LEMMA 2.3 (Alon [1]). *Let $G = (V, E)$ be a graph with m edges. Suppose $U \subset V$ and let G' be the induced subgraph of G on U. If G' has m' edges, then*

$$f(G) \geq f(G') + \frac{m-m'}{2}.$$

LEMMA 2.4 (Alon *et al.* [4]). *There exist two absolute constants $\epsilon, \delta \in (0, 1)$ such that the following holds. Let G be a graph on n vertices with m edges and degree sequence $d_1, d_2, \ldots, d_n$. Suppose, further, that, for each i, the induced subgraph on all the d_i neighbours of vertex number i contains at most $\epsilon d_i^{3/2}$ edges. Then*

$$f(G) \geq \frac{m}{2} + \delta \sum_{i=1}^{n} \sqrt{d_i}.$$

LEMMA 2.5 (Erdős *et al.* [10]). *Let G be a graph on n vertices with m edges and positive minimum degree. Then*

$$f(G) \geq \frac{m}{2} + \frac{n}{6}.$$

Finally, we use the following result proved in [20] (see also [4]), which gives the existence of a randomised induced subgraph in a graph with relatively large minimum degree and sparse neighbourhood.

THEOREM 2.6 (Zeng and Hou [20]). *Let $G = (V, E)$ be a graph on n vertices with m edges and minimum degree at least m^θ for some fixed real $\theta \in (0, 1)$. Suppose that m is sufficiently large and the induced subgraph on the neighbourhood of any vertex $v \in V$ of degree d contains fewer than $sd^{3/2}$ edges for some positive constant s. Then, for every constant $\eta \in (0, 1)$, there exists an induced subgraph $G' = (V', E')$ of G with the following properties.*

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(i) *G' contains at least $\eta^2 m/2$ edges.*
(ii) *Every vertex v of degree d in G that lies in V' has degree at least $\eta d/2$ in G'.*
(iii) *Every neighbourhood of the vertex v in V' contains at most $2\eta^2 sd^{3/2}$ edges in G'.*

Proof of Theorem 1.3(i). Let $G = (V, E)$ be a W_4-free graph with n vertices and m edges. In view of (1.1), we will assume throughout the proof that m is sufficiently large. Define $q = m^{1/2}$. The proof proceeds by considering two possible cases depending on the existence of dense subgraphs in G.

Case 1. Suppose that there is a subset W of N vertices of G such that the induced subgraph $G' = G[W]$ has minimum degree greater than q. Clearly, $e(G') > qN/2$. We first prove that there exists a subset $W' \subseteq W$ such that G' contains an induced subgraph $G'' = G[W']$ with at least $qN/4$ edges, which is $2r$-colourable for $r = \lceil 2N^2/q^2 \rceil$. Choose uniformly at random, r pairs of vertices $\{x_1, y_1\}, \ldots, \{x_r, y_r\}$ from W, with repetitions allowed. Let T be the set of these pairs and note that $|T| \le r$. Let W' be the set of all vertices w of W such that there exists a pair $\{x_i, y_i\}$ of T satisfying $wx_i, wy_i \in E(G')$. Let $G'' = G[W']$.

Claim 2.7. G'' spans at least $qN/4$ edges.

Let w be a fixed vertex of W and let $\{x_i, y_i\}$ be a randomised pair of T. The probability that both wx_i and wy_i are in $E(G')$ is given by $\binom{d_{G'}(w)}{2}/\binom{N}{2}$, where $d_{G'}(w)$ denotes the degree of w in G'. This, together with the definition of r, implies that the probability that there does not exist a pair $\{x_i, y_i\}$ of T satisfying $wx_i, wy_i \in E(G')$ is at most

$$\left(1 - \frac{\binom{d_{G'}(w)}{2}}{\binom{N}{2}}\right)^r \le \exp\left\{-\frac{d_{G'}(w)(d_{G'}(w)-1)}{N(N-1)} \cdot \frac{2N^2}{q^2}\right\} \le e^{-2} < \frac{1}{4}.$$

It follows that $\mathbb{P}(w \in W') \ge 3/4$. Hence, for every fixed edge vw of G',

$$\mathbb{P}(vw \in E(G'')) = \mathbb{P}(v \in W') + \mathbb{P}(w \in W') - \mathbb{P}(v \in W' \text{ or } w \in W') > \tfrac{3}{4} + \tfrac{3}{4} - 1 = \tfrac{1}{2}.$$

By linearity of expectation and using the fact that $e(G') > qN/2$, we obtain

$$\mathbb{E}(e(G'')) = \sum_{vw \in E(G')} \mathbb{P}(vw \in E(G'')) \ge \frac{1}{2} e(G') \ge \frac{1}{4} qN.$$

This implies that there exists a particular set T of at most r pairs of vertices such that the corresponding graph G'' spans at least $qN/4$ edges, which proves Claim 2.7.

Claim 2.8. G'' is $2r$-colourable.

Fix such a T and, for each pair $\{x_i, y_i\}$ of T, let G_i denote the subgraph induced by the common neighbours of x_i and y_i in G'. Since G is W_4-free, we conclude that every G_i cannot contain a path of length two. Thus every G_i is 2-colourable. Because there are at most r pairs of vertices in T, the induced subgraph G'' is $2r$-colourable, which proves Claim 2.8.

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Next, note that if $N \geq m^{3/4}$, then it follows from Lemma 2.5 that

$$f(G) \geq \frac{m}{2} + \frac{n}{6} \geq \frac{m}{2} + \frac{N}{6} \geq \frac{m}{2} + \frac{1}{6}m^{3/4}.$$

Assume that $N < m^{3/4}$. Combining Lemma 2.1 and Claims 2.7 and 2.8,

$$f(G'') \geq \frac{e(G'')}{2} + \frac{e(G'')}{4r} \geq \frac{e(G'')}{2} + \frac{qN}{16} \cdot \left\lceil \frac{2N^2}{q^2} \right\rceil^{-1} \geq \frac{e(G'')}{2} + \frac{q^3}{32N}.$$

The above inequality, together with Lemma 2.3, implies that

$$f(G) \geq f(G'') + \frac{m - e(G'')}{2} \geq \frac{m}{2} + \frac{q^3}{32N} \geq \frac{m}{2} + \frac{1}{32}m^{3/4}.$$

Case 2. Suppose that G is q-degenerate, that is, it contains no subgraph with minimum degree greater than q. If $n \geq \frac{1}{2}m^{3/4}$, the desired result follows immediately from Lemma 2.5. Thus, we assume that $n < \frac{1}{2}m^{3/4}$ and aim to employ Lemma 2.4 to get the desired result. We first show that there exists an induced subgraph G' of G satisfying the assumptions of that lemma.

Claim 2.9. There exists an induced subgraph G' of G with at least $\frac{1}{4}\eta^2 m$ edges such that the induced subgraph on all the neighbours of any vertex of degree d' in G' contains at most $\epsilon(d')^{3/2}$ edges in G', where $\eta \in (0, 1)$ is a fixed constant and ϵ is the constant from Lemma 2.4.

As long as there is a vertex of degree smaller than $m^{1/4}$ in G, we delete it. Since $n < \frac{1}{2}m^{3/4}$, this process terminates after deleting fewer than $m^{1/4}n < \frac{1}{2}m$ edges. It then terminates with an induced subgraph $G^* = (V^*, E^*)$ of G with at least $\frac{1}{2}m$ edges and minimum degree at least $m^{1/4}$. Clearly, G^* is also W_4-free. It follows that the induced subgraph (of G^*) on the neighbourhood of any vertex v of degree d in G^* is C_4-free. As is well known, there exists a constant $c_1 > 1$ such that this induced subgraph spans at most $c_1 d^{3/2}$ edges. Then we apply Theorem 2.6 on G^* by choosing $\eta < \varepsilon^2/32c_1^2$, and hence we obtain an induced subgraph $G' = (V', E')$ of G^* (and hence of G) satisfying the required properties. (Indeed, consider a random subset V' of V^* obtained by picking each vertex of V^* randomly and independently, with probability η. Let G' be the induced subgraph on V'. One can see that G' satisfies the properties of Claim 2.9.) This completes the proof of Claim 2.9.

By Claim 2.9, the assumptions of Lemma 2.4 hold for G'. By Lemma 2.2, there exists a labelling $v_1, v_2, \ldots, v_{n'}$ of the vertices of G' such that $d_i^+ \leq q$ for every i, where d_i^+ denotes the number of neighbours v_j of v_i with $j < i$ in G'. Clearly, $\sum_{i=1}^{n'} d_i^+ = e(G')$. Let d_i be the degree of v_i in G' for each $1 \leq i \leq n'$. From Lemma 2.4,

$$\begin{aligned} f(G') &\geq \frac{e(G')}{2} + \delta \sum_{i=1}^{n'} \sqrt{d_i} \geq \frac{e(G')}{2} + \delta \sum_{i=1}^{n'} \sqrt{d_i^+} \\ &\geq \frac{e(G')}{2} + \frac{\delta}{\sqrt{q}} \sum_{i=1}^{n'} d_i^+ \geq \frac{e(G')}{2} + \frac{\delta\eta^2}{4} m^{3/4}, \end{aligned}$$

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where $\delta = \delta(G')$ is a constant, as needed. The above inequality together with Lemma 2.3 yields

$$f(G) \geq f(G') + \frac{m - e(G')}{2} \geq \frac{m}{2} + \frac{\delta\eta^2}{4} m^{3/4}.$$

The desired result follows from Cases 1 and 2 by setting $c = \min\{\frac{1}{32}, \frac{1}{4}\delta\eta^2\}$. This completes the proof of Theorem 1.3(i). □

3. W_{2k}-free graphs

In this section, we prove Theorem 1.3(ii). By Lemma 2.1, graphs with small chromatic number must have large cuts. Thus, our goal is to show that the chromatic number of a W_{2k}-free graph is relatively small.

Let G be a graph. We use $\chi(G)$ and $\alpha(G)$ to denote the chromatic number and the independence number of G, respectively. A graph property is called *monotone* if it holds for all subgraphs of a graph which has this property, that is, it is preserved under deletion of edges and vertices. We require a general lemma on monotone properties, which appears in [11, 12].

Lemma 3.1 (Jensen and Toft [11, Section 7.3]). *For $s \geq 1$, let $\psi : [s, \infty) \to (0, \infty)$ be a positive continuous nondecreasing function. Suppose that $\mathcal{P}$ is a monotone class of graphs such that $\alpha(G) \geq \psi(|V(G)|)$ for every $G \in \mathcal{P}$ with $|V(G)| \geq s$. Then, for every such G with $|V(G)| \geq s$,*

$$\chi(G) \leq s + \int_s^{|V(G)|} \frac{1}{\psi(x)}\, dx.$$

In order to bound $\chi(G)$ by Lemma 3.1, we find a lower bound for $\alpha(G)$ on a W_{2k}-free graph G in terms of $|V(G)|$. We need the following well-known lower bound of Turán from [19] and a lemma proved in [13].

Lemma 3.2 (Turán's lower bound, [19]). *Let G be a graph on n vertices with average degree at most d. Then*

$$\alpha(G) \geq \frac{n}{1+d}.$$

Lemma 3.3 (Li *et al.* [13]). *Let G be a graph on n vertices with average degree at most d. If the average degree of the subgraph induced by the neighbourhood of any vertex is at most a, then*

$$\alpha(G) \geq nF_{a+1}(d),$$

where

$$F_a(x) = \int_0^1 \frac{(1-t)^{1/a}}{a + (x-a)t}\, dt > \frac{\log(x/a) - 1}{x} \quad \textit{for all } x > 0.$$

Next, we shall use the following upper bound, which was proved by Erdős and Gallai [9], on the maximum number of edges in P_t-free graphs, where P_t stands for a simple path with t vertices.

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Lemma 3.4 (Erdős and Gallai [9]). *Let $t \geq 2$ be an integer and let G be a graph on n vertices. If G is P_t-free, then $e(G) \leq (t-2)n/2$.*

We also need the following result, proved by Pikhurko [15], on the maximum number of edges in graphs without cycles of a given even length.

Lemma 3.5 (Pikhurko [15]). *Let $k \geq 2$ be an integer and let G be a graph on n vertices. If G is C_{2k}-free, then*

$$e(G) \leq (k-1)n^{1+1/k} + 16(k-1)n.$$

Finally, we prove a lemma that gives a lower bound for the independence number of a C_{2k}-free graph.

Lemma 3.6. *Let $k \geq 2$ be a fixed integer and let G be a C_{2k}-free graph on n vertices. Then*

$$\alpha(G) \geq \frac{1}{40k^2} n^{(k-1)/k} \log n.$$

Proof. Since G is C_{2k}-free, it follows from Lemma 3.5 that $e(G) \leq 20kn^{1+1/k}$ and hence the average degree of G is at most $40kn^{1/k}$. Since the neighbourhood of any vertex of G contains no copy of P_{2k}, Lemma 3.4 implies that the average degree of the subgraph induced by the neighbourhood of any vertex is at most $2k-2$. Hence, by Lemma 3.3,

$$\alpha(G) \geq nF_{2k-1}(40kn^{1/k}) \geq n \cdot \frac{\log(40kn^{1/k}) - \log(2k-1) - 1}{40kn^{1/k}} \geq \frac{1}{40k^2} n^{(k-1)/k} \log n.$$

This completes the proof of Lemma 3.6. □

We now give a lower bound for the independence number of a W_{2k}-free graph.

Lemma 3.7. *Let $k \geq 2$ be a fixed integer and let G be a W_{2k}-free graph on n vertices. Then*

$$\alpha(G) \geq \frac{1}{80k^2} n^{(k-1)/(2k-1)} \log n.$$

Proof. Let G be a graph with maximum degree Δ. Let G' be the subgraph of G induced by the neighbourhood of any vertex of G with maximum degree Δ, and let G'' be the subgraph of G' induced by the neighbourhood of any vertex of G' with maximum degree Δ' in G'.

If $\Delta > n^{k/(2k-1)}$, then the fact that G is W_{2k}-free means that G' is C_{2k}-free. From Lemma 3.6,

$$\alpha(G) \geq \alpha(G') \geq \frac{1}{40k^2} \Delta^{(k-1)/k} \log \Delta \geq \frac{1}{80k^2} n^{(k-1)/(2k-1)} \log n.$$

Suppose now that $\Delta \leq n^{k/(2k-1)}$. Clearly, the average degree of G is at most $n^{k/(2k-1)}$.

Claim 3.8.

$$\Delta' \leq \frac{1}{3k} n^{(k-1)/(2k-1)} \log n.$$

Otherwise, $\Delta' > (1/3k)n^{(k-1)/(2k-1)}\log n$. Note that G'' is P_{2k-1}-free. By Lemma 3.4, $e(G'') \le (2k-3)\Delta'/2$ and hence the average degree of G'' is at most $2k-3$. From Lemma 3.2,

$$\alpha(G) \ge \alpha(G'') \ge \frac{\Delta'}{2k-3+1} > \frac{1}{6k^2}n^{(k-1)/(2k-1)}\log n.$$

This gives the desired result and completes the proof of Claim 3.8.

Claim 3.9.

$$\frac{k}{2k-1}\log n - \log(\Delta'+1) - 1 \ge \frac{1}{4k^2}\log n.$$

It is trivial when $\Delta' \le 1$. So assume that $\Delta' \ge 2$. It follows that $\Delta'+1 \le 3\Delta'/2$. Therefore, it suffices to show that

$$\log n^{k/(2k-1)} - \log n^{1/(4k^2)} \ge \log(3e\Delta'/2).$$

For any real number $x > 0$, we have $x \ge e\log x$. It follows that $n^{1/(2k)} \ge (e/2k)\log n$. This, together with Claim 3.8, yields

$$\begin{aligned} n^{k/(2k-1)-1/(4k^2)} &= n^{(k-1)/(2k-1)} \cdot n^{1/(2k-1)-1/(4k^2)} \ge n^{(k-1)/(2k-1)} \cdot n^{1/(2k)} \\ &\ge n^{(k-1)/(2k-1)} \cdot \frac{e}{2k}\log n \ge \frac{3}{2}e\Delta', \end{aligned}$$

giving the desired result. This proves Claim 3.9.

Now, combining Lemma 3.3 and Claim 3.9,

$$\begin{aligned} \alpha(G) \ge nF_{\Delta'+1}(n^{k/(2k-1)}) &\ge n \cdot \frac{(k/(2k-1))\log n - \log(\Delta'+1) - 1}{n^{k/(2k-1)}} \\ &\ge \frac{1}{4k^2}n^{(k-1)/(2k-1)}\log n. \end{aligned}$$

This completes the proof of Lemma 3.7. □

With the help of Lemmas 3.1 and 3.7, we immediately have the following result.

Lemma 3.10. *Let $k \ge 2$ be a fixed integer and let G be a W_{2k}-free graph with n vertices. Then*

$$\chi(G) \le 240k^2 \cdot \frac{n^{k/(2k-1)}}{\log n}.$$

Proof. This is trivial for $n < e^6$ as $\chi(G) \le n < e^6$. Assume that $n \ge e^6$. For $x \ge e^6$, define

$$\gamma(x) = \frac{k}{2k-1} - \frac{1}{\log x} \quad \text{and} \quad \psi(x) = \frac{1}{80k^2}x^{(k-1)/(2k-1)}\log x.$$

Obviously, $\gamma(x)$, $\psi(x)$ are positive continuous and nondecreasing, and $\gamma(x) \ge 1/3$ for $x \ge e^6$. It follows from Lemma 3.7 that $\alpha(G) \ge \psi(n)$. Then, by Lemma 3.1,

$$\begin{aligned} \chi(G) &\le e^6 + \int_{e^6}^{n} \frac{1}{\psi(x)}\,dx \le e^6 + \frac{80k^2}{\gamma(e^6)}\int_{e^6}^{n} \frac{\gamma(x)}{x^{(k-1)/(2k-1)}\log x}\,dx \\ &= e^6 + 240k^2\Big(\frac{x^{k/(2k-1)}}{\log x}\Big)\Big|_{e^6}^{n} \le 240k^2 \cdot \frac{n^{k/(2k-1)}}{\log n}. \end{aligned}$$

This completes the proof of Lemma 3.10. □

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We wish to bound the chromatic number $\chi(G)$ of a W_{2k}-free graph G in terms of its number of edges. Thus we also need the following lemma.

Lemma 3.11. *Let $k \geq 3$ be a fixed integer. There exists a constant $b > 0$ such that if G is a W_{2k}-free graph with n vertices and average degree $d > 0$, then*

$$\alpha(G) \geq \frac{n \log d}{bk^2 d}.$$

Proof. Let G be a W_{2k}-free graph with n vertices and average degree d. If $d > e^2(bk^2 + 1)$, then $k < d^{1/2}/(\sqrt{b}e)$. At most half of the vertices of G have degree greater than $2d$. Let G' be G with these vertices deleted. Then G' has at least $n/2$ vertices and maximum degree at most $2d$. Clearly, G' is W_{2k}-free. Thus the neighbourhood of any vertex of G' cannot contain a cycle of length $2k$. Hence, by Lemma 3.5, the average degree of the subgraph induced by the neighbourhood of any vertex of G' is at most $40k(2d)^{1/k} < 60kd^{1/k}$. From Lemma 3.3 and the fact that $k < d^{1/2}/(\sqrt{b}e)$,

$$\begin{aligned}\alpha(G) \geq \alpha(G') &\geq \frac{n}{2} F_{60kd^{1/k}+1}(2d) \geq \frac{n(\log 2d - \log(61kd^{1/k}) - 1)}{4d} \\ &\geq \frac{n}{4d}\Big(\log(2\sqrt{b}) - \log 61 + \frac{k-2}{2k}\log d\Big) \geq \frac{n \log d}{k^2 d},\end{aligned}$$

where the last inequality holds by choosing $b > 930$, since $k \geq 3$.

So we assume that $d \leq e^2(bk^2 + 1)$. It is easy to see that $x \geq \log(x + 3) + 1/2$ for any real number $x \geq 2$ and that the function $g(x) = \log x/x$ is monotonically increasing over the interval $(0, e]$ and decreasing over the interval $[e, +\infty)$. Hence,

$$bk^2 \geq \log(bk^2 + 1) + \frac{5}{2} \geq \log d + \frac{1}{2} > \log d + \frac{1}{e} \geq \log d + \frac{\log d}{d} = \frac{(1 + d)\log d}{d}.$$

This, together with Lemma 3.2, implies that

$$\alpha(G) \geq \frac{n}{1 + d} \geq \frac{n \log d}{bk^2 d},$$

which completes the proof of Lemma 3.11. □

Now, we establish the following theorem, which plays a key role in the proof of Theorem 1.3(ii). The approach we take is an extension of that by Poljak and Tuza [16].

Theorem 3.12. *Let $k \geq 3$ be a fixed integer and let b be an integer given by Lemma 3.11. Suppose that G is a W_{2k}-free graph with $m > 1$ edges. Then*

$$\chi(G) \leq 97bk^2 \cdot \frac{m^{k/(3k-1)}}{\log m}.$$

Proof. Let G be a W_{2k}-free graph on n vertices with $m > 1$ edges. If $n < 8$, then $m < 28$ and the desired result follows from $\chi(G) \leq n < 8$. Suppose that $n \geq 8$.

Define $n^* = m^{(2k-1)/(3k-1)}$. The function $g(x) = x^{k/(2k-1)}/\log x$ is monotonically increasing over the interval $(e^{(2k-1)/k}, +\infty)$. If $n \le n^*$, then Lemma 3.10 implies that

$$\chi(G) \le 240k^2 \cdot \frac{n^{k/(2k-1)}}{\log n} \le 240k^2 \cdot \frac{(n^*)^{k/(2k-1)}}{\log n^*} \le 720k^2 \cdot \frac{m^{k/(3k-1)}}{\log m},$$

giving the desired result. Hence assume that $n > n^*$. We can delete all the vertices with degree zero or one in G, that is, we can assume that $m \ge n$. Now we construct a graph sequence $\{G_i\}_{i\ge 0}$ on G. Start with $i = 0$, $G_0 = G$ and $n_0 = |V(G_0)|$. If $n_i > n^*$, we carry out the following iterative procedure; otherwise, we stop. Choose S_i to be the maximum independent set of G_i. Then set $G_{i+1} = G_i \backslash S_i$, $n_i = |V(G_i)|$, and increment i.

Let G_ℓ be the graph at the end of the process. Clearly, $n_\ell \le n^*$ and G_ℓ is W_{2k}-free. By Lemma 3.10,

$$\chi(G_\ell) \le 240k^2 \cdot \frac{(n^*)^{k/(2k-1)}}{\log n^*} \le 720k^2 \cdot \frac{m^{k/(3k-1)}}{\log m}. \tag{3.1}$$

Note that $\chi(G) \le \chi(G_\ell) + \ell$. In the following, it is sufficient to bound the value of ℓ.

We first bound the value of $|S_i|$. Let $t = \lceil n/n^* \rceil$ and note that $t \ge 2$ because $n > n^*$. Let $I = \{0, 1, \dots, \ell - 1\}$. For each $i \in I$, we have $n_i > n^* \ge n/t$, by the definition of t. Let $v_1, \dots, v_{n_0}$ be a labelling of the vertices of G_0 such that $S_i = \{v_p : n_{i+1} < p \le n_i\}$ for each $i \in I$. Let S be the union of S_i for all $i \in I$ and let $J = \{2, 3, \dots, t\}$. For each $j \in J$, define

$$V_j = \left\{v_p \in S : \frac{n}{j} < p \le \frac{n}{j-1}\right\} \quad \text{and} \quad I_j = \left\{i \in I : n_i > \frac{n}{j}\right\}.$$

Observe that $S \backslash S_{\ell-1} \subseteq \bigcup_{j\in J} V_j \subseteq S$ and $I_2 \subseteq I_3 \subseteq \cdots \subseteq I_t$. Hence, for each $v \in V_j$, there exists an $i \in I_j$ such that $v \in S_i$. In addition,

$$|V_j| \le \left\lceil \frac{n}{j-1} - \frac{n}{j} \right\rceil \le \frac{4n}{j^2}. \tag{3.2}$$

Claim 3.13. For each $i \in I_j \neq \emptyset$,

$$|S_i| \ge \frac{n^2 \log(2jm/n)}{2bk^2 j^2 m}.$$

For each $i \in I$, we let d_i denote the average degree of G_i. For each $i \in I_j$, observe that $d_i \le 2m/n_i \le 2jm/n$. If $d_i > e$, the function $g(x) = \log x/x$ is decreasing over the interval $(e, +\infty)$. From Lemma 3.11 and the fact that $d_i \le 2jm/n$,

$$|S_i| \ge \frac{n_i \log d_i}{bk^2 d_i} \ge \frac{n^2 \log(2jm/n)}{2bk^2 j^2 m}.$$

If $d_i \le e$, then, by Lemma 3.2, $|S_i| \ge n_i/(1+e) \ge n_i/(bk^2) \ge n/(bk^2 j)$. Since $x \ge \log x$, this gives the required lower bound and completes the proof of Claim 3.13.

Then, for each $v \in S_i$ and $i \in I$, let $w(v) = |S_i|^{-1}$. Hence, for each $v \in S_i \subset V_j$, Claim 3.13 gives

$$w(v) = |S_i|^{-1} \le \frac{2bk^2 j^2 m}{n^2 \log(2jm/n)} \le \frac{2bk^2 j^2 mn^{-2}}{\log j + \log(m/n)}.$$

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Combining the above inequality, the definition of $w(v)$ and (3.2),

$$\ell - 1 = \sum_{i \in I \setminus \{\ell-1\}} \sum_{v \in S_i} w(v) \le \sum_{j \in J} \sum_{v \in V_j} w(v) \le \sum_{j=2}^{t} |V_j| w(v) \le \sum_{j=2}^{t} \frac{8bk^2mn^{-1}}{\log j + \log(m/n)}. \tag{3.3}$$

If $t < m/n$, then delete the first term of the denominator of (3.3). Since $t - 1 < n/n^*$ by the definition of t,

$$\ell - 1 \le \sum_{j=2}^{t} \frac{8bk^2mn^{-1}}{\log(m/n)} = \frac{(t-1)8bk^2mn^{-1}}{\log(m/n)} \le \frac{8bk^2m}{n^* \log(m/n)}. \tag{3.4}$$

Because $t - 1 < n/n^* \le t$ and by the definition of n^*,

$$t \cdot \frac{m}{n} \ge \frac{n}{n^*} \cdot \frac{m}{n} = \frac{m}{n^*} = m^{k/(3k-1)}. \tag{3.5}$$

It follows that $\max\{t, m/n\} \ge m^{k/(2(3k-1))}$ and thus

$$\max\left\{\log t, \log \frac{m}{n}\right\} \ge \frac{k}{2(3k-1)} \log m. \tag{3.6}$$

Combining (3.4)–(3.6), we conclude that $\ell - 1 \le 48bk^2m^{k/(3k-1)}/\log m$.

If $t \ge m/n$, noting that

$$\sum_{j=2}^{t} \frac{1}{\log j} \le \int_2^t \frac{1}{\log x}\,dx \le \frac{2(t-1)}{\log t} < \frac{2n}{n^* \log t},$$

we delete the second term of the denominator in (3.3) and obtain

$$\ell - 1 \le \frac{8bk^2m}{n} \sum_{j=2}^{t} \frac{1}{\log j} \le \frac{16bk^2m}{n^* \log t} \le 96bk^2 \cdot \frac{m^{k/(3k-1)}}{\log m},$$

where the last inequality follows from (3.5) and (3.6).

Hence we can conclude that $\ell - 1 \le 96bk^2m^{k/(3k-1)}/\log m$. This, together with (3.1), implies that

$$\chi(G) \le \chi(G_\ell) + \ell \le (720 + 96b)k^2 \cdot \frac{m^{k/(3k-1)}}{\log m} + 1 \le 97bk^2 \cdot \frac{m^{k/(3k-1)}}{\log m}.$$

This completes the proof of Theorem 3.12. □

Proof of Theorem 1.3(ii). Let $r + 1 = 2k \ge 6$ be a fixed even integer and let G be a W_{2k}-free graph with m edges. The desired result follows immediately for $m = 1$. For $m > 1$, we set $c(r) = 2(r+1)^{-2}/(97b)$. Then the desired result follows from Lemma 2.1 and Theorem 3.12. □

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4. W_{2k+1}-free graphs

In this section, we present a proof of Theorem 1.3(iii). The proof is similar to that of Theorem 1.3(ii). The following result bounds the independence number of a C_{2k+1}-free graph for every $k \geq 2$.

LEMMA 4.1 (Zeng and Hou [21]). *Let $k \geq 2$ be a fixed integer and let G be a C_{2k+1}-free graph on n vertices. Then*

$$\alpha(G) \geq \frac{1}{5k^2}(n^k \log n)^{1/(k+1)}.$$

PROOF OF THEOREM 1.3(iii). Let $r = 2k+1 \geq 5$ be a fixed integer and let G be a W_{2k+1}-free graph with m edges and maximum degree Δ. Let G' be the graph induced by the neighbourhood of any vertex of G with maximum degree Δ. Clearly, G' is C_{2k+1}-free. We first give a lower bound of $\alpha(G)$. If $\Delta \geq (n^{k+1}/\log n)^{1/(2k+1)}$, then, by Lemma 4.1,

$$\alpha(G) \geq \alpha(G') \geq \frac{1}{5k^2}(\Delta^k \log \Delta)^{1/(k+1)} \geq \frac{1}{5k^4}(n^k \log n)^{1/(2k+1)}.$$

Otherwise, $\Delta < (n^{k+1}/\log n)^{1/(2k+1)}$, and from Lemma 3.2,

$$\alpha(G) \geq \frac{n}{1+\Delta} \geq \frac{n}{2n^{(k+1)/(2k+1)}} \cdot (\log n)^{1/(2k+1)} \geq \frac{1}{5k^4}(n^k \log n)^{1/(2k+1)}.$$

Next, we bound $\chi(G)$. Let

$$\psi(x) = \frac{1}{5k^4}(x^k \log x)^{1/(2k+1)} \quad \text{and} \quad \gamma(x) = \frac{k+1}{2k+1}\Big(1 - \frac{1}{(k+1)\log x}\Big).$$

Note that $\psi(x)$, $\gamma(x)$ are positive continuous and nondecreasing and $\gamma(x) \geq 1/3$ for $x \geq e$. Moreover, $\alpha(G) \geq \psi(n)$. From Lemma 3.1,

$$\chi(G) \leq e + \frac{5k^4}{\gamma(e)} \int_e^n \frac{\gamma(x)}{(x^k \log x)^{1/(2k+1)}}\, dx \leq 15k^4\Big(\frac{n^{k+1}}{\log n}\Big)^{1/(2k+1)}, \tag{4.1}$$

where the last inequality holds because an antiderivative for the integrand is exactly $(x^{k+1} \log^{-1} x)^{1/(2k+1)}$.

Finally, if $n \geq (m^{2k+1} \log m)^{1/(3k+2)}$, then Lemma 2.5 implies that

$$f(G) \geq \frac{m}{2} + \frac{n}{6} \geq \frac{m}{2} + \frac{1}{6}(m^{2k+1} \log m)^{1/(3k+2)}.$$

If $n < (m^{2k+1} \log m)^{1/(3k+2)}$, then, by Lemma 2.1 and (4.1), we conclude that

$$f(G) \geq \frac{m}{2} + \frac{m}{30k^4} \cdot \Big(\frac{\log n}{n^{k+1}}\Big)^{1/(2k+1)} \geq \frac{m}{2} + \frac{1}{60k^4}(m^{2k+1} \log m)^{1/(3k+2)},$$

where the last inequality holds because the function $g(x) = ((\log x)/x^{k+1})^{1/(2k+1)}$ is monotonically decreasing over the interval $(e^{1/(k+1)}, +\infty)$. We get the desired result by noting that $r = 2k+1$ and by setting $c'(r) = 4/(15(r-1)^4)$. This completes the proof of Theorem 1.3(iii). □

https://doi.org/10.1017/S0004972718001259 Published online by Cambridge University Press

Acknowledgements

The authors are indebted to the anonymous referees for many valuable comments and constructive suggestions. The first author would also like to thank Genghua Fan, Jianfeng Hou and Huawen Ma for insightful discussions following an earlier version of this paper.

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