THE G-FUNCTION OF MACRAE

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Introduction. Let R be a commutative ring with identity. A finitely generated R-module M is called a torsion module if the annihilator of M contains a non zero-divisor. In [**18**] MacRae proved the following

THEOREM. *If R is a noetherian ring then there is a map G with the following properties from the class of torsion R-modules of finite homological dimension to the set of integral invertible ideals of R.*

(i) *If M is a finitely generated torsion R-module with homological dimension $\leqq 1$ then $G(M) = F(M)$, the first Fitting ideal of M.*

(ii) *If S is a multiplicative subset of R then $G(M_S) = G(M)_S$.*

(iii) *If $0 \to L \to M \to N \to 0$ is an exact sequence of torsion modules of finite homological dimension then $G(M) = G(L)G(N)$.*

MacRae also showed that the function G, when applied to a cyclic torsion module R/I, gives the greatest common divisor of I, whenever R/I has a finite resolution by finitely generated free modules, and observed that this implies unique factorization in regular local rings. In sections 1 and 2 of this note we obtain a version of the above theorem for arbitrary commutative rings with identity. In section 3 we show that when $G(M)$ is defined it is the smallest invertible ideal containing $F(M)$, and therefore $G(R/I)$ is the greatest common divisor of I whenever R/I is a torsion module having a finite free resolution.

In section 4 we use the function G to obtain information about certain types of resolutions. In particular we obtain a version of Burch's theorem on the structure of cyclic modules M of homological dimension 2 over a local (noetherian) ring R, for the case that R is an arbitrary domain and M is a finitely generated torsion R-module having a finite free resolution of length 2. We conclude with some remarks generalizing the unique factorization of regular local rings.

Throughout this note R denotes a commutative ring with identity and R-mod denotes the category of R-modules. We write $T(R)$ for the total quotient ring of R. A torsion R-module is an R-module M such that $T(R) \otimes_R M = 0$. If M is finitely generated this means that $rM = 0$ for some non zero-divisor r of R. We let $(A : B)_C = \{x \in C \mid xB \subset A\}$ whenever this makes sense. If A is a subset of an R-module then Ann (A) denotes the annihilator of A. By a fractional ideal of R we mean an R-submodule A of $T(R)$ containing a regular element of R and such that $rA \subset R$ for some regular $r \in R$. If A is fractional ideal of R, $(R : A)_{T(R)}$ is also and is denoted A^{-1}. A fractional ideal I of R is

called integral if $I \subset R$. We let $\mathscr{F}(R)$, $\mathscr{I}(R)$, and $\mathscr{P}(R)$ denote the sets of fractional ideals, invertible fractional ideals, and principal fractional ideals of R respectively.

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1. Euler maps. Let R be a ring and let $\mathscr{O}$ be a full subcategory of the category of projective R-modules. Assume that $\mathscr{O}$ is closed under finite direct sums and contains the zero-module. We say that an R-module M has an $\mathscr{O}$-resolution if there exists an exact sequence $\ldots \to P_n \to \ldots \to P_0 \to M \to 0$ with each $P_i \in \mathscr{O}$. If there exists such a resolution with $P_i = 0$ for every $i > n$, we say that M has an $\mathscr{O}$-resolution of length n. Let

$$\mathrm{Res}_\infty(\mathscr{O}) \supset \mathrm{Res}(\mathscr{O}) \supset \mathrm{Res}_n(\mathscr{O})$$

be the full subcategories of R-modules having $\mathscr{O}$-resolutions, finite $\mathscr{O}$-resolutions, and $\mathscr{O}$-resolutions of length n respectively. If $M \in \mathrm{Res}_\infty(\mathscr{O})$ we define the $\mathscr{O}$-dimension of M, $d_\mathscr{O}(M)$, to be the minimal length (possibly infinite) of an $\mathscr{O}$-resolution of M.

We can define an equivalence relation $\sim$ on the objects of R-mod by $M_1 \sim M_2$ if there exists $P_1, P_2 \in \mathscr{O}$ such that $M_1 \oplus P_1 \cong M_2 \oplus P_2$. Let $\mathscr{O}^*$ be the full subcategory of R-mod with objects $\{M \in R\text{-mod} | M \sim 0\}$. Then $\mathscr{O} \subset \mathscr{O}^*$ and $\mathscr{O}^{**} = \mathscr{O}^*$.

It follows that $\mathrm{Res}(\mathscr{O}) = \mathrm{Res}(\mathscr{O}^*)$; and if $M \in \mathrm{Res}(\mathscr{O})$ then $d_\mathscr{O}(M) = d_{\mathscr{O}^*}(M)$ if $d_{\mathscr{O}^*}(M) \neq 0$, and $0 \leqq d_\mathscr{O}(M) \leqq 1$ if $d_{\mathscr{O}^*}(M) = 0$ [**21**, 2.2 and 2.3]. Also if

$$\ldots \longrightarrow P_{n+1}{}^* \xrightarrow{f_{n+1}} P_n{}^* \longrightarrow \ldots P_0{}^* \longrightarrow M \longrightarrow 0$$

is an $\mathscr{O}^*$-resolution for M, and $M \in \mathrm{Res}_n(\mathscr{O}^*)$, then $\ker f_{n-1} \in \mathscr{O}^*$ ([**21**, Corollary 2.5] or [**1**, p. 37]). The following two theorems show that $\mathscr{O}^*$-dimension behaves like the usual homological dimension, and in fact is the usual homological dimension whenever the $\mathscr{O}^*$-dimension is defined and finite.

1.1 DIMENSION THEOREM. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. If two of the modules A, B, and C, have $\mathscr{O}^*$-resolutions, then so does the third and in this case $d_{\mathscr{O}^*}(B) \leqq \max \{d_{\mathscr{O}^*}(A), d_{\mathscr{O}^*}(C)\}$. If this inequality is strict then*

$$d_{\mathscr{O}^*}(C) = d_{\mathscr{O}^*}(A) + 1.$$

Proof. See [**1**, p. 39, Proposition 6.8].

1.2 THEOREM. *If $M \in \mathrm{Res}(\mathscr{O}^*)$, then $d_{\mathscr{O}^*}(M) = d(M)$, where $d(M)$ denotes the usual homological dimension of M.*

Proof. The proof is essentially the same as [**19**, p. 162, Lemma 4.3], so we omit it.

Let $\mathscr{C}$ be a subcategory of R-mod. An *Euler map* from $\mathscr{C}$ to an abelian group G is a map f from the objects of $\mathscr{C}$ to G such that $f(A) + f(C) = f(B)$ for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{C}$.

Let $\mathscr{U}$ and $\mathscr{T}$ be the full subcategories of R-mod whose objects are the finitely generated projective modules and the torsion R-modules respectively. Let Res $(\mathscr{U}, \mathscr{T}) = $ Res $(\mathscr{U}) \cap \mathscr{T}$ and $\mathrm{Res}_n(\mathscr{U}, \mathscr{T}) = \mathrm{Res}_n(\mathscr{U}) \cap \mathscr{T}$. The following theorem is found in [**21**, Theorems 3.3 and 3.4], and is basic to what follows.

1.3 THEOREM. *If $f : \mathrm{Res}_1(\mathscr{U}, \mathscr{T}) \to G$ is an Euler map, then f extends uniquely to an Euler map $f : $ Res $(\mathscr{U}, \mathscr{T}) \to G$.*

Let $K(\mathrm{Res}_1(\mathscr{U}, \mathscr{T}))$ denote the Grothendieck group of $\mathrm{Res}_1(\mathscr{U}, \mathscr{T})$, i.e. the group with generators $\{[M] | M \in \mathrm{Res}_1(\mathscr{U}, \mathscr{T})\}$, and relations

$\{[A] + [C] = [B] | 0 \to A \to B \to C \to 0$ is an exact sequence in $\mathrm{Res}_1(\mathscr{U}, \mathscr{T})\}$.

Then the Euler maps $f : \mathrm{Res}_1(\mathscr{U}, \mathscr{T}) \to G$ correspond to the group homomorphisms $f' : K(\mathrm{Res}_1(\mathscr{U}, \mathscr{T})) \to G$ and it is easily seen that the above theorem is equivalent to the assertion that the canonical map $K(\mathrm{Res}_1(\mathscr{U}, \mathscr{T})) \to K(\mathrm{Res}(\mathscr{U}, \mathscr{T}))$ is a group isomorphism. Further, $K(\mathrm{Res}_1(\mathscr{U}, \mathscr{T}))$ becomes a pre-ordered group if we take for the positive elements

$\{[M] | M \in \mathrm{Res}_1(\mathscr{U}, \mathscr{T})\}$

(and similarly for $K(\mathrm{Res}(\mathscr{U}, \mathscr{T}))$), and the isomorphism $K(\mathrm{Res}_1(\mathscr{U}, \mathscr{T})) \to K(\mathrm{Res}(\mathscr{U}, \mathscr{T}))$ is order preserving (but not in general an isomorphism of ordered groups).

2. The MacRae function. In [**18**], MacRae obtained for any noetherian ring R, an Euler map G from $\mathrm{Res}(\mathscr{U}, \mathscr{T})$ into the group $\mathscr{I}(R)$ of invertible ideals of R and showed that $G(M)$ is always an integral ideal of R. He then showed that if M is cyclic then $G(M)$ is the greatest common divisor of the annihilator of M. In this section we apply Theorem 1.3 to obtain this Euler map for an arbitrary commutative ring. In the next section we discuss the relationship of G to greatest common divisors.

Let $\mathscr{V}$ be the full subcategory of R-mod whose objects are the finitely generated free modules. If M is a finitely generated R-module let $F(M)$ denote the first Fitting ideal of M. (We are using Kaplansky's numbering here [**11**, p. 145]. This is called the zero-th Fitting ideal in [**18**].) We collect the facts we need about F in the following theorem.

2.1 THEOREM. *Let M, A, B, C be finitely generated R-modules, and let S be a multiplicative subset of R.*
 (a) $F(M)_S = F(M_S)$.
 (b) *If M can be generated by n elements, then $\mathrm{Ann}(M)^n \subset F(M) \subset \mathrm{Ann}(M)$.*

(c) If $0 \to A \to B \to C \to 0$ is exact and $C \in \mathrm{Res}_1(\mathscr{U}, \mathscr{T})$, then $F(A)F(C) = F(B)$.

(d) If $M \in \mathrm{Res}_1(\mathscr{U}, \mathscr{T})$, then $F(M)$ is an (integral) invertible ideal of R.

(e) If $M \in \mathrm{Res}_1(\mathscr{V}, \mathscr{T})$, then $F(M) \in \mathscr{P}(R)$.

Proof. (a), (b), and (c) are proved in [**18**]. To prove (d) note that the hypothesis implies that $F(M)$ contains a regular element and is finitely generated. Thus (d) follows from [**18**, p. 158, Lemma 2.9]. (e) is straightforward.

2.2 THEOREM. *There exists a unique Euler map* $G : \mathrm{Res}(\mathscr{U}, \mathscr{T}) \to \mathscr{I}(R)$ *such that if* $M \in \mathrm{Res}_1(\mathscr{U}, \mathscr{T})$, *then* $G(M) = F(M)$. *Further, if* $M \in \mathrm{Res}(\mathscr{V}, \mathscr{T})$, *then* $G(M) \in \mathscr{P}(R)$.

2.3 COROLLARY. *If* $M \in \mathrm{Res}(\mathscr{U}, \mathscr{T})$ *and* S *is a multiplicative subset of* R, *then* $G(M)_S = G(M_S)$.

Proof. This is immediate from the uniqueness of G.

2.4 *Note.* The proof of [**15**, Satz 5] extends easily to show that, if R is a domain, the group homomorphism $G : K(\mathrm{Res}(\mathscr{V}, \mathscr{T})) \to \mathscr{P}(R)$, induced by G, is an isomorphism (of ordered groups).

In order to show that $G(M)$ is actually an integral ideal of R for each $M \in \mathrm{Res}(\mathscr{U}, \mathscr{T})$ we use the following lemma. If $A \subset R$ we write $A^{\frac{1}{2}}$ for the intersection of the prime ideals of R containing A.

2.5 LEMMA. *Let* R *be a quasi-local ring with maximal ideal* $m = ((a_1, \ldots, a_n) : b)^{\frac{1}{2}}$ *where* $a_1, \ldots, a_n$ *is an* R-*sequence. Then* $\mathrm{Res}(\mathscr{U}) = \mathrm{Res}_n(\mathscr{U})$.

Proof. We use induction on n. If $n = 0$ then each finite subset of m has non-zero annihilator [**16**, p. 93, Corollary 1.14] and the assertion follows by [**11**, Theorem 191]. If $n > 0$, let $M \in \mathrm{Res}(\mathscr{U})$ and let $0 \to K \to F \to M \to 0$ be exact with $F \in \mathscr{U} (= \mathscr{V})$. For each R-module A let $A^* = A/a_1 A$ and let $\mathscr{U}^* = \{\text{finitely generated projective } R^*\text{-modules}\}$. Since a_1 is a regular element it follows that $K^* \in \mathrm{Res}(\mathscr{U}^*)$, and $d_{\mathscr{U}^*}(K^*) = d_{\mathscr{U}}(K)$ by the same argument as [**26**, p. 663]. Also, R^* is quasi-local with maximal ideal $m^* = ((a_2^*, \ldots, a_n^*) : b^*)^{\frac{1}{2}}$ and it follows from [**11**, Theorem 116] that $a_2^*, \ldots, a_n^*$ forms an R^*-sequence (where x^* is the image of x in $R/a_1 R$). Thus by induction we have that $d_{\mathscr{U}}(K) = d_{\mathscr{U}^*}(K^*) \leqq n - 1$ and therefore $d_{\mathscr{U}}(M) \leqq n$.

2.6 THEOREM. $G(M) \subset R$ *for each* $M \in \mathrm{Res}(\mathscr{U}, \mathscr{T})$.

Proof. We use induction on $n = d_{\mathscr{U}}(M)$. If $n = 1$ then $G(M) = F(M) \subset R$; so assume $n > 1$. Then there exists an exact sequence $0 \to K \to L \to M \to 0$ with $K, L \in \mathrm{Res}_{n-1}(\mathscr{U}, \mathscr{T})$. Then $G(M) = G(L)G(K)^{-1}$ and by the induction hypothesis $G(L), G(K) \subset R$. Let $G(L) = I$ and $G(K) = J$. To show that $IJ^{-1} \subset R$ it suffices to show that $(J : I)_R = R$. If $(J : I) \neq R$ let P be a minimal prime divisor of $J : I$. Then $J_P : I_P = (J : I)R_P \neq R_P$ and we may reduce to the case that R is quasi-local with maximal ideal $(J : I)^{\frac{1}{2}}$. But in

this case I and J are principal, and thus by Lemma 2.5 $d_{\mathcal{U}}(M) \leqq 1$. But then $G(M) = F(M)$ implies $I = JF(M) \subset J$, a contradiction.

3. Greatest common divisors. Let A be a fractional ideal of R. We define the divisorial ideal $\tilde{A}$ corresponding to A as

$$\tilde{A} = \cap\, \{I \in \mathscr{P}(R) | A \subset I\}.$$

Let $A^{-1} = (R : A)_{T(R)}$ and let $\bar{A} = \{x \in A | x$ is regular in $T(R)\}$. If $A \subset R$ and $\tilde{A}$ is principal, then $\tilde{A}$ is the greatest common divisor of A. For domains we always have $\tilde{A} = (A^{-1})^{-1}$. Thus for domains, invertible ideals are divisorial and hence $\tilde{A} = \cap\, \{I \in \mathscr{I}(R) | A \subset I\}$. For general commutative rings we have the following

3.1 PROPOSITION. $\tilde{A} = R : \overline{R : A}$ *for each fractional ideal A of R.*

Proof. For $x \in T(R)$ we have $x \in R : \overline{R : A}$ if and only if $xp \in R$ for every regular $p \in R : A$ if and only if $x \in p^{-1}R$ for every regular $p \in T(R)$ such that $A \subset p^{-1}R$ if and only if $x \in \tilde{A}$.

It follows from 3.1 that $\tilde{A} = (A^{-1})^{-1}$ for every fractional ideal A of R if R satisfies the following property.

(P) *Every regular ideal of R is generated by regular elements.*

Rings with this property have been studied by Jean Marot who has shown that R satisfies (P) if $T(R)/0^{\frac{1}{2}}$ is absolutely flat [**19**]. Thus coherent rings of finite weak global dimension [**22**, p. 270, Corollary 3] as well as noetherian rings and domains satisfy (P). The following result includes [**18**, p. 167, Proposition 5.5]. [**15**, p. 480, Satz 1], and [**23**, p. 881, Theorem].

3.2 THEOREM. *If $M \in \mathrm{Res}(\mathscr{U}, \mathscr{T})$, then $G(M)$ is an invertible ideal containing $F(M)$, and is contained in every other invertible ideal containing $F(M)$.*

Proof. To show $F(M) \subset G(M)$ suppose there exists $a \in F(M) \backslash G(M)$, and let P be a minimal prime divisor of $(G(M) : a)_R$. Then $PR_P = (G(M)_P : aR_P)^{\frac{1}{2}}$, and since $G(M)_P$ is R_P-free, $M_P \in \mathrm{Res}_1(\mathscr{U}, \mathscr{T})$ by Lemma 2.5. Hence $aR_P \subset F(M)_P = G(M)_P$, a contradiction. Thus $F(M) \subset G(M)$.

Now suppose J is an invertible ideal of R containing $F(M)$. If $G(M) \not\subset J$ let P be a minimal prime divisor of $(J : G(M))_R$. Since J and $G(M)$ are invertible, it follows as above that $M_P \in \mathrm{Res}_1(\mathscr{U}, \mathscr{T})$, and hence $J_P \supset F(M)_P = G(M)_P$, a contradiction. Thus $G(M) \subset J$.

It follows that if $M \in \mathrm{Res}(\mathscr{V}, \mathscr{T})$, then $G(M)$ is the greatest common divisor of the elements of $F(M)$, and in particular, if $R/I \in \mathrm{Res}(\mathscr{V}, \mathscr{T})$, then $G(R/I)$ is the greatest common divisor of I.

If R satisfies (P) and $M \in \mathrm{Res}(\mathscr{U}, \mathscr{T})$, then $G(M) = \widetilde{F(M)} = (F(M)^{-1})^{-1}$, and hence $\widetilde{F(M)} = (F(M)^{-1})^{-1}$ is invertible. In fact, in this case $F(M)^{-1}$ is invertible.

3.3 COROLLARY. *If R is a ring satisfying* (P), *and* $M \in \mathrm{Res}(\mathscr{V}, \mathscr{T})$, *then* $F(M)^{-1}$ *is an invertible fractional ideal of* R.

Proof. $F(M)^{-1} = ((F(M)^{-1})^{-1})^{-1} = G(M)^{-1}$.

3.4 COROLLARY. *If R satisfies* (P) *and* $M \in \mathrm{Res}(\mathscr{U}, \mathscr{T})$, *then the following properties of M are equivalent*:
  (i) $F(M)$ *is divisorial.*
  (ii) $F(M)$ *is invertible.*
  (iii) $M \in \mathrm{Res}_1(\mathscr{U}, \mathscr{T})$.

Proof. The hypotheses imply that $G(M) = \widetilde{F(M)} = (F(M)^{-1})^{-1}$ so (i) $\Leftrightarrow$ (ii) is clear. The implication (ii) $\Rightarrow$ (iii) follows from [**17**, p. 423, Lemma 1], and (iii) $\Rightarrow$ (ii) since (iii) implies $F(M) = G(M)$.

4. Applications. The first result in this section was shown by D. Buchsbaum in the case that R is a unique factorization domain [**4**, Lemma 3.3], and its Corollary 4.2 was proved by H. Kramer for noetherian domains [**15**, Satz 2]. The proof given here of 4.1 is a modification of Kramer's argument.

4.1 THEOREM. *Let R be a ring and let*

$$R^{n+1} \xrightarrow{f} R^n \to M \to 0$$

be exact with M a torsion module. If (a_{ij}) is the matrix representing f with respect to the standard bases of R^{n+1} and R^n, and if Δ_k is the determinant of the matrix obtained by deleting the k-th column from (a_{ij}), then $(-\Delta_1, \Delta_2, \ldots, (-1)^{n+1}\Delta_{n+1})$ generates a free submodule of R^{n+1} and

$$\ker f = F(M)^{-1}(-\Delta_1, \Delta_2, \ldots, (-1)^{n+1}\Delta_{n+1}).$$

Proof. Let $\{e_1, \ldots, e_{n+1}\}$ and $\{e_1', \ldots, e_n'\}$ be the canonical bases for R^{n+1} and R^n respectively. Let

$$h_j = f(e_j) = \sum_{i=1}^{n} a_{ij}e_i'.$$

Then

$$\ker f = \left\{ (\lambda_1, \ldots, \lambda_{n+1}) \in R^{n+1} \mid \sum_{j=1}^{n+1} \lambda_j h_j = 0 \right\},$$

and

$$F(M) = \sum_{i=1}^{n+1} \Delta_i R.$$

Thus the first assertion follows from 2.1(b).

Let j_1 and j_2 be integers satisfying $1 \le j_1 < j_2 \le n + 1$ and let $\wedge R^n$ be the exterior algebra of R^n. Multiplying the relation $\sum_{j=1}^{n+1}\lambda_j h_j = 0$ in $\wedge R^n$ by

$h_1 \wedge \ldots \wedge \hat{h}_{j_1} \wedge \ldots \wedge \hat{h}_{j_2} \ldots \wedge h_{n+1}$ (where $\hat{h}_i$ means delete h_i) we get

$$(-1)^{j_1}\lambda_{j_1}(h_1 \wedge \ldots \wedge \hat{h}_{j_2} \wedge \ldots \wedge h_{n+1})$$
$$+ (-1)^{j_2-1}\lambda_{j_2}(h_1 \wedge \ldots \wedge \hat{h}_{j_1} \wedge \ldots \wedge h_{n+1}) = 0.$$

Writing the h_j in terms of the e_i' and simplifying we get

(*) $(-1)^{j_1}\lambda_{j_1}\Delta_{j_2} + (-1)^{j_2-1}\lambda_{j_2}\Delta_{j_1} = 0.$

Let L be the submodule of R^{n+1} consisting of all solutions to (*). Thus $L = \{(\lambda_1, \ldots, \lambda_{n+1}) \in R^{n+1} | (-1)^{j_1}\lambda_{j_1}\Delta_{j_2} + (-1)^{j_2-1}\lambda_{j_2}\Delta_{j_1} = 0$ for all j_1, j_2 such that $1 \leqq j_1 < j_2 \leqq n + 1\}$. Now we show that $L = \ker f$. Since $\ker f \subset L$ it suffices to show the reverse inclusion. Let $(\lambda_1, \ldots, \lambda_{n+1}) \in L$ and let $h = \sum_{j=1}^{n+1}\lambda_j h_j$. We must show $h = 0$. But

$$h = \sum_{j=1}^{n+1} \lambda_j h_j = \sum_{j=1}^{n+1} \lambda_j \left(\sum_{i=1}^{n} a_{ij}e_i' \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n+1} \lambda_j a_{ij} \right)e_i',$$

and so it suffices to show

$$b_i = \sum_{j=1}^{n+1} \lambda_j a_{ij} = 0 \quad \text{for } i = 1, 2, \ldots n.$$

Since $(\lambda_1, \ldots, \lambda_{n+1}) \in L$ we have $(-1)^j\lambda_j\Delta_k = (-1)^k\lambda_k\Delta_j$ for every pair $j, k \in \{1, 2, \ldots, n+1\}$. Thus

$$b_i\Delta_k = \sum_{j=1}^{n+1} a_{ij}\lambda_j\Delta_k = \sum_{j=1}^{n+1} (-1)^{|j-k|}a_{ij}\Delta_j\lambda_k = \lambda_k \sum_{j=1}^{n+1} (-1)^{|j-k|}a_{ij}\Delta_j.$$

But $\sum_{j=1}^{n+1}(-1)^{|j-k|}a_{ij}\Delta_j$ is, except possibly for sign, the determinant of the matrix

$$\begin{bmatrix} a_{i1} & a_{i2} & \ldots & a_{in+1} \\ a_{11} & a_{12} & \ldots & a_{1n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn+1} \end{bmatrix}$$

and hence is zero. Thus $b_i \in \text{Ann } (F(M))$ for each i, and since $F(M)$ contains a regular element by 2.1(b) we get $b_i = 0$ for each i. Therefore $h = 0$ and $L = \ker f$.

If $x \in F(M)^{-1}$ then $(-x\Delta_1, x\Delta_2, \ldots, (-1)^{n+1}x\Delta_{n+1}) \in L$ since if $1 \leqq j_1 < j_2 \leqq n + 1$, then

$$(-1)^{j_1}((-1)^{j_1}x\Delta_{j_1})\Delta_{j_2} + (-1)^{j_2-1}((-1)^{j_2}x\Delta_{j_2})\Delta_{j_1} = 0.$$

Now choose a regular element $b = \sum_{i=1}^{n+1}t_i\Delta_i$ in $F(M)$. Then for $(\lambda_1, \ldots, \lambda_{n+1}) \in L$ let $u = \sum_{i=1}^{n+1}(-1)^i t_i\lambda_i$. Then

$$u\Delta_j = \sum_{i=1}^{n+1} (-1)^i t_i\lambda_i\Delta_j = \sum_{i=1}^{n+1} (-1)^j t_i\Delta_i\lambda_j = (-1)^j\lambda_j b.$$

Therefore with $x = u/b$ we have

$$(\lambda_1, \ldots, \lambda_{n+1}) = (-x\Delta_1, \ldots, (-1)^{n+1}x\Delta_{n+1}).$$

4.2 THEOREM. *Let $R^{n+1} \to R^n \to M \to 0$ be exact with M a torsion R-module. If R satisfies* (P) *then the following properties of M are equivalent.*
  (i) *$F(M)^{-1}$ is an invertible fractional ideal.*
  (ii) *$M \in \text{Res}_2(\mathscr{U}, \mathscr{T})$.*
  (iii) *$M \in \text{Res}(\mathscr{U}, \mathscr{T})$.*

Proof. (i) $\Rightarrow$ (ii) by 4.1, and (ii) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ (i) by 3.3.

Remark. Property (P) was needed only for the implication (iii) $\Rightarrow$ (i) above.

The noetherian case of the following corollary was shown by MacRae in [**18**] although it was not explicitly stated.

4.3 COROLLARY. *Let R be a ring satisfying property* (P) *and let A be an ideal of R that can be generated by 2 elements. If $R/A \in \text{Res}(\mathscr{U}, \mathscr{T})$ then $R/A \in \text{Res}_2(\mathscr{U}, \mathscr{T})$.*

Theorem 4.1 is related to the following result which is sometimes called Burch's theorem. Theorem 4.4 was first stated in this generality by Kaplansky [**11**, p. 148, Exercise 8], though several special cases have been known previously [**10**, pp. 239-240; **7**, Theorem 5; **24**, Proposition 1; **14**, Lemma 2; **4**, Theorem 3.4]. We have added the last sentence to Kaplansky's statement for completeness. The hypothesis in Theorem 4.4 that I is regular is redundant if R is noetherian [**11**, p. 141, Theorem 196], though not in general [**27**, Example].

4.4 THEOREM. *Let I be a regular ideal of R with a resolution*

$$0 \to R^n \xrightarrow{g} R^{n+1} \xrightarrow{f} I \to 0.$$

Let (b_{ij}) be the matrix of g with respect to the canonical bases of R^n and R^{n+1} and let x_i be the image under f of the i-th basis element of R^{n+1}. If d_i is the determinant of the matrix obtained by deleting the i-th row from (b_{ij}), then $x_i = (-1)^i d_i d$ for $i = 1, 2, \ldots, n + 1$ for some $d \in R$. Further, if some x_i is regular then $dR = G(R/I)$.

Proof. We need only prove the last statement. Assume x_1 is regular and apply the function G to the sequence $0 \to I/x_1R \to R/x_1R \to R/I \to 0$ and get $G(I/x_1R)G(R/I) = x_1R$. It follows that $G(I/x_1R) = F(I/x_1R) = d_1R$, and hence $x_1R = d_1G(R/I) = dd_1R \Rightarrow G(R/I) = dR$.

The following result generalizes 4.4 in the case that R is a domain by replacing the cyclic torsion module R/I with an arbitrary torsion module M. Much more can be said if R is noetherian [**5**, Theorem 1]. The proof of 4.5 is similar to an argument of S. Moen [**20**, Proposition 1.2].

We need the following notation of [**5**]. We use a non-negative integer n to denote the set $\{1, 2, \ldots, n\}$. If $\varphi : R^{n_1} \to R^{n_2}$ is a homomorphism then we identify φ with the matrix of φ with respect to the canonical bases of R^{n_1} and R^{n_2}. If $u \subset n_1$ and $v \subset n_2$, then we write $[\varphi, u, v]$ for the minor of φ with columns u and rows v. Of course we write $[\varphi, u, v]$ only if u and v have the same number of elements. Also, if $w \subset n$ we let $\|w\|$ denote the sum of elements of w.

4.5 THEOREM. *Let R be a domain and let*

$$0 \to R^{n_2} \xrightarrow{\varphi_2} R^{n_1} \xrightarrow{\varphi_1} R^{n_0} \xrightarrow{\varphi_0} M \to 0$$

be exact with M a torsion R-module. Then there exists $d \in R$ such that

$$[\varphi_1, u, n_0] = (-1)^{\|n_1 \backslash u\|} d [\varphi_2, n_2, n_1 \backslash u]$$

for each subset u of n_1 having n_0 elements. Further $dR = G(M)$ and thus $F(M) = G(M)F_{n_0+1}(\ker \varphi_0)$.

Proof. Since M is a torsion module, tensoring with the quotient field $T(R)$ shows that $n_1 = n_0 + n_2$. Let $\{e_j{}^i\}_{j=1}^{n_i}$ be the canonical basis for R^{n_i}, $i = 0, 1, 2$, and let $\varphi_1 = (\alpha_{ij})$, and $\varphi_2 = (\beta_{kl})$. First we show that $[\varphi_1, u, n_0]R = G(M)[\varphi_2, n_2, n_1 \backslash u]$ for each subset u of n_1 having n_0 elements. For simplicity assume $u = \{1, 2, \ldots, n_0\}$, and let $\Delta = [\varphi_1, u, n_0] = [\varphi_1, n_0, n_0]$. If $\Delta \neq 0$ let N be the submodule of $\varphi_1(R^{n_1})$ generated by $\varphi_1(e_1{}^1), \varphi_1(e_2{}^1), \ldots, \varphi_1(e_{n_0}{}^1)$, and let $\bar{\varphi}_1$ be the composite map

$$R^{n_1} \to \varphi_1(R^{n_1}) \to \varphi_1(R^{n_1})/N.$$

Then

$$\ker \bar{\varphi}_1 = \ker \varphi_1 + \sum_{i=1}^{n_0} Re_i{}^1.$$

This sum is direct for if $x + \sum_{i=1}^{n_0} r_i e_i{}^1 = 0$ with $x \in \ker \varphi_1$ and $r_i \in R$, then $\sum_{i=1}^{n_0} r_i \varphi_1(e_i{}^1) = 0$, and $\Delta \neq 0$ implies by Cramer's rule that $r_1 = r_2 = \ldots = r_{n_0} = 0$. Thus $\ker \bar{\varphi}_1$ is free of rank n_1 and it follows that $G(\varphi_1(R^{n_1})/N) = F(\varphi_1(R^{n_1})/N) = [\varphi_2, n_2, n_1 \backslash n_0]R$. Now from the exact sequence of torsion modules

$$0 \to \varphi_1(R^{n_1})/N \to R^{n_0}/N \to R^{n_0}/\varphi_1(R^{n_1}) \to 0,$$

we get $[\varphi_2, n_2, n_1 \backslash n_0]$ $G(M) = G(R^{n_0}/N) = F(R^{n_0}/N) = \Delta R$. If $\Delta = 0$, then there exists $r_1, r_2, \ldots, r_{n_0} \in R$, not all zero, such that $\sum_{j=1}^{n_0} r_j \varphi_1(e_j{}^1) = 0$. Thus

$$\sum_{j=1}^{n_0} r_j e_j{}^1 \in \operatorname{Im} \varphi_2; \quad \sum_{j=0}^{n_0} r_j e_j{}^1 = \sum_{l=1}^{n_2} s_l \varphi(e_l{}^2)$$

$$= \sum_{l=1}^{n_2} s_l \left(\sum_{k=1}^{n_1} \beta_{kl} e_k{}^1 \right) = \sum_{k=1}^{n_1} \left(\sum_{l=1}^{n_2} s_l \beta_{kl} \right) e_k{}^1.$$

Therefore $[\varphi_2, n_2, n_1 \backslash n_0] = 0$.

Now let d generate $G(M)$. We have shown that there exist units $s_u \in R$ such that $[\varphi_1, u, n_0] = s_u d[\varphi_2, n_2, n_1 \backslash u]$ for each subset u of n_1 having n_0 elements. To show that $s_u = (-1)^{||n \backslash u||} s$ for some $s \in R$ which does not depend on u, we may pass to the quotient field $T(R)$ of R, and then the lemma [**20**, p. 5, Lemma] applies.

The following corollary is due to H. Kramer in the case that R is a noetherian domain [**15**, Satz 4].

4.6 COROLLARY. *If R, m is a quasi-local domain and $0 \to R^{n_2} \to R^{n_1} \to R^{n_0} \to M \to 0$ is exact with the n_i minimal, and M is a torsion module, then $F(M) \subset m^{\max(n_0, n_2)}$.*

Proof. This is immediate from 4.5.

Remark. If M in 4.6 is cyclic, then the assumption that R is a domain is not needed; for then 4.4 applies.

We conclude this paper by using the function G to obtain characterizations of rings in terms of the homological properties of certain ideals.

If $M \in \operatorname{Res}(\mathscr{V})$, then M has a finite free resolution $0 \to F_n \to \ldots \to F_0 \to M \to 0$, and the integer

$$\chi(M) = \sum_{i=0}^{n} (-1)^i rk(F_i)$$

is independent of the resolution of M and is called the Euler characteristic of M [**11**, p. 137]. We need a result of Vasconcelos [**27**], which we generalize slightly. For an R-module M let $\alpha_k(M) = \operatorname{Ann}(\wedge^k M)$, the kth invariant factor of M. The proof of the following theorem and corollary are similar to those in [**27**] (of the case $k = 1$), so we omit them. We also note that both of these results hold (same proofs) with the invariant factors replaced by the Fitting ideals, thus furnishing a non-noetherian version of [**11**, p. 146, Theorem].

4.7 THEOREM. *Let $M \in \operatorname{Res}(\mathscr{V})$ with $\chi(M) = m$.*
(a) *If $k \leqq m$ then $\alpha_k(M) = 0$.*
(b) *If $k > m$ then $\operatorname{Ann}(\alpha_k(M)) = 0$.*

4.8 COROLLARY. *If $M \in \operatorname{Res}(\mathscr{U})$, then $\operatorname{Ann}(\alpha_k(M))$ is generated by an idempotent for each $k \geqq 1$.*

Applying 4.7 to R-modules of the form R/aR we get that a ring R is an integral domain if and only if $aR \in \operatorname{Res}(\mathscr{V})$ for each $a \in R$. Similarly, applying 4.8 to such R-modules we get the following. Recall that a ring R is called a p.p. ring if each principal ideal is projective, or equivalently, if $T(R)$ is absolutely flat and R_m is a domain for every maximal ideal m of R [**8**].

4.9 THEOREM. *A ring R is a p.p. ring if and only if $aR \in \operatorname{Res}(\mathscr{V})$ for every $a \in R$.*

Some special cases of 4.9 are found in [**22**, p. 270, Corollary 3], and [**2**, Theorem A].

We switch attention to ideals generated by two elements. Recall that a GCD-domain is an integral domain in which any two elements have a greatest common divisor [**11**, p. 32].

4.10 THEOREM. *The following properties of a ring R are equivalent.*

(i) *R is a GCD-domain;*

(ii) *R is a domain in which the intersection of any two principal ideals is principal;*

(iii) *(x, y) has an FFR of length one for every $x, y \in R$;*

(iv) *$(x, y) \in \text{Res} (\mathscr{V})$ for every $x, y \in R$.*

Proof. (i) $\Rightarrow$ (ii) Let d be the greatest common divisor of x and y. Then $x = dx'$ and $y = dy'$ with $x', y' \in R$ and x' and y' have greatest common divisor 1. We will show that $(x) \cap (y) = (c)$ where $c = dx'y'$. Since $c = dx'y' = xy' = x'y$, then $(c) \subset (x) \cap (y)$. Let $ax = by \in (x) \cap (y)$. Then $ax' = by'$ and thus $b \in (x')$ since 1 is the greatest common divisor of x' and y' [**11**, p. 41, Exercise 7]. Writing $b = x'b'$ with $b' \in R$ we get $ax = by = b'x'y = b'c$, and therefore $(x) \cap (y) \subset (c)$.

(ii) $\Rightarrow$ (iii) Consider the exact sequence $0 \to K \to R^2 \to (x, y) \to 0$. Then $K \cong (x) \cap (y)$ (unless $x = 0 = y$) and the result follows.

(iii) $\Rightarrow$ (iv) This is trivial.

(iv) $\Rightarrow$ (i) This follows from the remarks following 4.8 and the remarks following 3.2.

The above theorem includes the unique factorization of regular local rings as well as the generalization of this fact given by Quentel [**23**]. If we switch from $\mathscr{V}$ to $\mathscr{U}$ we have the following.

4.11 COROLLARY. *If R is a ring with $(x, y) \in \text{Res} (\mathscr{U})$ for every $(x, y) \subset R$, then R is integrally closed.*

Proof. Since $T(R)$ is absolutely flat by 4.9, we may check integral closure of R locally [**9**, p. 112, Proposition 5]. But R is locally a GCD-domain by 4.10, and GCD domains are integrally closed [**11**, p. 33, Theorem 50].

An analogue of (iii) $\Leftrightarrow$ (iv) of 4.10 does not hold for ideals generated by three elements. A theorem originally due to Burch [**6**] and extended in [**13**, Theorem A] says that if A is any finitely generated module over a noetherian ring R, then there is a triply generated ideal I or R such that $d(R/I) = d(A)$.

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