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A PARABOLIC SINGULAR INTEGRAL OPERATOR WITH ROUGH KERNEL

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Abstract

Let $\Omega(y')$ be an $H^1(S^{n-1})$ function on the unit sphere satisfying a certain cancellation condition. We study the L^p boundedness of the singular integral operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\Omega(y')\rho(y)^{-\alpha}\, dy,$$

where $\alpha \geq n$ and ρ is a norm function which is homogeneous with respect to certain nonistropic dilation. The result in the paper substantially improves and extends some known results.

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1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the routine norm $|x|$ for each $x \in \mathbb{R}^n$. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere on $\mathbb{R}^n$ equipped with the Lebesgue measure $\sigma(x')$, where we use x' to denote the unit vector in the direction of x. To study the existence and regularity results for an elliptic differential operator, that is,

$$D = \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j},$$

with constant coefficients $\{a_{i,j}\}$, among some other estimates, one needs to study the singular integral operator T with a convolution kernel K (see [1] or [2]) satisfying:

(a) $K(\mu x_1, \ldots, \mu x_n) = \mu^{-n} K(x)$, for any $\mu > 0$;
(b) $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
(c) $\int_{S^{n-1}} K(x')\, d\sigma(x') = 0$.

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Similarly, for the heat operator

$$D = \frac{\partial}{\partial x_1} - \sum_{j=2}^{n} \frac{\partial^2}{\partial x_j^2},$$

the corresponding singular integral operator T have a kernel K satisfying:

(a′) $K(\mu^2 x_1, \ldots, \mu x_n) = \mu^{-n-1} K(x)$, for any $\mu > 0$;
(b′) $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
(c′) $\int_{S^{n-1}} K(x')(2{x'_1}^2 + {x'_2}^2 + \cdots + {x'_n}^2)\, d\sigma(x') = 0.$

To study a more general parabolic differential operator with constant coefficients, in 1966, Fabes and Rivière [6] defined the parabolic singular integral operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(y) f(x-y)\, dy,$$

with K satisfying:

(A) $K(\mu^{\alpha_1} x_1, \ldots, \mu^{\alpha_n} x_n) = \mu^{-\alpha} K(x_1, \ldots, x_n)$, $\mu > 0$, $\alpha = \sum_{i=1}^{n} \alpha_i$;
(B) $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
(C) $\int_{S^{n-1}} K(x') J(x')\, d\sigma(x') = 0$;

where $\alpha_i \geq 1$ $(i = 1, 2, \ldots, n)$ and $J(x') = \alpha_1 x_1'^2 + \cdots + \alpha_n x_n'^2$ is shown as follows. For any $x \in \mathbb{R}^n$, set

$$\begin{aligned}
x_1 &= \rho^{\alpha_1} \cos\varphi_1 \cdots \cos\varphi_{n-2} \cos\varphi_{n-1} \\
x_2 &= \rho^{\alpha_2} \cos\varphi_1 \cdots \cos\varphi_{n-2} \sin\varphi_{n-1} \\
&\vdots \\
x_{n-1} &= \rho^{\alpha_{n-1}} \cos\varphi_1 \sin\varphi_2 \\
x_n &= \rho^{\alpha_n} \sin\varphi_1.
\end{aligned}$$

Then $dx = \rho^{\alpha-1} J(x')\, d\rho\, d\sigma(x')$ and $\rho^{\alpha-1} J(x')$ is the Jacobian of the above transformation. One may check that $J(x')$ is a C^∞ function on S^{n-1}, that is, bounded below uniformly by 1. Moreover, without loss of generality, we may assume that $\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq 1$. Note that the above condition (A) can be written as:

(A′) $K(A_\mu x) = |\det(A_\mu)|^{-1} K(x)$;

where $A_\mu = \text{diag}[\mu^{\alpha_1}, \ldots, \mu^{\alpha_n}]$ is a diagonal matrix.

For each fixed $x \in \mathbb{R}^n$, the function

$$F(x, \rho) = \sum_{i=1}^{n} \frac{{x_i}^2}{\rho^{2\alpha_i}},$$

is a strictly decreasing function of $\rho > 0$. Therefore, there exists an unique $\rho = \rho(x)$ for which $F(x, \rho) = 1$. It was proved in [6] that ρ is a metric on $\mathbb{R}^n$. Furthermore, we

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observe that $(\mathbb{R}^n, \rho)$ is a homogeneous group that admits a family of dilations $\delta_\mu = \exp(A \log \mu)$ for which $\rho(\delta_\mu x) = \mu\rho(x)$, $\mu > 0$, where A is a diagonalizable linear operator with positive eigenvalues.

By the above coordinates of polar type, one now has that

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\Omega(y')\rho(y)^{-\alpha}\, dy,$$

where $\Omega(y') = K(y')$ satisfying

$$\int_{S^{n-1}} \Omega(y')J(y')\, d\sigma(y') = 0. \tag{1.1}$$

The following theorem was proved by Fabes and Riviere in [6].

THEOREM A. *If $\Omega \in C^1(S^{n-1})$ and satisfies (1.1), then the operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

Later, the above theorem was improved by Nagel *et al.* [9] and the regularity condition on Ω was removed. The result is the following.

THEOREM B. *If $\Omega \in L \log^+ L(S^{n-1})$ and satisfies (1.1), then the operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

On the other hand, it was shown in [10] that in the case $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, the condition $\Omega \in L \log^+ L(S^{n-1})$ can be replaced further by the weaker condition $\Omega \in H^1(S^{n-1})$, where $H^1(S^{n-1})$ is the Hardy space that contains $L \log^+ L(S^{n-1})$ as a proper subspace on the unit sphere. Thus, a natural question is if one can use a weaker condition in Theorem B. The main purpose of this paper is to establish the following theorem.

THEOREM 1. *If $\Omega \in H^1(S^{n-1})$ and satisfies (1.1), then the operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

Before proving the theorem, we want to say a few words. First, we understand that the underlying space is a special homogeneous group (see [8]). Thus, many standard results might be adapted. For instance, we can use the Littlewood–Paley theory without any modifications. Second, although we follow the ideas in [7], we find that it is not an easy process of copy and paste. We must obtain some nontrivial estimates in our proof.

The reader can find these new estimates in Section 3. In Section 2 we present some basic definitions and known lemmas. The letter C in the paper denotes positive constant independent of essential variables.

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2. Definitions and lemmas

The Poisson kernel on S^{n-1} is defined by $P_{\mathrm{t}y'}(x') = (1-\mathrm{t}^2)/|\mathrm{t}y' - x'|^n$ with $0 \le \mathrm{t} < 1$, $x', y' \in S^{n-1}$. For any $\Omega \in L^1(S^{n-1})$, we define the radial maximal function

$$P^+\Omega(x') = \sup_{0\le \mathrm{t}<1} \left| \int_{S^{n-1}} \Omega(y') P_{\mathrm{t}x'}(y')\, d\sigma(y') \right|.$$

The Hardy space $H^1(S^{n-1})$, is a subspace of $L^1(S^{n-1})$ which contains all L^1 functions Ω with the finite norms $\|\Omega\|_{H^1(S^{n-1})} = \|P^+\Omega\|_{L^1(S^{n-1})} < \infty$.

An important property of $H^1(S^{n-1})$ is the atomic decomposition, which is reviewed in the following. An exceptional atom $E(x')$ is an $L^\infty(S^{n-1})$ function bounded by 1. A regular $H^1(S^{n-1})$ atom is an $L^\infty(S^{n-1})$ function $a(x')$ satisfying the following conditions:

$$\operatorname{supp}(a) \subset S^{n-1} \cap \{y \in \mathbb{R}^n : |y - \xi'| < r \text{ for some } \xi' \in S^{n-1} \text{ and } r \in (0, 1]\}; \tag{2.1}$$

$$\int_{S^{n-1}} a(x') Y_m(x')\, d\sigma(x') = 0 \tag{2.2}$$

for any spherical harmonic polynomial Y_m with degree $m \le N$, where N is any fixed integer;

$$\|a\|_{L^\infty(S^{n-1})} \le r^{1-n}. \tag{2.3}$$

From [3], we find that any $\Omega \in H^1(S^{n-1})$ has an atomic decomposition

$$\Omega = \sum_{j=1}^{\infty} \lambda_j a_j + \sum_{i=1}^{\infty} u_i E_i,$$

where each a_j is a regular $H^1(S^{n-1})$ atom and each E_i is an exceptional atom. Moreover,

$$\sum_{j=1}^{\infty} |\lambda_j| + \sum_{i=1}^{\infty} |u_i| \le C\|\Omega\|_{H^1(S^{n-1})}.$$

We note that for any $x' \in S^{n-1}$,

$$\left| \sum_{i=1}^{\infty} u_i E_i(x') \right| \le \sum_{i=1}^{\infty} |u_i|.$$

Without loss of generality, we can assume

$$\left| \sum_{i=1}^{\infty} u_i E_i(x') \right| \le \|\Omega\|_{H^1(S^{n-1})}.$$

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Thus, we write

$$\sum_{i=1}^{\infty} u_i E_i(x') = \|\Omega\|_{H^1(S^{n-1})}\omega(x'),$$

with

$$\omega(x') = \sum_{i=1}^{\infty} u_i E_i(x')/\|\Omega\|_{H^1(S^{n-1})}.$$

In this new definition,

$$\Omega = \sum_{j=1}^{\infty} \lambda_j a_j + \|\Omega\|_{H^1(S^{n-1})}\omega \quad \text{and} \quad \|\omega\|_{L^\infty(S^{n-1})} \le 1.$$

Noting that $J(x/|x|)|x|^2$ is a homogeneous polynomial of degree 2, by [11, Theorem 2.1], we can write $J(x/|x|)|x|^2 = P_2(x) + |x|^2 P_0(x)$, where $P_k(x)$ is a harmonic polynomial of degree k $(k = 0, 2)$. Then $J(x') = P_2(x') + P_0(x')$, where $P_k(x')$ is a spherical harmonic polynomial of degree k $(k = 0, 2)$. So by (2.2), we have

$$\int_{S^{n-1}} a_j(y')J(y')\,d\sigma(y') = \int_{S^{n-1}} a_j(y')P_2(y')\,d\sigma(y') + \int_{S^{n-1}} a_j(y')P_0(y')\,d\sigma(y') = 0, \tag{2.4}$$

for all $j = 1, 2, \ldots$. Thus, if Ω satisfies the cancellation condition (1.1), then

$$\int_{S^{n-1}} \omega(y')J(y')\,d\sigma(y') = 0. \tag{2.5}$$

The following Lemmas 2.1 and 2.2 can be found in [7].

LEMMA 2.1. *Suppose that $n \ge 3$ and b satisfies (2.1), (2.3) and*

$$\int_{S^{n-1}} b(y')\,d\sigma(y') = 0. \tag{2.6}$$

Let

$$F_b(s) = (1 - s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}} b(s, (1 - s^2)^{1/2}\tilde{y})\,d\sigma(\tilde{y}),$$

and

$$G_b(s) = (1 - s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}} |b(s, (1 - s^2)^{1/2}\tilde{y})|\,d\sigma(\tilde{y}).$$

Then there exists a constant C, independent of b, such that

$$\operatorname{supp}(F_b) \subset (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')); \tag{2.7}$$

$$\operatorname{supp}(G_b) \subset (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')); \tag{2.8}$$

$$\|F_b\|_\infty \le C/r(\xi'); \qquad \|G_b\|_\infty \le C/r(\xi'); \tag{2.9}$$

$$\int_{\mathbb{R}} F_b(s)\,ds = 0, \tag{2.10}$$

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where $r(\xi') = |\xi|^{-1}|L_r\xi|$ *and* $L_r\xi = (r^2\xi_1, r\xi_2, \ldots, r\xi_n)$.

LEMMA 2.2. *Suppose that* $n = 2$ *and* b *satisfies (2.1), (2.3) and (2.6). Let*

$$F_b(s) = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)(b(s, (1-s^2)^{1/2}) + b(s, -(1-s^2)^{1/2})),$$

and

$$G_b(s) = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)(|b(s, (1-s^2)^{1/2})| + |b(s, -(1-s^2)^{1/2})|).$$

Then $F_b(s)$ *satisfies (2.7), (2.10) and*

$$\|F_b\|_q \le C|L_r(\xi')|^{-1+1/q},$$

and $G_b(s)$ *satisfies (2.8) and*

$$\|G_b\|_q \le C|L_r(\xi')|^{-1+1/q},$$

for some $q \in (1, 2)$.

LEMMA 2.3. *Fix any function* $\psi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\operatorname{supp}(\psi) \subset \{x : 1/2 \le \rho(x) \le 2\}$. *Let* $\widehat{\Psi}(\xi) = \psi(\rho(\xi))$, $\Psi_t(\xi) = t^{-\alpha}\Psi(A_{t^{-1}}\xi)$ *for* $t > 0$. *For* $j \in \mathbb{Z}$, *define the multiplier* S_j *by* $\widehat{S_j f}(\xi) = \psi(2^j\rho(\xi))\widehat{f}(\xi)$. *Then for* $1 < p < \infty$, *we have*

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \le C\|f\|_p.$$

where C *is a constant independent of* $f \in L^p(\mathbb{R}^n)$.

Lemma 2.3 is a discrete version of a more general theorem in [8]. One can prove Lemma 2.3 easily following the idea in [8].

Next, we let $\Phi(t) = e^{-\pi t^2}$ $(t \in \mathbb{R})$. Then $\widehat{\Phi}(t) = e^{-\pi t^2}$. Define a radial function Φ_k on $\mathbb{R}^n$ by $\widehat{\Phi_k}(\xi) = \widehat{\Phi}(|L_r A_{2^k}\xi|)$. We have the following lemma.

LEMMA 2.4. *The maximal operator* $f \to \sup_k |\Phi_k * f|$ *is bounded on* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$.

PROOF. It is easy to check that

$$\Phi_k(x) = r^{-2}2^{-k\alpha_1}\Phi(x_1 r^{-2}2^{-k\alpha_1}) \prod_{j=2}^{n}\{r^{-1}2^{-k\alpha_j}\Phi(x_j r^{-1}2^{-k\alpha_j})\}.$$

Thus

$$\sup_k |\Phi_k * f(x)| \le CM_1M_2\cdots M_n(f)(x),$$

where M_j is the one-dimensional Hardy–Littlewood maximal operator acting on x_j variable. So the lemma follows easily by the L^p boundedness of the Hardy–Littlewood maximal function. □

The following lemma is a variation of a lemma in [5].

LEMMA 2.5. *Let $r > 0$. Suppose that there exist positive numbers γ and β such that*

$$|\widehat{\sigma_k}(\xi)| \le C \min\{|L_r A_{2^k}\xi|^{-\beta}, |L_r A_{2^k}\xi|^{\gamma}\}, \tag{2.11}$$

where $L_r A_{2^k}\xi = (r^2 2^{k\alpha_1}\xi_1, r2^{k\alpha_2}\xi_2, \ldots, r2^{k\alpha_n}\xi_n)$. Moreover, suppose also that, for some $q > 1$,

$$\|\sigma^*(f)\|_q = \left\| \sup_k ||\sigma_k| * f| \right\|_q \le C\|f\|_q, \tag{2.12}$$

where $C > 0$ is independent of $k \in \mathbb{Z}$, ξ and r. Then the following two operators are bounded on $L^p(\mathbb{R}^n)$ uniformly for $r > 0$, whenever $|1/p - 1/2| < 1/2q$:

$$B(f) = \sum_k \sigma_k * f, \quad g(f) = \left(\sum_k |\sigma_k * f|^2 \right)^{1/2}.$$

PROOF. If (2.12) holds and $1/2q = |1/2 - 1/p_0|$, then by [5], for arbitrary vector $\{h_k\}$ in $L^{p_0}(l^2)$, the following vector valued inequality holds

$$\left\| \left(\sum_k |\sigma_k * h_k|^2 \right)^{1/2} \right\|_{p_0} \le C \left\| \left(\sum_k |h_k|^2 \right)^{1/2} \right\|_{p_0}. \tag{2.13}$$

Choose a C_0^∞ function ψ such that $0 \le \psi \le 1$ and $\mathrm{supp}(\psi) \subset \{y : 1/2 \le \rho(y) \le 2\}$ and $\sum_j \psi(2^j \rho(L_r\xi))^2 = 1$. Define Υ and Δ by $\widehat{\Upsilon}(\xi) = \psi(\rho(L_r\xi))$ and $\widehat{\Delta}(\xi) = \psi(\rho(\xi))$. Denote $\Upsilon_j(x) = 2^{-j\alpha}\Upsilon(A_{2^{-j}}x)$ and $\Delta_j(x) = 2^{-j\alpha}\Delta(A_{2^{-j}}x)$ for $j \in \mathbb{Z}$. Then it is easy to check $\widehat{\Upsilon_j}(\xi) = \psi(2^j\rho(L_r\xi))$ and $\widehat{\Delta_j}(\xi) = \psi(2^j\rho(\xi))$ and $\Upsilon_j(x) = (1/r^{n+1})2^{-j\alpha}\Delta(L_{1/r}A_{2^{-j}}x)$. Define the multiplier S_j on $\mathbb{R}^n$ by $\widehat{(S_j f)}(\xi) = \psi(2^j\rho(L_r\xi))\widehat{f}(\xi)$. Then we know $S_j f(x) = \Upsilon_j * f(x)$. Now we claim that

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \le C\|f\|_p \quad \text{for all } 1 < p < \infty, \tag{2.14}$$

with C independent of r. In fact, by the definition of Υ_j, we have

$$\begin{aligned} \Upsilon_j * f(x) &= \frac{1}{r^{n+1}} 2^{-j\alpha} \int_{\mathbb{R}^n} \Delta(L_{1/r}A_{2^{-j}}y) f(x-y)\, dy \\ &= 2^{-j\alpha} \int_{\mathbb{R}^n} \Delta(A_{2^{-j}}y) f(L_r(L_{1/r}x - y))\, dy \\ &= \Delta_j * U(L_{1/r}x), \end{aligned}$$

where $U(x) = f(L_r x)$. Then, by Lemma 2.3, we obtain

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$$
\begin{aligned}
\left\| \left(\sum_j |S_j f(x)|^2 \right)^{1/2} \right\|_p &= \left\{ \int_{\mathbb{R}^n} \left(\sum_j |\Delta_j * U(L_{1/r}x)|^2 \right)^{p/2} dx \right\}^{1/p} \\
&= \left\{ r^{n+1} \int_{\mathbb{R}^n} \left(\sum_j |\Delta_j * U(x)|^2 \right)^{p/2} dx \right\}^{1/p} \\
&\le C r^{(n+1)/p} \|U\|_p \\
&= C \left(r^{n+1} \int_{\mathbb{R}^n} |f(L_r x)|^p \, dx \right)^{1/p} = C \|f\|_p.
\end{aligned}
$$

Thus, (2.14) is proved. By the definition of S_j, we have, for any $f \in \mathcal{S}(\mathbb{R}^n)$, $\sum_j S_j^2 f(x) = \sum_j S_j(S_j f)(x) = f(x)$. We decompose operator B as follows:

$$
B(f) = \sum_k \sigma_k * \left(\sum_j S_{j+k} S_{j+k} f \right) = \sum_j \sum_k (S_{j+k}(\sigma_k * S_{j+k} f)) = \sum_j \mathcal{B}_j(f).
$$

First we estimate $\mathcal{B}_j$ in L^{p_0}. By (2.13) and (2.14), we have

$$
\begin{aligned}
\|\mathcal{B}_j(f)\|_{p_0} &\le C \left\| \left(\sum_k |\sigma_k * S_{j+k} f|^2 \right)^{1/2} \right\|_{p_0} \\
&\le C \left\| \left(\sum_k |S_{j+k} f|^2 \right)^{1/2} \right\|_{p_0} \\
&\le C \|f\|_{p_0}. \qquad (2.15)
\end{aligned}
$$

Now we compute the L^2-norm of $\mathcal{B}_j(f)$. When $j < 0$, by using the estimate $|\widehat{\sigma_k}(\xi)| \le C|L_r A_{2^k} \xi|^{-\beta}$ we have

$$
\begin{aligned}
\|\mathcal{B}_j(f)\|_2^2 &\le \sum_k \int_{2^{-j-k-1} \le \rho(L_r \xi) \le 2^{-j-k+1}} |\widehat{f}(\xi)|^2 |L_r A_{2^k} \xi|^{-2\beta} \, d\xi \\
&= \frac{1}{r^{n+1}} \sum_k \int_{2^{-j-k-1}}^{2^{-j-k+1}} \int_{S^{n-1}} J(\xi') |\widehat{f}(L_{r^{-1}} A_\rho \xi')|^2 |A_{2^k} A_\rho \xi'|^{-2\beta} \rho^{\alpha-1} \, d\sigma(\xi') \, d\rho \\
&\le C 2^{2\beta j \alpha_1} \frac{1}{r^{n+1}} \sum_k \int_{2^{-j-k-1}}^{2^{-j-k+1}} \int_{S^{n-1}} J(\xi') |\widehat{f}(L_{r^{-1}} A_\rho \xi')|^2 \rho^{\alpha-1} \, d\sigma(\xi') \, d\rho \\
&\le C 2^{2\beta j} \frac{1}{r^{n+1}} \int_{\mathbb{R}^n} |\widehat{f}(L_{r^{-1}} \xi)|^2 \, d\xi \\
&= C 2^{2\beta j} \|f\|_2^2.
\end{aligned}
$$

So we obtain

$$
\|\mathcal{B}_j(f)\|_2 \le C 2^{\beta j} \|f\|_2 \quad \text{for all } j < 0. \qquad (2.16)
$$

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If $j > 0$, using the estimate $|\widehat{\sigma_k}(\xi)| \le |L_r A_{2^k}\xi|^{\gamma}$ and the same idea of proving (2.16), we have

$$\|\mathcal{B}_j(f)\|_2 \le C2^{-\gamma j}\|f\|_2 \quad \text{for all } j > 0, \tag{2.17}$$

where C is independent of j and f. Thus, by (2.16) and (2.17) we obtain

$$\|\mathcal{B}_j(f)\|_2 \le C2^{-\delta|j|}\|f\|_2 \quad \text{for all } \delta > 0. \tag{2.18}$$

Now, if $|1/p - 1/2| < 1/2q$, we have $1/p = \theta/2 + (1-\theta)/p_0$ for some $0 < \theta < 1$. By interpolating between (2.15) and (2.18), we obtain

$$\|B(f)\|_p \le \sum_j \|\mathcal{B}_j(f)\|_p \le C\sum_j 2^{-\delta\kappa|j|}\|f\|_p \le C\|f\|_p \quad \text{for all } 0 < \kappa < 1.$$

The inequality $\|g(f)\|_p \le C\|f\|_p$ can be proved by essentially the same argument. □

LEMMA 2.6 (See [5]). *Let $\{\eta_k\}$ be a lacunary sequence of positive numbers* ($\inf_k(\eta_{k+1}/\eta_k) = \eta > 1$). *Suppose that $\{\lambda_k\}$ is a sequence of nonnegative functions satisfying, for some $\theta > 0$,*

$$|\widehat{\lambda_k}(\xi) - 1| \le C|\eta_{k+1}\xi|^{\theta}, \quad |\widehat{\lambda_k}(\xi)| \le C|\eta_k\xi|^{-\theta},$$

for all $k \in \mathbb{Z}$. Then, the maximal operator $(\lambda^* f)(x) = \sup_k |\lambda_k * f(x)|$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$.

LEMMA 2.7 (See [4]). *Suppose that m denotes the distinct numbers of $\{\alpha_j\}$. Then for any $x, y \in \mathbb{R}^n$, $0 \le \beta \le 1$*

$$\left|\int_1^2 e^{-iA_\lambda x\cdot y}\frac{d\lambda}{\lambda}\right| \le C|x\cdot y|^{-(\beta/m)},$$

where $C > 0$ is independent of x and y.

3. Proof of Theorem 1

Noting that

$$Tf(x) = \int_0^\infty \int_{S^{n-1}} \Omega(y')J(y')f(x - A_\rho y')\, d\sigma(y')\frac{d\rho}{\rho}.$$

Since $\Omega \in H^1(S^{n-1})$ satisfying the cancellation condition (1.1), we can write

$$\Omega = \sum_{j=1}^{\infty} \lambda_j a_j + \|\Omega\|_{H^1(S^{n-1})}\omega,$$

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where ω satisfies (2.5) and $\|\omega\|_{L^\infty(S^{n-1})} \le 1$, each a_j is a regular $H^1(S^{n-1})$ atom and $\sum_{j=1}^{\infty} |\lambda_j| \le C\|\Omega\|_{H^1(S^{n-1})}$. So

$$\|Tf\|_p \le C \sum_{j=1}^{\infty} |\lambda_j| \|B_j(f)\|_p + \|\Omega\|_{H^1(S^{n-1})} \|T_\omega f\|_p, \tag{3.1}$$

where

$$B_j(f)(x) = \int_0^\infty \int_{S^{n-1}} a_j(y')J(y')f(x - A_\rho y')\, d\sigma(y') \frac{d\rho}{\rho},$$

and

$$T_\omega(f)(x) = \int_{\mathbb{R}^n} f(x-y)\omega(y')\rho(y)^{-\alpha}\, dy.$$

Since $\omega(y') \in L^\infty(S^{n-1}) \subset L\log^+ L(S^{n-1})$ and satisfies (2.5), by Theorem B, we obtain

$$\|T_\omega f\|_p \le C\|f\|_p, \tag{3.2}$$

where C is independent of ω and f. Therefore, to prove Theorem 1, it suffices by (3.1) and (3.2) to show that

$$\|B_j(f)\|_p \le C\|f\|_p, \quad j = 1, 2, \ldots, \tag{3.3}$$

where C is independent of the atoms a_j and f. By (2.4) and by the observation that $J(y') \in C_0^\infty(S^{n-1})$, it is easy to check that $\widetilde{a}_j(y') = a_j(y')J(y')/\|J\|_{L^\infty(S^{n-1})}$ satisfies (2.1), (2.3) and (2.6). Thus,

$$B_j(f)(x) = \|J\|_{L^\infty(S^{n-1})} \int_0^\infty \int_{S^{n-1}} \widetilde{a}_j(y')f(x - A_\rho y')\, d\sigma(y') \frac{d\rho}{\rho}.$$

For simplicity in our argument, we denote $\widetilde{a}_j$ by a and $B_j(f)/\|J\|_{L^\infty(S^{n-1})}$ by $B(f)$ from now. Without loss of generality, we may assume that $\mathrm{supp}(a) \subset B(\iota, r) \cap S^{n-1}$, where $\iota = (1, 0, \ldots, 0)$ and $B(\iota, r) = \{y : |y - \iota| < r\}$. Let $I_k = (2^k, 2^{k+1})$. Then $B(f)(x)$ is equal to

$$\int_0^\infty \int_{S^{n-1}} \rho^{-1} a(y') \sum_k \chi_{I_k}(\rho) f(x - A_\rho y')\, d\sigma(y')\, d\rho = \sum_k \sigma_k * f(x),$$

where

$$\widehat{\sigma_k}(\xi) = \int_{I_k} \int_{S^{n-1}} a(y') e^{-2\pi i A_\rho y' \cdot \xi}\, d\sigma(y') \frac{d\rho}{\rho}. \tag{3.4}$$

Let

$$\widehat{\mu_k}(\xi) = \int_{I_k} \int_{S^{n-1}} |a(y')| e^{-2\pi i A_\rho y' \cdot \xi}\, d\sigma(y') \frac{d\rho}{\rho}, \tag{3.5}$$

and

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$$\sigma^*(f)(x) = \sup_k |\mu_k * f(x)|. \tag{3.6}$$

Then it is easy to verify that $\|\widehat{\mu_k}\|_\infty < C$, $\|\sigma_k\|_1 < C$ uniformly for $k \in \mathbb{Z}$ and for all $k \in \mathbb{Z}$. Since $a(y')$ satisfies (2.6), then

$$\widehat{\sigma_k}(0) = \int_{2^k}^{2^{k+1}} \frac{d\rho}{\rho} \int_{S^{n-1}} a(y')\, d\sigma(y') = 0.$$

By Lemma 2.5, if we can show that $\{\sigma_k\}$ satisfies condition (2.11) and

$$\|\sigma^*(f)\|_p \le C\|f\|_p \quad \text{for all } 1 < p < \infty, \tag{3.7}$$

where C is independent of a and f. Thus (3.3) is obtained. We first verify that $\{\sigma_k\}$ satisfies (2.11). We only prove the case $n > 2$, since the proof for $n = 2$ is essentially the same (using Lemma 2.2 instead of Lemma 2.1). For any $\xi \ne 0$, we choose a rotation $\mathcal{O}$ such that $\mathcal{O}(A_\rho \xi) = |A_\rho \xi|\iota = |A_\rho \xi|(1, 0, \ldots, 0)$. Let $y' = (s, y_2', y_3', \ldots, y_n')$. Then it is easy to see that

$$\widehat{\sigma_k}(\xi) = \int_{I_k} \int_{S^{n-1}} a(\mathcal{O}^{-1}(y'))e^{-2\pi i|A_\rho\xi|(\iota\cdot y')}\, d\sigma(y') \frac{d\rho}{\rho},$$

where $\mathcal{O}^{-1}$ is the inverse of $\mathcal{O}$. Now $a(\mathcal{O}^{-1}(y'))$ also satisfies (2.3), (2.6) and is supported in $B(\zeta, r) \cap S^{n-1}$, where $\zeta = A_\rho \xi/|A_\rho \xi|$. Thus,

$$\widehat{\sigma_k}(\xi) = \int_{I_k} \int_{\mathbb{R}} F_a(s)e^{-2\pi i|A_\rho\xi|s}\, ds \frac{d\rho}{\rho},$$

where $F_a(s)$ is the function defined in Lemma 2.1. By Lemma 2.1, we know $\mathrm{supp}(F_a) \subset (-2r(\zeta) + \zeta_1, 2r(\zeta) + \zeta_1)$, where $r(\zeta) = |L_r A_\rho \xi|/|A_\rho \xi|$ and $\zeta_1 = \rho^{\alpha_1}\xi_1/|A_\rho \xi|$. Thus, $N(s) = r(\zeta)F_a(r(\zeta)s)$ is a function supported in the interval $(-2 + \zeta_1/r(\zeta), 2 + \zeta_1/r(\zeta))$, and $\|N\|_\infty < C$ (C is independent of s and ρ), $\int_{\mathbb{R}} N(s)\, ds = 0$. After changing variables

$$\widehat{\sigma_k}(\xi) = \int_{I_k} \int_{\mathbb{R}} N(s)e^{-2\pi i s|L_r A_\rho\xi|}\, ds \frac{d\rho}{\rho}.$$

So by the cancellation property of $N(\cdot)$, we obtain that

$$\begin{aligned}
|\widehat{\sigma_k}(\xi)| &\le \int_{I_k} \left| \int_{\mathbb{R}} N(s)\, (e^{-2\pi i|L_r A_\rho \xi|s} - e^{-2\pi i \rho^{\alpha_1}\xi_1}) ds \right| \frac{d\rho}{\rho} \\
&\le C \int_{I_k} \int_{|s - \frac{\zeta_1}{r(\zeta)}| \le 2} |N(s)||L_r A_\rho \xi| \left| s - \frac{\zeta_1}{r(\zeta)} \right| ds \frac{d\rho}{\rho} \\
&\le C \int_1^2 |L_r A_{2^k\rho}\xi| \frac{d\rho}{\rho} \\
&\le C|L_r A_{2^k}\xi|.
\end{aligned} \tag{3.8}$$

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On the other hand, by (3.4) and Hölder's inequality,

$$|\widehat{\sigma_k(\xi)}|^2 \le C H_k(\xi), \tag{3.9}$$

where

$$H_k(\xi) = \int_{I_k} \left| \int_{S^{n-1}} a(y') e^{-2\pi i A_\rho y' \cdot \xi} \, d\sigma(y') \right|^2 \frac{d\rho}{\rho}.$$

Then we obtain

$$\begin{aligned} H_k(\xi) &= \int_{2^k}^{2^{k+1}} \iint_{S^{n-1} \times S^{n-1}} a(y') \overline{a(x')} e^{-2\pi i A_\rho (y'-x') \cdot \xi} \, d\sigma(y') \, d\sigma(x') \frac{d\rho}{\rho} \\ &\le C \iint_{S^{n-1} \times S^{n-1}} |a(y')| |a(x')| \left| \int_{2^k}^{2^{k+1}} e^{-2\pi i A_\rho (y'-x') \cdot \xi} \frac{d\rho}{\rho} \right| d\sigma(y') \, d\sigma(x'). \end{aligned}$$

By Lemma 2.7, we know

$$\begin{aligned} \left| \int_{2^k}^{2^{k+1}} e^{-2\pi i A_\rho (y'-x') \cdot \xi} \frac{d\rho}{\rho} \right| &= \left| \int_1^2 e^{-2\pi i A_{2^k \rho} (y'-x') \cdot \xi} \frac{d\rho}{\rho} \right| \\ &\le C(|(y'-x') \cdot A_{2^k} \xi|)^{-2\beta/m}. \end{aligned}$$

where $0 \le \beta < 1/2$ and m denotes the distinct numbers of $\{\alpha_j\}$. Then by the above inequality we obtain

$$H_k(\xi) \le C \iint_{S^{n-1} \times S^{n-1}} |a(y')| |a(x')| (|(y'-x') \cdot A_{2^k} \xi|)^{-2\beta/m} \, d\sigma(y') \, d\sigma(x'). \tag{3.9$'$}$$

Denote

$$I_1(\xi) = \iint_{S^{n-1} \times S^{n-1}} |a(y')| |a(x')| (|(y'-x') \cdot A_{2^k} \xi|)^{-2\beta/m} \, d\sigma(y') \, d\sigma(x').$$

For any $\xi \ne 0$, we choose a rotation $\mathcal{O}$ such that

$$\mathcal{O}(A_{2^k} \xi) = |A_{2^k} \xi| \iota = |A_{2^k} \xi| (1, 0, \ldots, 0).$$

Let

$$y' = (s, y_2', y_3', \ldots, y_n') \quad \text{and} \quad x' = (t, x_2', x_3', \ldots, x_n').$$

Then it is easy to see that

$$\begin{aligned} I_1(\xi) = \iint_{S^{n-1} \times S^{n-1}} & |a(\mathcal{O}^{-1}(y'))| |a(\mathcal{O}^{-1}(x'))| \\ & \times (|(y'-x') \cdot |A_{2^k} \xi| \iota|)^{-2\beta/m} \, d\sigma(y') \, d\sigma(x'), \end{aligned}$$

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where $\mathcal{O}^{-1}$ is the inverse of $\mathcal{O}$. Now $a(\mathcal{O}^{-1}(y'))$ also satisfies (2.3), (2.6) and is supported in $B(\vartheta, r) \cap S^{n-1}$, where $\vartheta = A_{2^k}\xi/|A_{2^k}\xi|$. Thus,

$$I_1(\xi) = \iint_{\mathbb{R}\times\mathbb{R}} G_a(s)G_a(t)(|A_{2^k}\xi||s-t|)^{-2\beta/m}\,ds\,dt,$$

where $G_a(s)$ is the function defined in Lemma 2.1. By Lemma 2.1, we know $\mathrm{supp}(G_a) \subset (-2r(\vartheta)+\vartheta_1, 2r(\vartheta)+\vartheta_1)$, where $r(\vartheta) = |L_r A_{2^k}\xi|/|A_{2^k}\xi|$ and $\vartheta_1 = 2^{k^{\alpha_1}}\xi_1/|A_{2^k}\xi|$. Thus,

$$\varphi(s) = r(\vartheta)G_a\biggl(r(\vartheta)\biggl(s - \frac{\vartheta_1}{r(\vartheta)}\biggr)\biggr)$$

is a function supported in the interval $(-2, 2)$, and $\|\varphi\|_\infty < C$, where C is independent of r, ϑ and k. Since $2\beta/m < 1$, we obtain

$$\begin{aligned} I_1(\xi) &= \int_{-2}^{2}\int_{-2}^{2} \varphi(s)\varphi(t)(|L_r A_{2^k}\xi||s-t|)^{-2\beta/m}\,ds\,dt \\ &\le C|L_r A_{2^k}\xi|^{-2\beta/m}\int_{-2}^{2}\int_{-2}^{2}|s-t|^{-2\beta/m}\,ds\,dt \\ &\le C|L_r A_{2^k}\xi|^{-2\beta/m}. \end{aligned}$$

This together with (3.9) and (3.9′) gives

$$|\widehat{\sigma_k}(\xi)| \le C|L_r A_{2^k}\xi|^{-\beta/m}. \tag{3.10}$$

By (3.8) and (3.10),

$$|\widehat{\sigma_k}(\xi)| \le C\min\{|L_r A_{2^k}\xi|, |L_r A_{2^k}\xi|^{-\beta/m}\}. \tag{3.11}$$

Inequality (3.11) shows that $\{\sigma_k\}$ satisfies (2.11). Hence, it remains to show (3.7). We define the measure sequences $\{\lambda_k\}$ on $\mathbb{R}$ by

$$\widehat{\lambda_k}(\xi_1) = \|a\|_{L^1(S^{n-1})}\int_{I_k} e^{-2\pi i\rho^{\alpha_1}\xi_1}\frac{d\rho}{\rho}. \tag{3.12}$$

Let δ be the Dirac delta function acting on $(x_2, \ldots, x_n)$. Now, choose the function Φ_k as in Lemma 2.4 and for each k define

$$\nu_k = \mu_k - \Phi_k * (\lambda_k \otimes \delta). \tag{3.13}$$

By (3.6) and (3.13),

$$\sigma^*(f) \le \biggl(\sum_k |\nu_k * f|^2\biggr)^{1/2} + \sup_k \Phi_k * \Bigl(\sup_k |(\lambda_k\otimes\delta) * f|\Bigr), \tag{3.14}$$

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and

$$||\nu_k| * f| \le \sigma^*(f) + \sup_k \Phi_k * \Big(\sup_k |(\lambda_k \otimes \delta) * f| \Big). \tag{3.15}$$

By (3.14), if we can prove

$$\Big\| \sup_k \Phi_k * \Big(\sup_k |(\lambda_k \otimes \delta) * f| \Big) \Big\|_p \le C \|f\|_p \quad \text{for all } 1 < p < \infty, \tag{3.16}$$

and

$$\Big\| \Big(\sum_k |\nu_k * f|^2 \Big)^{1/2} \Big\|_p \le C \|f\|_p \quad \text{for all } 1 < p < \infty, \tag{3.17}$$

where C is independent of f, then we obtain (3.7). We first verify (3.16). By the definition of λ_k, it is easy to see that, for each k, $|\widehat{\lambda_k}(\xi_1)| \le C$, and λ_k is a positive measure on $\mathbb{R}$:

$$\begin{aligned} |\widehat{\lambda_k}(\xi_1) - \widehat{\lambda_k}(0)| &\le \|a\|_{L^1(S^{n-1})} \int_{I_k} |e^{-2\pi i \rho^{\alpha_1} \xi_1} - 1| \frac{d\rho}{\rho} \\ &\le C 2^{k\alpha_1} |\xi_1|. \end{aligned}$$

Using integration by parts,

$$|\widehat{\lambda_k}(\xi_1)| \le C (2^{k\alpha_1} |\xi_1|)^{-1}.$$

By Lemma 2.6, we know that $\sup_k |\lambda_k * f|$ is bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$. Since δ is the Dirac delta function acting on $(x_2, \ldots, x_n)$, we see $\sup_k |(\lambda_k \otimes \delta) * f|$ is bounded on $L^p(\mathbb{R}^n)$. So by Lemma 2.4, we obtain (3.16). Hence, it remains to show (3.17). We still use Lemma 2.5 to do this. We first show that $\{\nu_k\}$ satisfies (2.11). By (3.13),

$$|\widehat{\nu_k}(\xi)| \le |\widehat{\mu_k}(\xi) - \widehat{\lambda_k}(\xi_1)| + |\widehat{\lambda_k}(\xi_1)| |\widehat{\Phi_k}(\xi) - 1|, \tag{3.18}$$

and

$$|\widehat{\nu_k}(\xi)| \le |\widehat{\mu_k}(\xi)| + |\widehat{\lambda_k}(\xi_1)| |\widehat{\Phi_k}(\xi)|. \tag{3.19}$$

By (3.5) and applying the method of rotation again

$$\widehat{\mu_k}(\xi) = C \int_{I_k} \int_{\mathbb{R}} G_a(s) e^{-2\pi i |A_\rho \xi| s} \, ds \frac{d\rho}{\rho}.$$

Note that $\operatorname{supp}(G_a) = \operatorname{supp}(F_a) = (\zeta_1 - 2r(\zeta), \zeta_1 + 2r(\zeta))$ by (2.7) and (2.8), and also note $\int_{\mathbb{R}} G_a(s) \, ds = \|a\|_{L^1(S^{n-1})}$, then by Lemma 2.1 we have

$$\begin{aligned} |\widehat{\mu_k}(\xi) - \widehat{\lambda_k}(\xi_1)| &\le \int_{I_k} \int_{\mathbb{R}} |G_a(s)| |e^{-2\pi i |A_\rho \xi| s} - e^{-2\pi i \rho^{\alpha_1} \xi_1}| \, ds \frac{d\rho}{\rho} \\ &\le C \int_{I_k} \frac{1}{r(\zeta)} \int_{|s - \zeta_1| \le 2r(\zeta)} |A_\rho \xi| |s - \zeta_1| \, ds \frac{d\rho}{\rho} \\ &\le C |L_r A_{2^k} \xi|. \end{aligned} \tag{3.20}$$

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From the proof of (3.10), we also have

$$|\widehat{\mu_k}(\xi)| \le C|L_r A_{2^k}\xi|^{-\beta/m}. \tag{3.21}$$

On the other hand,

$$\widehat{\Phi_k}(\xi) = \widehat{\Phi}(|L_r A_{2^k}\xi|) = e^{-\pi|L_r A_{2^k}\xi|^2} = \int_{\mathbb{R}} e^{-\pi t^2} e^{-2\pi i t|L_r A_{2^k}\xi|}\,dt.$$

Thus,

$$\begin{aligned} |\widehat{\Phi_k}(\xi) - 1| &= \left| \int_{\mathbb{R}} e^{-\pi t^2}(e^{-2\pi i t|L_r A_{2^k}\xi|} - 1)\,dt \right| \\ &\le C|L_r A_{2^k}\xi| \int_0^\infty e^{-\pi t^2} t\,dt \le C|L_r A_{2^k}\xi|. \end{aligned} \tag{3.22}$$

Using integration by parts,

$$|\widehat{\Phi_k}(\xi)| \le \frac{C}{|L_r A_{2^k}\xi|} \int_0^\infty e^{-\pi t^2} t\,dt \le C|L_r A_{2^k}\xi|^{-1}. \tag{3.23}$$

By (3.18), (3.20), (3.22) and $|\widehat{\lambda_k}(\xi_1)| \le C$, we have

$$|\widehat{\nu_k}(\xi)| \le C|L_r A_{2^k}\xi|. \tag{3.24}$$

On the other hand, by (3.19), (3.21), (3.23) and $|\widehat{\lambda_k}(\xi_1)| \le C$, it is easy to see that

$$|\widehat{\nu_k}(\xi)| \le C|L_r A_{2^k}\xi|^{-\eta}, \tag{3.25}$$

where $|L_r A_{2^k}\xi|^{-\eta} = \max\{|L_r A_{2^k}\xi|^{-1}, |L_r A_{2^k}\xi|^{-\beta/m}\}$ and $\eta = \beta/m$ or 1. Thus, (3.24) and (3.25) yield

$$|\widehat{\nu_k}(\xi)| \le C \min\{|L_r A_{2^k}\xi|, |L_r A_{2^k}\xi|^{-\eta}\}. \tag{3.26}$$

So $\{\nu_k\}$ satisfies (2.11). By (3.26) and the same idea of proving (2.18) in Lemma 2.5, we obtain

$$\left\| \left(\sum_k |\nu_k * f|^2 \right)^{1/2} \right\|_2 \le C\|f\|_2,$$

where C is independent of f. By (3.14) and (3.16), we see $\|\sigma^*(f)\|_2 \le C\|f\|_2$. Therefore, by (3.15), (3.16), we obtain $\|\sup_k ||\nu_k| * f|\|_2 \le C\|f\|_2$. Applying Lemma 2.5 with $q = 2$ again, we obtain

$$\left\| \left(\sum_k |\nu_k * f|^2 \right)^{1/2} \right\|_p \le C\|f\|_p \quad \text{for all } 4/3 < p < 4. \tag{3.27}$$

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and

$$\|\sigma^*(f)\|_p \le C\|f\|_p \quad \text{for all } 4/3 < p < 4. \tag{3.28}$$

Thus, a bootstrap argument by reiterating application of Lemma 2.5 gives us

$$\left\| \left(\sum_k |\nu_k * f|^2 \right)^{1/2} \right\|_p \le C\|f\|_p,$$

for all $1 < p < \infty$. Thus, (3.17) is proved and Theorem 1 follows.

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