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QUANTITIES RELATED TO UPPER AND LOWER SEMI-FREDHOLM TYPE LINEAR RELATIONS

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Certain norm related functions of linear operators are considered in the very general setting of linear relations in normed spaces. These are shown to be closely related to the theory of strictly singular, strictly cosingular, F_+ and F_- linear relations. Applications to perturbation theory follow.

1. INTRODUCTION

Several authors ([**11, 15, 19, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33**]) have introduced operational quantities in order to obtain characterisations and perturbation results for various classes of operators of Fredholm theory. We remark that all the above authors considered only the case of bounded linear operators in Banach spaces. It is the purpose of this paper to consider these quantities in the more general setting of linear relations between normed spaces.

In Section 2 we define some quantities associated with an arbitrary quantity f, and in particular, with the measures of nonprecompactness ρ and $\mathcal{K}$ and the measure of non strict singularity $\mathcal{SS}$. These quantities are related to certain quantities generated by the norm and these relations will be applied to obtain characterisations and perturbation results for F_+ and strictly singular linear relations. Cross [**8**] has proved analogous results derived from the injection modulus j; these results are included in this section for completeness.

In Section 3 we analyze the F_- and strictly cosingular linear relations in a similar way. First, we consider quantities derived from an arbitrary quantity f, and in particular, from the measures ρ and $\mathcal{K}$ and the measure of non strict cosingularity $\mathcal{SC}$, and exhibit several equalities used to deduce characterisations and perturbation results for F_- and strictly cosingular linear relations.

In Section 4 we study the surjection modulus q for linear relations between normed spaces. Various quantities generated by q are defined and use to obtain properties of F_- and strictly cosingular linear relations.

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NOTATIONS. We recall some basic definitions from the theory of linear relations in normed spaces. Let X and Y denote normed spaces. A linear relation or multivalued linear operator ([**1, 20**]) T in $X \times Y$ is a mapping from a subspace $D(T) \subset X$, called the domain of T, into $P(Y) \setminus \{\emptyset\}$ such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all scalars α, $\beta \in \mathbf{K}$ and x_1, $x_2 \in D(T)$. The class of such relations T is denoted by $LR(X,Y)$. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply linear operator.

The graph $G(T)$ of $T \in LR(X,Y)$ is $G(T) := \{(x,y) \in X \times Y : x \in D(T), y \in Tx\}$. T is closed if its graph is a closed subspace. The closure $\overline{T}$ of T is defined by $G(\overline{T}) := \overline{G(T)}$. Let M be a subspace of $D(T)$. Then the restriction $T|_M$ is defined by $G(T|_M) := \{(m,y) : m \in M,\ y \in Tm\}$. For any subspace M of X such that $M \cap D(T) \neq \emptyset$, we write $T|_M = T|_{M \cap D(T)}$. The inverse of T is the linear relation T^{-1} defined by $G(T^{-1}) := \{(y,x) \in Y \times X : (x,y) \in G(T)\}$. If T^{-1} is single valued, then T is called injective, that is, T is injective if and only if its null space $N(T) := T^{-1}(0) = \{0\}$, and T is said to be surjective if its range $R(T) := T(D(T)) = Y$.

We shall adopt the following notation: If M and N are subspaces of X and X' respectively, then $M^{\perp} := \{x' \in X' : x'(x) = 0 \text{ for all } x \in M\}$, $N^{\top} := \{x \in X : x'(x) = 0 \text{ for all } x' \in N\}$. The adjoint or conjugate T' of T is defined by $G(T') := G(-T^{-1})^{\perp} \subset Y' \times X'$ where $\langle (y,x),(y',x') \rangle := \langle x,x' \rangle + \langle y,y' \rangle$. This means that $(y',x') \in G(T')$ if and only if $y'(y) - x'(x) = 0$ for all $(x,y) \in G(T)$.

For a given closed subspace E of X let Q_E^X (or simply, Q_E) denote the natural quotient map from X onto X/E. We shall denote $Q^Y_{\overline{T(0)}}$ by Q_T, or simply by Q when T is understood. Clearly QT is single valued. For $x \in D(T)$, $\|Tx\| := \|QTx\|$ and the norm of T is defined by $\|T\| := \|QT\|$. We note that this quantity is not a true norm since $\|T\| = 0$ does not imply $T = 0$.

$T \in LR(X,Y)$ is said to be continuous if for each neighbourhood V in $R(T)$, $T^{-1}(V)$ is a neighbourhood in $D(T)$ and T is called open if its inverse is continuous. For linear relations, it can be shown that T is continuous if and only if $\|T\| < \infty$ (see [**8**, II. 3.2]). This definition of a norm agrees with the standard definition of the norm of a single valued linear operator. It also agrees with the norm defined for convex processes (see Aubin and Frankowska [**3**]). We note that the notion for continuity adopted here refers to the property of lower semi-continuity in set-valued analysis.

The minimum modulus of $T \in LR(X,Y)$, $\gamma(T)$ is defined by $\gamma(T) := \sup \{\lambda \geqslant 0 : \|Tx\| \geqslant \lambda d(x, N(T)) \text{ for } x \in D(T)\}$ and T is open if and only if $\gamma(T) > 0$ ([**8**, II. 3.2]).

We denote the set $\{x \in X : \|x\| \leqslant 1\}$ by B_X and we shall write J_X for the injection of X into its completion $\widetilde{X}$. The families of infinite dimensional, closed infinite codimensional, finite dimensional, finite codimensional and closed finite codimensional subspaces of X are denoted by $\mathcal{I}(X)$, $\mathcal{E}(X)$, $\mathcal{F}(X)$, $\mathcal{C}(X)$ and $\mathcal{P}(X)$ respectively.

A linear relation $T \in LR(X,Y)$ is said to be partially continuous if there exists $M \in \mathcal{C}(X)$ such that the restriction $T|_M$ is continuous, precompact if $QTB_{D(T)}$ is totally bounded, strictly singular if there is no $M \in \mathcal{I}(D(T))$ such that $T|_M$ is injective and open, upper semi-Fredholm if there exists a finite codimensional subspace M of X for which $T|_M$ is injective and open, lower semi-Fredholm if T' is upper semi-Fredholm, strictly cosingular if $\sup\{\Gamma'(Q_M T) : M \in \mathcal{E}(Y)\} = 0$, where $\Gamma'(T) := \inf\{\|Q_M J_Y T\| : M \in \mathcal{E}(\widetilde{Y})\}$, (see [**8**]).

The class of partially continuous, precompact, strictly singular, strictly cosingular, upper semi-Fredholm, lower semi-Fredholm linear relations in $LR(X,Y)$ will be denoted by $PB(X,Y)$, $K(X,Y)$, $SS(X,Y)$, $SC(X,Y)$, $F_+(X,Y)$ and $F_-(X,Y)$ respectively.

Continuous everywhere defined linear operators referred to as bounded linear operators.

In the sequel X and Y will be denote normed spaces and T will always denote an element of $LR(X,Y)$ except where stated otherwise. A quantity will be a procedure which determines for any pair X, Y of normed spaces, a map from $LR(X,Y)$ into $[0,+\infty]$. Let $\mathcal{A}$ be a subclass of the class of all linear relations between normed spaces. Then, a quantity f is called a measure of non $\mathcal{A}$ if every $T \in LR(X,Y)$, $T \in \mathcal{A}$ if and only if $f(T) = 0$.

Linear relations were introduced into Functional Analysis by J. von Neumann [**21**], motivated by the need to consider adjoints of non-densely defined linear differential operators which are considered by Coddington [**5**], Coddington and Dijksma [**6**], Dikjsma, Sabbah and De Snoo [**10**], among others.

Other recent works on multivalued mappings include the treatise on partial differential relations by Gromov [**16**] and the application of multivalued methods to solving differential equations by Favini and Yagi [**12**].

Problems in optimisation and control also led to the study of set-valued maps and differential inclusions (see for example, Aubin and Cellina [**2**], Clarke [**4**], among others). Studies of convex processes, tangent cones, subgradients and epiderivatives et cetera, form part of the theory of convex analysis developed to deal with non-smooth problems in viability and control theory, for example. Some of the basic topological properties from this area coincide with the core of the topological results for multivalued linear operators.

2. F_+ AND STRICTLY SINGULAR LINEAR RELATIONS

Some quantities derived from certain measures of non-precompactness and non-strict-singularity will now be introduced.

DEFINITION 1: Let $T \in LR(X,Y)$. The Hausdorff measure of non-precompactness of T is $\rho(T) := \inf\{\lambda \geqslant 0\colon$ There exists a finite set $\{y_1,\dots,y_n\}$ in Y such that $TB_{D(T)} \subset \bigcup_{i=1}^{i=n} y_i + \lambda B_Y + \overline{T(0)}\}$. The measure of non-precompactness $\mathcal{K}$ of T is defined by $\mathcal{K}(T) := \inf\{\|T - S\| : S \in K(X,Y), D(T) \subset D(S), S(0) \subset \overline{T(0)}\}$. Analogously, we define the measure of non-strict-singularity $\mathcal{SS}$ of T by $\mathcal{SS}(T) := \inf\{\|T - S\| : S \in SS(X,Y), D(T) \subset D(S), S(0) \subset \overline{T(0)}\}$.

For bounded linear operators in Banach spaces, ρ appears in [**15**], $\mathcal{K}$ in [**18**] and $\mathcal{SS}$ in [**23**].

Noting that if E is a closed subspace of X then $Q_E B_X = B_{R(Q_E)}$, it follows immediately from the definition that $\rho(T)$ coincides with the Hausdorff measure of non-precompactness of QT. Moreover, since if $\|T\| < \infty$, then $\|T\| = \inf\{\lambda > 0 : TB_{D(T)} \subset \lambda B_{R(T)} + T(0)\}$ [**8**, II. 1.10], we obtain that $f \leqslant \|\ \|$ for $f \in \{\rho, \mathcal{K}, \mathcal{SS}\}$.

DEFINITION 2: Let f be any quantity. Then the quantities Γ_f and $\Delta\rho$ are defined by $\Gamma_f(T) := \inf\{f(T|_M) : M \in \mathcal{I}(D(T))\}, \Delta_f(T) := \sup\{\Gamma_f(T|_M) : M \in \mathcal{I}(D(T))\}$, with the convention that both quantities are zero when $D(T)$ is finite dimensional.

For bounded linear operators in Banach spaces, the quantities Γ_ρ and Δ_ρ were introduced (independently) by Rakočević [**24**] and Tylli [**27**]; and $\Gamma_\mathcal{K}$ and $\Delta_\mathcal{K}$ by Lebow and Schechter [**18**] (with a different notation).

In the particular case when $f(T)$ is the norm function $n(T) := \|T\|$, we obtain the quantities $\Gamma(T) := \Gamma_n(T)$ and $\Delta(T) := \Delta_n(T)$ which were introduced for bounded linear operators by Gramsch [**13**] and Schechter [**25**] respectively, and generalised to unbounded linear operators and linear relations in [**7**] and [**8**], where they are used to characterise F_+, unbounded strictly singular, and other classes of unbounded linear operators and linear relations.

We shall also utilise the quantity $\overline{(\Gamma_0)}(T) := \inf\{\|T|_M\| : M \in \mathcal{P}(X)\}$, (see [**8**]).

PROPOSITION 3. *Let $T, S \in LR(X,Y)$ with $S(0) \subset \overline{T(0)}$. Then:*

(i) $\overline{(\Gamma_0)}(T) \leqslant \overline{(\Gamma_0)}(T+S) + \overline{(\Gamma_0)}(S)$.

(ii) $\Delta(T) \leqslant \Delta(T+S) + \Delta(S)$.

(iii) $\rho(T) \leqslant \rho(T+S) + \rho(S)$.

PROOF:

(i) Since $S(0) \subset \overline{T(0)}$ it follows that $\overline{T(0)} = \overline{(T+S)(0)}$ and hence $\|Tx\| = \|Tx + Sx - Sx\| \leqslant \|Tx + Sx\| + \|Sx\|$, $x \in D(T+S)$. Then, for $M \in \mathcal{P}(X)$ we have $\|T|_M\| \leqslant \|T|_M + S|_M\| + \|S|_M\|$ and taking the infimum over $M \in \mathcal{P}(X)$ we obtain $\overline{(\Gamma_0)}(T) \leqslant \overline{(\Gamma_0)}(T+S) + \overline{(\Gamma_0)}(S)$.

(ii) The proof is similar to the preceding inequality.

(iii) Combining [**8**, IV. 5.2] with the fact in this setting $\overline{T(0)} = \overline{(T+S)(0)}$ gives $Q_T = Q^{Y/\overline{S(0)}}_{\overline{T(0)}/\overline{S(0)}} Q_S = Q_{T+S}$. Now, from [**7**] we have $\rho(T) = \rho(T+S-S) = \rho(Q_{T+S}(T+S) - Q_{T+S}S) \leqslant \rho(T+S) + \rho(S)$ as required. ∎

PROPOSITION 4. *We have:*

(i) $\overline{(\Gamma_0)}(T) \leqslant \mathcal{K}(T) \leqslant \|T\|$.

(ii) $\Delta(T) \leqslant \mathcal{SS}(T) \leqslant \|T\|$.

PROOF:

(i) Let $S \in K(X,Y)$ such that $D(T) \subset D(S)$ and $S(0) \subset \overline{T(0)}$. Then $\overline{(\Gamma_0)}(S) = 0$ [**8**, V. 2.2] and so by Proposition 3, $\overline{(\Gamma_0)}(T) \leqslant \overline{(\Gamma_0)}(T+S) \leqslant \|T+S\|$. Hence $\overline{(\Gamma_0)}(T) \leqslant \mathcal{K}(T) \leqslant \|T\|$.

(ii) Let $S \in SS(X,Y)$ with $D(T) \subset D(S)$ and $S(0) \subset \overline{T(0)}$. Then, since $\Delta(S) = 0$ [**8**, V. 2.6] from above Proposition we deduce that $\Delta(T) \leqslant \Delta(T+S) \leqslant \|T+S\|$. Therefore $\Delta(T) \leqslant \mathcal{SS}(T) \leqslant \|T\|$. ∎

COROLLARY 5. *The quantities ρ and $\mathcal{K}$ are measures of non-precompactness and $\mathcal{SS}$ is a measure of non-strict-singularity.*

PROOF: Combine [**7**, 5.2], [**8**, V. 2.2 and V. 2.6] with Proposition 4. ∎

PROPOSITION 6. *Let $T \in LR(X,Y)$. Then:*

(i) $\Gamma_\rho(T) \leqslant \Gamma(T) \leqslant 2\Gamma_\rho(T)$.

(ii) $\Delta_\rho(T) \leqslant \Delta(T) \leqslant 2\Delta_\rho(T)$.

(iii) $\Gamma(T) = \Gamma_{\mathcal{K}}(T) = \Gamma_{\mathcal{SS}}(T)$.

(iii) $\Delta(T) = \Delta_{\mathcal{K}}(T) = \Delta_{\mathcal{SS}}(T)$.

PROOF: Let f be either of the quantities $\overline{(\Gamma_0)}$ or Δ. Then by [**8**, IV. 4.15]

$$\Gamma(T) = \inf\Big\{f(T\,|_M) : M \in \mathcal{I}(D(T))\Big\} \tag{6.1}$$

Now from $\rho \leqslant \|\ \|$, we have $\Gamma_\rho(T) \leqslant \Gamma(T)$. Therefore substituting $f = \overline{(\Gamma_0)}$ in (6.1) and observing that $\overline{(\Gamma_0)}(QT) \leqslant 2\rho(QT)$ [**7**, 5.1], we obtain (i) and hence (ii).

Observing that $\overline{(\Gamma_0)} \leqslant \mathcal{K} \leqslant \|\ \|$ (Proposition 4), and again substituting $f = \overline{(\Gamma_0)}$ in (6.1), we have $\Gamma(T) = \Gamma_{\mathcal{K}}(T)$. From (6.1) with $f = \Delta$ and Proposition 4 it follows that $\Gamma(T) = \Gamma_{\mathcal{SS}}(T)$, proving (iii) and hence (iv). ∎

Parts (i) and (ii) of Proposition 6 are generalisations from Rakočević [**24**].

From previous Proposition we can formulate extended versions of some earlier results.

Theorem 7. *Let $T \in LR(X,Y)$ and let $f \in \{n, \rho, \mathcal{K}, \mathcal{SS}\}$. Then:*

(i) *T is strictly singular if and only if $\Delta_f(T) = 0$.*

(ii) *If $D(T)$ is infinite dimensional, then $T \in F_+$ if and only if $\Gamma_f(T) > 0$.*

(iii) *If $S \in LR(X,Y)$ satisfies $S(0) \subset \overline{T(0)}$ and $\Delta_f(S) < \Gamma_f(T)$, then $T + S \in F_+$.*

Proof: Properties (i) and (ii) follow from above Proposition combined with [**8**, V. 2.6 and V. 2.4].

(ii) If $f = n, \mathcal{K}$ or $\mathcal{SS}$, then the result is an immediate consequence of the Proposition 6. On the other hand, if $f = \rho$, suppose that $T + S \notin F_+$. Then $\dim D(T+S) = \infty$ and according to [**8**, V. 2.4], $\Gamma(T+S) = 0$. Hence given $\varepsilon > 0$ there exists $M \in \mathcal{I}\big(D(T+S)\big)$ such that $\rho(T|_M + S|_M) \leqslant \|T|_M + S|_M\| < \varepsilon$. Choose $\varepsilon > 0$ such that $\varepsilon < \Gamma_\rho(T) - \Delta_\rho(S)$. From Proposition 3 it follows that for every infinite dimensional subspace N of M, $\rho(T|_N) - \rho(S|_N) \leqslant \rho(T|_N + S|_N) < \varepsilon$ whence $\rho(S|_N) > \rho(T|_N) - \varepsilon$. Therefore $\Delta_\rho(S) \geqslant \Gamma_\rho(T) - \varepsilon$ and so $\Gamma_\rho(T) - \Delta_\rho(S) \leqslant \varepsilon$, a contradiction.

Part (iii) of Theorem 7 generalises the result for bounded upper semi-Fredholm linear operators in Banach spaces of Tylli [**27**]. ∎

We recall some quantities based on the injection modulus j. (See [**8**].)

Definition 8: Let $T \in LR(X,Y)$. Define

$$j(T) := \sup\{\lambda \geqslant 0 : \|Tx\| \geqslant \lambda\|x\| \ \text{ for all } \ x \in D(T)\},$$

$$\tau(T) := \sup\Big\{j(T|_M) : M \in \mathcal{I}\big(D(T)\big)\Big\},$$

$$\tau_0(T) := \sup\{j(T|_M) : M \in \mathcal{C}(X)\},$$

(again with the convention $\tau(T) = \tau_0(T) = 0$ if $\dim D(T) < \infty$).

The injection modulus has been considered in [**17, 22, 27, 29, 32**] in the context of bounded linear operators in Banach sspaces.

It is clear that $\Gamma_j(T) = \Gamma(T)$ whenever $\dim D(T) = \infty$ and $\Delta_j(T) = \tau(T)$.

Since T is strictly singular if and only if $\Delta(T) = 0$ if and only if $(\tau T) = 0$ [**8**, V. 2.6] the quantities τ and Δ_j are measures of non-strict-singularity.

It is shown in [**8**, V. 3.2 and V. 2.4] that if $\tau(S) < \tau_0(T)$ with $T, S \in LR(X,Y), S(0) \subset \overline{T(0)}$, then $T + S \in F_+$, and that if $\dim D(T) = \infty$, then $T \in F_+$ if and only if $\tau_0(T) > 0$.

The class of strict singular linear relations S with $\dim S(0) < \infty$ coincides with the perturbation class of F_+, that is, the class of linear relations $P(F_+)$ such that if $T \in F_+$ and $S \in P(F_+)$, then $T + S \in F_+$ [**8**, V. 7.10]. We remark that the statement of [**8**, V. 7.10] is know to be false in the context of bounded linear operators in Banach spaces, (for details see [**8**, p. 175]).

3. F_- AND STRICT COSINGULAR LINEAR RELATIONS

In this section we study the F_--relations, paralleling our investigation of F_+-relations of the previous section. To this end, some new quantities will be introduced.

First we recall the following:

DEFINITION 9: If Y is infinite dimensional, define

$$\Gamma'(T) := \inf\{\|Q_M J_Y T\| : M \in \mathcal{E}(\widetilde{Y})\},$$
$$\Delta'(T) := \sup\{\Gamma'(Q_M T) : M \in \mathcal{E}(Y)\},$$
$$\Gamma'_0(T) := \inf\{\|Q_M T\| : M \in \mathcal{F}(Y)\}.$$

If Y is finite dimensional then all the quantities are defined to be zero.

For bounded linear operators in Banach spaces, Weis [**28**] define the quantities Γ' and Δ' which also appear in [**31**]. (Here, Γ' is denoted by K because of its relation with the Kolmogorov numbers). In this context, Γ'_0 was introduced by Fajnshtejn [**11**], and also by Zemánek [**31**]. These quantities are generalised to arbitrary linear operators and linear relations in normed spaces in [**9**] and [**8**], where they serve to characterise F_- and strictly cosingular linear operators and linear relations.

These quantities are generalised as follows:

DEFINITION 10: Let f be a quantity. If $\dim Y = \infty$, we define

$$\Gamma'_f(T) := \inf\{f(Q_M J_Y T) : M \in \mathcal{E}(\widetilde{Y})\},$$
$$\Delta'_f(T) := \sup\{\Gamma'_f(Q_M T) : M \in \mathcal{E}(Y)\},$$
$$\Gamma'_{0f}(T) := \inf\{f(Q_M T) : M \in \mathcal{F}(Y)\}.$$

Again, the quantities are defined to be zero if Y is finite dimensional.

The measure of non-strict-cosingularity $\mathcal{SC}$ will now be introduced.

DEFINITION 11: For $T \in LR(X,Y)$, we define $\mathcal{SC}(T) := \inf\{\|T - S\| : S \in SC(X,Y), D(T) \subset D(S), S(0) \subset \overline{T(0)}\}$.

The authors believe that the quantities $\Gamma'_{\mathcal{K}}$, $\Delta'_{\mathcal{K}}, \Gamma'_{\mathcal{SC}}$ and $\Delta'_{\mathcal{SC}}$ are new, even in the case of bounded linear operators in Banach spaces. However, Γ'_ρ and Δ'_ρ have been considered in the latter context by Tylli [**27**] (with a different notation).

We prove a generalisation of Fajnshtejn [**11**] (see Martinón [**19**, 23.1]).

PROPOSITION 12: Let T be single valued. Then $\Gamma'_0(T) \leqslant \rho(T)$ with the equality if T is continuous.

PROOF: Suppose that $\rho(T) < \alpha$. Then there exists a finite set $\{y_1, \dots, y_n\}$ in Y which is an α-net in $TB_{D(T)}$. Let $x \in B_{D(T)}$. Then there exists y_j $(1 \leqslant j \leqslant n)$ such that $\|Tx - y_j\| < \alpha$.

Let F denote the linear span of the set $\{y_1, \dots, y_n\}$. We have $d(Tx, F) = d(Tx - y_j, F) < \alpha$. Hence $\|Q_F Tx\| < \alpha$, whence $\Gamma_0'(T) \leqslant \rho(T)$.

Now assume that T is continuous. Let $\Gamma_0'(T) < \alpha$. Then there exists a finite dimensional subspace F of Y for which $\|Q_F T\| < \alpha$. Let $\varepsilon > 0$ be arbitrary. Since $\|T\| < \infty$, the set $(\|T\| + d)B_F$ is totally bounded and hence contains an ε-net $\{y_1, \dots, y_n\}$. We shall verify that it is an $(\alpha + \varepsilon)$-net for $TB_{D(T)}$. Indeed, for each $x \in B_{D(T)}$ we have $d(Tx, F) = \|Q_F Tx\| < \alpha$. Hence there exists $y \in F$ such that $d(Tx, F) < \alpha$, and consequently we have $\|y\| \leqslant d(Tx, y) + \|Tx\| < \alpha + \|T\|$. Hence $y \in \big(\|T\| + \alpha\big)B_F$. Now choose y_j $(1 \leqslant j \leqslant n)$ so that $\|y - y_j\| < \varepsilon$. Then $\|Tx - y_j\| \leqslant \|Tx - y\| + \|y - y_j\| \leqslant \alpha + \varepsilon$ as required. Hence $\rho(T) \leqslant \alpha + \varepsilon$. It follows that $\rho(T) \leqslant \Gamma_0'(T)$. ∎

In general the inequality in above Proposition is strict; it suffices to consider an operator T for which $\Gamma_0'(T) < \infty$ but not continuous, for example a discontinuous finite rank operator.

Proposition 13. *We have:*

- (i) $\Gamma'(T) \leqslant \Delta'(T) \leqslant \Gamma_0'(T) \leqslant \|T\|$.
- (ii) $\Gamma'(T) \leqslant SC(T) \leqslant \|T\|$.
- (iii) $\Gamma'(T) \leqslant \rho(T) \leqslant \|T\|$.

Proof:

- (i) See [**8**, IV. 5.9].
- (ii) Let $S \in SC(X, Y)$ such that $D(T) \subset D(S)$ and $S(0) \subset \overline{T(0)}$. Then $\Delta'(T) = \Delta'(T + S - S) \leqslant \Delta'\big(J_Y(T + S)\big) \leqslant \|T + S\|$ [**8**, IV. 5.2 and V. 5.18]. Hence (ii) holds.
- (iii) Clearly T is discontinuous if and only if $\rho(T) = \infty$. Accordingly we can assume that $\|T\| < \infty$.

Let us consider the various cases for $T(0)$:

(a) $\overline{T(0)}$ infinite codimensional. Then by [**8**, IV. 5.4], part (i) and Proposition 12, we have $\Gamma'(T) \leqslant \Gamma'(QT) \leqslant \Gamma_0'(QT) \leqslant \rho(QT) \leqslant \|T\|$.

(b) $\overline{T(0)}$ finite codimensional. In that case QT is a continuous linear operator with finite dimensional range and applying [**8**, V. 1.3] we deduce that T is precompact, that is, $\rho(T) = 0$ and also T is strict cosingular [**8**, V. 5.19], that is, $\Delta'(T) = 0$. Consequently $\Gamma'(T) = 0$.

(c) $\overline{T(0)}$ finite dimensional. This is covered by (a) if we assume without loss of generality that $\dim Y = \infty$.

Corollary 14. *The quantity SC is a measure of non-strict-cosingularity.*

Lemma 15. *Let f be any quantity satisfying $\Gamma' \leqslant f \leqslant \|\ \|$. Then $\Gamma' = \Gamma_f'$.*

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PROOF: We have for $M \in \mathcal{E}(\widetilde{Y})$, $f(Q_M J_Y T) \leqslant \|Q_M J_Y T\|$ and we have $\Gamma'_f(T) \leqslant \Gamma'(T)$. But $\Gamma'(T) = \Gamma'(J_Y T)$ ([**8**, IV. 5. 12]) $\leqslant \Gamma'(Q_M J_Y T)$ ([**8**, IV. 5.4]). Thus $\Gamma'(T) \leqslant f(Q_M J_Y T)$ and hence $\Gamma'(T) \leqslant \Gamma'_f(T)$.

For the case $f = \Gamma'_0$, this Lemma was obtained by [**4**, IV. 5.4]. ∎

PROPOSITION 16. *We have:*

(i) $\Gamma'(T) = \Gamma'_\rho(T) = \Gamma'_\mathcal{K}(T) = \Gamma'_{SC}(T)$.
(ii) $\Delta'(T) = \Delta'_\rho(T) = \Delta'_\mathcal{K}(T) = \Delta'_{SC}(T)$.

PROOF:

(i) It suffices to apply Proposition 13 and Lemma 15.
(ii) This is an immediate consequence of (i). ∎

Using Proposition 16 we can now derive generalised versions of some known results from the perturbation theory of bounded lower semi-Fredholm linear operators in Banach spaces.

THEOREM 17. *For any quantity f such that $\Gamma' \leqslant f \leqslant \|\ \|$ we have:*

(i) *If $\dim Y/\overline{T(0)} = \infty$, then $T \in F_-$ if and only if $\Gamma'_f(QT) > 0$.*
(ii) *T is strictly cosingular if and only if $\Delta'_f(T) = 0$.*
(iii) *Let $T \in F_-$ and let $S \in LR(X,Y)$ with $D(T) \subset D(S)$, $\Delta'_f(J_Y S) < \Gamma'_f(T)$ and $\dim Q_T S(0) < \infty$. Then $T + S \in F_-$.*

PROOF: By Lemma 15 we need only establish the result for $f = \|\ \|$.

The properties (i) and (ii) were proved by Cross [**8**, V. 5.16 and V. 5.18].

(iii) Since $J_Y T \in F_-$ if and only if $T \in F_-$, we may suppose that Y is complete. Moreover, by the equivalence $T \in F_- \Leftrightarrow Q_T T \in F_-$ [**8**, V. 5.2] it is sufficient to show that $Q_{T+S}(T+S) \in F_-$.

First, assume that $\dim S(0) = 0$, that is, S is single valued. Then since $T = T + S - S$, and $T(0) = (T+S)(0)$, we have $Q_T T = Q_{T+S}(T+S-S)$ and so from [**8**, IV. 5.11] we obtain that

$$\Gamma'(Q_T T) = \Gamma'(Q_{T+S}(T+S) - Q_{T+S}S) \leqslant \Gamma'\big(Q_{T+S}(T+S)\big) + \Delta'(Q_T S). \tag{17.1}$$

Let us consider three possibilities for $T(0)$:

(a) $\dim Y/\overline{T(0)} = \infty$. Then $T + S \in F_-$ if and only if $\Gamma'\big(Q_{T+S}(T+S)\big) > 0$ [**8**, V. 5.16]. Moreover, $\Gamma'(T) \leqslant \Gamma'(Q_T T)$; $\Delta'(Q_T S) \leqslant \Delta'(S)$ [**8**, IV. 5.4]. Now from (17.1) we obtain that $\Gamma'\big(Q_{T+S}(T+S)\big) \geqslant \Gamma'(Q_T T) - \Delta'(Q_T S) \geqslant \Gamma'(T) - \Delta'(S) > 0$.

(b) $\dim Y/\overline{T(0)} < \infty$. In that case $Q_T S$ is a linear relation with finite dimensional range and since $T \in F_-$, $Q_{T+S}(T+S) \in F_-$ by [**8**, V. 5.12].

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(c) $\dim \overline{T(0)} < \infty$. This is covered by (a) if we assume without loss of generality that Y is infinite dimensional.

For the general case, let $\dim Q_T S(0) < \infty$. For any $N \in \mathcal{F}(Y)$, $T \in F_-$ if and only if $Q_N T \in F_-$ (as $(Q_N T)' = T' J_{N^\perp}$). Hence with $F := Q_T S(0) \in \mathcal{F}(Y/\overline{T(0)})$ and combining [**8**, IV. 5.2] with the fact that in this setting $\overline{T(0)} + S(0) = \overline{(T+S)(0)}$ we have the chain of implications:

$T \in F_- \Rightarrow Q_T T \in F_- \Rightarrow Q_F Q_T T = Q_{T+S} \in F_-$.

But, since $\Gamma'(Q_F Q_T T) = \Gamma'(Q_T T)$; $\Delta'(Q_F Q_T S) = \Delta'(Q_T S)$ [**8**, IV. 5.6] it follows from what has been shown that $Q_{T+S}(T+S) \in F_-$.

For the case $f = \rho$ and bounded linear operators in Banach spaces, the statements (i) and (iii) of Theorem 17 were obtained by Tylli [**27**], and (ii) by Martinón [**19**]. ∎

4. Quantities derived from the surjection modulus

The notion of surjection modulus (see [**22, 27, 30, 31**]) is generalised to linear relations as follows:

Definition 18: For $T \in LR(X,Y)$, the surjection modulus $q(T)$ is the quantity $q(T) := \sup\{\alpha \geqslant 0 : TB_{D(T)} \supset \alpha B_Y\}$.

The relationship between q and the minimum modulus γ is described in the following proposition.

Proposition 19. *Let $T \in LR(X,Y)$. Then $q(T) = 0$ if T is not surjective and $q(T) = \gamma(T)$ if T is surjective.*

Proof: From the definition of surjection modulus it is clear that $q(T) = 0$ whenever T is not surjective.

Suppose that T is surjective. Then $q(T) = \gamma(T)$ since by [**8**, II. 2.4] we obtain that $\gamma(T) = \sup\{\lambda : TB_{D(T)} \supset \lambda B_{R(T)}\}$. ∎

Proposition 20. *We have:*

(i) $q(T') = j(T)$.
(ii) *If $q(T) > 0$, then $q(T) = j(T')$.*
(iii) *If X is a Banach space, then $q(\overline{T}) = j(T')$.*

Proof: The linear relation T' is surjective if and only if T is injective and open (see [**8**, II. 3.2 and III. 6.2]). We thus have the chain of implications, $q(T') > 0 \Rightarrow T'$ surjective and $q(T') = \gamma(T')$ by Proposition 19 $\Rightarrow T$ is injective and open, and $0 < j(T) = \gamma(T) = \gamma(T') = q(T')$ [**8**, III. 4.6].

Now suppose that $q(T') = 0$. Then either T is not injective (in which case $j(T) = 0$), or T is not open, that is, $\gamma(T) = 0$. It thus suffices to consider the case when T

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is injective. Then $j(T) = \gamma(T)$, while $\gamma(T) = 0$ (as T is not open). Thus $j(T) = 0$. Hence (i) is true.

(ii) Let $q(T) > 0$. Then $0 < q(T) = \gamma(T) = \gamma(T')$ [**8**, III. 4.6]. Now from the equality $N(T') = R(T)^{\perp} = \{0\}$, the linear relation T' is injective. Hence $j(T') = \gamma(T') = q(T)$.

(iii) By (ii) we may assume that $q(\overline{T}) = 0$ (as $T' = \overline{T}'$). If $j(T') > 0$, then $j(T') = \gamma(T') > 0$, and so T' is open and injective. Let X be complete. Then $\gamma(\overline{T}) = \gamma(T')$ and since $\gamma(T') > 0$, $R(\overline{T})$ is closed [**8**, III. 5.3]. Now, we have $R(\overline{T}) = R(\overline{T})^{\perp\top} = N(\overline{T}')^{\top} = Y$, that is, T is surjective. Therefore $q(\overline{T}) = \gamma(\overline{T}) = \gamma(T') = j(T') = 0$. ∎

For a simple example with $j(T') \neq q(T)$, let Y be a Banach space, X a proper dense subspace of Y and T the identity injection operator of X into Y.

Recall the following Lemma (see [**14**, V. 1.1]).

Lemma 21. *Let M and N be subspaces of X with $\dim M > \dim N$. Then there exists $m \neq 0$ in M such that $\|m\| = d(m, N)$.*

Proposition 22. *Let $D(T)$ be infinite dimensional. Then:*

(i) $j(T) \leqslant \Gamma_0'(T)$.

(ii) *If $\dim N(T) < \infty$, then $q(T) \leqslant \gamma(T) \leqslant \Gamma_0(T)$.*

Proof: (i) First assume that T is single valued. If $j(T) = 0$, then the inequality holds trivially. Accordingly we suppose that T is injective. Let $F \in \mathcal{F}(Y)$. By Lemma 21, there exists $x \in D(T)$, $\|x\| = 1$, such that $\|Tx\| = d(Tx, F)$. Thus $j(T) \leqslant \|Tx\| = \|Q_F Tx\| \leqslant \|Q_F T\|$.

Taking the infimum over $F \in \mathcal{F}(Y)$ gives $j(T) \leqslant \Gamma_0'(T)$.

For the general case, let us consider three possibilities for $T(0)$:

(a) $\overline{T(0)}$ infinite codimensional. Then by single valued case and [**8**, IV. 5.4] we have $j(T) = j(QT) \leqslant \Gamma_0'(QT) \leqslant \Gamma_0'(T)$.

(b) $\overline{T(0)}$ finite codimensional. In that case QT is a finite rank linear operator such that $\dim D(QT) = \infty$ and consequently QT is not injective, that is, $j(T) = j(QT) = 0 \leqslant \Gamma_0'(T)$.

(c) $\overline{T(0)}$ finite dimensional. This case is covered by (a) if we assume without loss of generality that $\dim Y = \infty$.

(ii) First we prove the property when T is single valued. Let $M \in \mathcal{C}(D(T))$. Clearly $\gamma(T) \leqslant \gamma(T|_{M+N(T)})$. Suppose that $\dim N(T) < \infty$. By Lemma 21, there exists $m_o \neq 0$ in M such that $\|m_o\| = d(m_o, N(T))$. We have

$$\gamma(T) \leqslant \gamma(T|_{M+N(T)}) = \inf\left\{\|Tm\| / d(m, N(T)) : m \in M\right\}$$
$$\leqslant \|Tm_o\| / \|m_o\| \leqslant \|T|_M\|.$$

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Taking the infimum over $M \in \mathcal{C}(D(T))$ gives the property required.

Let $T \in LR(X,Y)$ with $N(T)$ finite dimensional.

(a) $q(T) > 0$, then $0 < q(T) = \gamma(T)$ (Proposition 19). Therefore T is open and its null space is closed. Hence $N(T) = N(QT)$; $\gamma(T) = \gamma(QT)$ [**8**, II. 3.9]. Now, the result follows by single valued case.

(b) $q(T) = 0$, Then if $\gamma(T) = 0$, the property holds trivially. Accordingly we suppose that $\gamma(T) > 0$. The rest of the proof now proceeds as for the previous case. ∎

Some further quantities derived from the surjection modulus will now be defined.

DEFINITION 23: For $T \in LR(X,Y)$ define

$$(q_0)'(T) := \sup\{q(Q_F T) : F \in \mathcal{F}(Y)\},$$
$$(q_1)'(T) := \sup\{q(Q_M J_Y T) : M \in \mathcal{E}(\widetilde{Y})\},$$
$$(q_2)'(T) := \inf\{(q_1)'(Q_M J_Y T) : M \in \mathcal{E}(\widetilde{Y})\},$$

with the convention that $(q_1)'(T) = (q_0)'(T) = \infty$ and $(q_2)'(T) = 0$ if $\dim Y < \infty$.

For bounded linear operators in Banach spaces, the quantities $(q_0)'$ and $(q_1)'$ were introduced by Zemánek [**31**] and $(q_0)'$ was denoted by M because of its relation with the Mityangin numbers; $(q_2)'$ was considered in [**19, 15**].

THEOREM 24. *If $(q_0)'(T) > 0$, then $T \in F_-$. The converse holds if T is closed and X is complete.*

PROOF: If $(q_o)'(T) > 0$, then there exists some $F \in \mathcal{F}(Y)$ such that $q(Q_F T) > 0$. Hence, since $(Q_F T)' = T' J_{F^\perp}$ (see [**8**, III. 1.6]), applying Proposition 20, we have $T' J_{F^\perp}$ is open and injective. Therefore $T' \in F_+$, that is, $T \in F_-$.

Now, let X be complete and suppose that T is a closed F_--relation. Then there exists $F \in \mathcal{F}(Y)$ such that $Q_F T$ is open and surjective [**8**, V. 5.21]. Hence $(Q_F T)' = T' J_{F^\perp}$ is open and injective. Moreover, since T is closed and Q_F is a quotient map with finite dimensional null space, the linear relation $Q_F T$ is closed by [**8**, II. 5.13] and so from Proposition 20 we deduce that $0 < j((Q_F T)') = \gamma((Q_F T)') = \gamma(Q_F T) = q(Q_F T)$ and consequently $(q_0)'(T) > 0$. ∎

This Theorem extends the classical result for bounded lower semi-Fredholm linear operators in Banach spaces of Zemánek [**31**].

THEOREM 25. *If $J_Y T$ is strictly cosingular, then $(q_1)'(T) = 0$. The converse is true whenever Y is complete and $T \in PB(X,Y)$ and single valued.*

PROOF: Suppose that $(q_1)'(T) > 0$. Then there exists $M \in \mathcal{E}(\widetilde{Y})$ such that $q(Q_M J_Y T) > 0$. By Proposition 20, $T' J_{M^\perp} = (Q_M J_Y T)'$ is open and injective.

Therefore $Q_M J_Y T \in F_-$. If $J_Y T$ is strictly cosingular, then $Q_M J_Y T$ is strictly cosingular (as $\Delta'(Q_M J_Y T) \leqslant \Delta'(J_Y T)$) and so from [**8**, V. 5.20] it follows that $Q_M J_Y T - Q_M J_Y T \in F_-$, a contradiction.

Now, let Y be complete. Since T is partially continuous and single valued, there exists some $F \in \mathcal{F}(Y)$ for which $Q_F T$ is continuous [**8**, V. 9.2 and V. 9.3]. Assume that T is not strictly cosingular. Then there exists $M \in \mathcal{E}(Y)$ such that $(Q_M T)'$ has a continuous inverse. Consequently, since $Q_{M+F}T = Q_{M+F/F}Q_F T$ [**8**, IV. 5.2], $Q_{M+F}T$ is continuous with $(Q_{M+F}T)'$ having a continuous inverse (as $T'|_{(M+F)^\perp}$ is a restriction of $T'|_{M^\perp}$). By Proposition 20, we now have $q(Q_{M+F}T) = j\big((Q_{M+F}T)'\big) > 0$. Therefore $(q_1)'(T) > 0$. ∎

Corollary 26. *Let Y be a Banach space and T a partially continuous and single valued. Then T is strictly cosingular if and only if $(q_1)'(T) = 0$.*

This Corollary generalises the corresponding result for bounded strictly cosingular linear operators in Banach spaces of Martinón [**19**]. We do not know whether non partially continuous strictly cosingular operators exist.

Proposition 27. $(q_2)'(T) \leqslant \Gamma'(T)$.

Proof: Since $\gamma(T) \leqslant \gamma(QT)$ [**8**, II. 3.4], from Proposition 19 we deduce that $q \leqslant \| \ \|$, so $(q_1)' \leqslant \| \ \|$, and it follows from the definitions that $(q_2)' \leqslant \| \ \|$ as required. ∎

Corollary 28. *Let $T \in F_-$ and let $S \in LR(X, Y)$ such that $D(T) \subset D(S)$ and $\dim Q_T S(0) < \infty$. If $\Gamma_0'(J_Y S) < (q_2)'(T)$, then $T + S \in F_-$.*

Proof: Since $\Delta'(J_Y S) \leqslant \Gamma_0'(J_Y S)$ (Proposition 13), the result follows immediately from Proposition 16 combined with Proposition 27. ∎

For bounded linear operators in Banach spaces, this result was obtained by Martinón [**19**].

References

[1] R. Arens, 'Operational calculus of linear relations', *Pacific J. Math.* **11** (1961), 9–23.

[2] J.P. Aubin and A. Cellina, *Differential inclusions. Set valued maps and viability theory.* (Springer-Verlag, Berlin, New York, 1984).

[3] J.P. Aubin and H. Frankowska, *Set Valued analysis* (Birkhauser, Boston, 1990).

[4] F.H. Clarke, *Optimization and nonsmooth analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts (J. Wiley and Sons, Toronto, 1983).

[5] E.A. Coddington, *Multivalued operators and boundary value problems*, Lecturs Notes in Math. **183** (Springer-Verlag, Berlin, 1971).

[6] E.A. Coddington and A. Dijksma, 'Selfadjoint subspaces and eigenfunction expansions for ordinary differential subspaces', *J. Differential Equations* **20** (1976), 473–526.

[7] R.W. Cross, 'Properties of some norm related functions of unbounded linear operators', *Math. Z.* **199** (1988), 285–302.

[8] R.W. Cross, *Multivalued linear operators*, Monographs and Textbooks in Pure and Applied Mathematics **213** (Marcel Dekker, New York, 1998).

[9] R.W. Cross and L.E. Labuschagne, 'Characterisations of operators of lower semi-Fredholm type in normed linear spaces', *Quaestiones Math.* **15** (1992), 151–173.

[10] A. Dijksma, A. El Sabbah and H.S.V. de Snoo, 'Selfadjoint extensions of regular canonical systems with Stieltjes boundary conditions', *J. Math. Anal. Appl.* **152** (1990), 546–583.

[11] A.S. Fajnshtejn, 'On measures of noncompactness of linear operators and anlogoues of the minimal modulus for semi-Fredholm operators', (Russian), *Spektr. Teor. Oper.* **6** (1985), 182-195.

[12] A. Favini and A. Yagi, 'Multivalued linear operators and degenerate evolution equations', *Ann. Mat. Pura. Appl. (4)* **163** (1993), 353–384.

[13] B. Gramsch, *Über analytische Störungen und den Index von Fredholmoperatoren auf* (Banachräumen, Dept. Mathematics, Univ. Maryland, 1969).

[14] S. Goldberg, *Unbounded linear operators*, Theory and Applications (McGraw-Hill, New York, 1966).

[15] M. González and A. Martinón, 'Operational quantities derived from the norm and measures of noncompactness', *Proc. Roy. Irish Acad. Sect. A* **91** (1991), 63–70.

[16] M. Gromov, *Partial differential relations* (Springer-Verlag, Berlin, 1986).

[17] T. Kato, *Perturbation theory for linear operators* (Springer-Verlag, Berlin, 1966).

[18] A. Lebow and M. Schechter, 'Semigroups of operators and measures of noncompactness', *J. Funct. Anal.* **7** (1971), 1–26.

[19] A. Martinón, *Operational quantities in Fredholm theory*, (Spanish) (Thesis, Univ. La Laguna, 1989).

[20] R. Mennicken and B. Sagraloff, 'Eine Verallgemeinerung des Satzes von abgeschlossenen Werterbereich in lokalkonvexen Räumen', *Manuscripta Math.* **18** (1976), 109–146.

[21] J. von Neumann, *Functional operators, II. The Geometry of Orthogonal spaces*, Annals of Math. Studies **22** (Princeton University Press, Princeton, N.J., 1950).

[22] A. Pietsch, *Operators ideals*, North-Holland Mathematical Library **20** (North-Holland, Amsterdam, 1980).

[23] V. Rakočević, 'On one subset of M. Schechter spectrum', *Mat. Vesnik* **5** (1981), 389–391.

[24] V. Rakočević, 'Measures of non-strict-singularity of operators', *Mat. Vesnik* **35** (1981), 79–82.

[25] M. Schechter, 'Quantities related to strictly singular operators', *Indiana Univ. Math. J.* **21** (1972), 1061–1071.

[26] A.A. Sedaev, 'The structure of certain linear operators', (Russian), *Mat. Issled.* **5** (1970), 166–175.

[27] H.O. Tylli, 'On the asymptotic behaviour of some quantities related to semi-Fredholm operators', *J. London Math. Soc. (2)* **31** (1985), 340–348.

[28] L. Weis, *Über strikt singularë und strikt cosingularë Operatoren in Banachräumen*, (Dis-

sertation) (Bonn, 1974).

[29] L. Weis, 'On perturbations of Fredholm operators in $L_p(\mu)$ spaces', *Proc. Amer. Math. Soc.* **67** (1977), 287–292.

[30] L. Weis, 'On the computation of some quantities in the theory of Fredholm operators', *Rend. Cir. Math. Palermo (2)* **5** (1984), 109–133.

[31] J. Zemánek, 'Geometric characteristics of semi-Fredholm operators and their asymptotic behaviour', *Studia Math.* **80** (1984), 219–234.

[32] J. Zemánek, 'The semi-Fredholm radius of a linear operator', *Bull. Polish Acad. Sci. Math.* **32** (1984), 67–76.

[33] J. Zemánek, 'On the Δ-characteristic of M. Schechter', in *Proceedings of the International Conference, Leipzig, 1983* (Teubner, Leipzig, 1984).

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