Can. J. Math., Vol. XXIII, No. 1, 1971, pp. 84–89

ON BRANCHED COVERINGS OF S^3

GERHARD BURDE

In [**3**] Fox studied a certain class of irregular coverings of S^3 branched along some knot or link which turned out to be homotopy spheres. By a simple geometric construction, it is shown in this paper that these homotopy spheres are just 3-spheres, provided that the group of the knot or link k in question cannot be generated by a number of Wirtinger generators† smaller than the minimal number of bridges of this knot or link. The knots and links with two bridges provide examples for such coverings. In the covering sphere there is a link $\hat{k}$ covering k. With the help of braid automorphisms, $\hat{k}$ can be determined. Figure 3 shows a link $\hat{k}$ in a 5-sheeted covering over $k = 4_1$. Links over 3_1 and 6_1 in 3-sheeted coverings were determined by Kinoshita [**5**] by a different method. The method used here is applicable to these cases and confirms his results.

1. Let k be a tame link in S^3 of multiplicity r . k can be presented as a plat [**8**], which consists of a $2m$-braid z whose upper end and lower end are joined by simple arcs in a certain way as indicated in Figure 1. We think of S^3 as being composed of two balls B_i, $i = 0, 1$, and a product $I \times S^2$, $I = \{t| \, 0 \leqq t \leqq 1\}$, such that z is contained in $I \times S^2$, the upper arcs in B_0, the lower arcs in B_1.

As a special case, take z to be the trivial braid whose strings can be chosen as fibres of the trivial fibration $I \times S^2 \to S^2$. Denote by η_i: $\partial B_i \to \{i\} \times S^2$ the matching homeomorphisms which produce the trivial plat, consisting of m unknotted components embedded in S^3 (Figure 2). Let these circles be spanned by disks which intersect B_0 and B_1 in disks F_i and F_i' $(i = 1, 2, \ldots, m)$, respectively. Any pair of orientation-preserving homeomorphisms $\bar{\eta}_i$: $\partial B_i \to \{i\} \times S^2$, $i = 0, 1$, which join the ends of z to those of the respective arcs will produce a plat in S^3. The knot type of the plat only depends on the isotopy classes $[\bar{\eta}_i]$ of the matching homeomorphisms. Furthermore, we can get all $2m$-braids by choosing $\bar{\eta}_0 = \eta_0$. The isotopy classes of the autohomeomorphisms $\eta = \bar{\eta}_1\eta_1^{-1}$ of the sphere $\{1\} \times S^2$ punctured in the $2m$ points where it is pierced by z induce classes of braid automorphisms of the free fundamental group of this punctured sphere modulo inner automorphisms. The corresponding classes of braids $[z]$ are the classes of braids modulo the centre of their group. For $[z_1] = [z_2]$, the plats derived from z_1 and z_2 represent obviously the same knot type or link type [**1**]. We call η a defining homeomorphism of k.

Let p: $\hat{S}^3 \to S^3$ be an n-sheeted covering branched along k, and $\hat{k} = p^{-1}(k)$.

Received March 4, 1970.

†A Wirtinger generator is an element of the knot group conjugated to a meridian.

We denote by k_i, $i = 1, 2, \ldots, r$, the components of k (which are also represented by plats). The braid corresponding to k_i may consist of $2\lambda_i$ strings. By μ_i we denote the multiplicity of $\hat{k}_i = p^{-1}(k_i)$. Observe that $\sum_{i=1}^{r} \lambda_i = m$, $1 \leqq \mu_i \leqq n$.

THEOREM. *Let $p: \hat{S}^3 \to S^3$ be a branched covering along k, and*

$$(*) \qquad mn - \sum_{i=1}^{r} \lambda_i\mu_i - n + 1 = 0.$$

Then

(1) *$\hat{S}^3$ is a 3-sphere,*

(2) *$\hat{k} = \bigcup \hat{k}_i$ is representable as a plat with a defining $(2 \cdot \sum \lambda_i\mu_i)$-braid $\hat{\mathfrak{z}}$. The minimal number of bridges of the components of $\hat{k}_i$ is at most λ_i,*

(3) *The defining homeomorphism $\hat{\eta}$ of $\hat{k}$ satisfies $p\hat{\eta} = \eta p$.*

(3) implies that for any knot k and covering p satisfying $(*)$, the braid automorphism $\hat{\xi}$ induced by $\hat{\eta}$ can be calculated. Hence $\hat{\mathfrak{z}}$ and, as the proof of the theorem will show, $\hat{k}$ can be determined in this case.

Proof. A simple calculation shows that the Euler characteristic of $p^{-1}(\{1\} \times S^2) = \hat{Q}$ is

$$\chi(\hat{Q}) = 2 \sum_{i=1}^{r} \lambda_i\mu_i + 2n - 2mn.$$

Since $\pi_1(S^3 - k)$ can be generated by paths in $\{1\} \times S^2$, the closed oriented surface $\hat{Q}$ is connected. $\chi(\hat{Q}) = 2 - 2g$, where g is the genus of $\hat{Q}$, whence

$$(**) \qquad mn - \sum \lambda_i\mu_i - n + 1 = g \geqq 0.$$

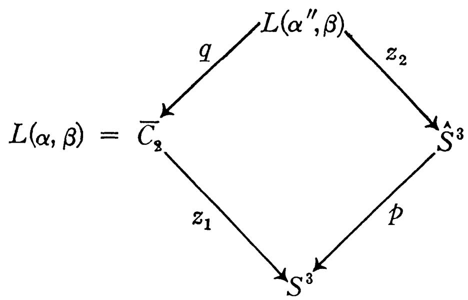
The class of coverings which Fox [**3**] showed to be homotopy spheres is characterized by the inequality $mn - \sum \lambda_i\mu_i - n + 1 \leqq 0$. The above equation shows that as long as there is a projection of k to realize geometrically the m Wirtinger generators, this inequality is always an equation equivalent to $g = 0$.

Now $\hat{S}^3$ can be pieced together in the same way as S^3. $I \times S^2$ will be covered by $I \times \hat{S}^2$, where $\hat{S}^2$ is another 2-sphere. The B_i are covered by handlebodies $\hat{B}_i$ which are also balls, since their boundaries are 2-spheres. Thus (1) is proved. The m arcs in B_i will be covered by $\sum \lambda_i\mu_i$ arcs in $\hat{B}_i$ ($i = 0, 1$), and the disks F_j and F_j' will be covered by disks $\hat{F}_{jk}$ and $\hat{F}_{jk}'$, respectively. This follows from the fact that $B_0 - \bigcup F_i$ and $B_1 - \bigcup F_i'$ are simply connected. Thus $\hat{S}^3$ is a 3-sphere and $\hat{k}$ is a plat whose defining homeomorphism $\hat{\eta}$ is just η lifted to $\hat{S}^3$, which proves (2) and implies (3).

2. Examples for knots and coverings satisfying $(*)$ have been considered by Reidemeister [**6**]. These are knots and links with two bridges ($m = 2$). Schubert who in [**9**] classified knots and links with two bridges associates to each of them a pair of integers (α, β), β odd, $|\beta| < \alpha$, $\gcd(\alpha, \beta) = 1$. α is called the "torsion", and is odd for knots and even for links. In the case of knots there is, for any $\alpha'|\alpha, \alpha' > 1$, an α'-sheeted irregular covering branched along k which satisfies $(*)$ [**6**; **7**].

Putting $\alpha' = 2s + 1$, one obtains $\mu = \mu_1 = s + 1$. If k is a link with two bridges, we obtain the irregular covering, satisfying $(*)$, for any odd $\alpha'|\alpha$. Both components of k are covered by $(s + 1) = \mu_1 = \mu_2$ components of $\hat{k}$.

From the theorem, it follows that in the case of knots with two bridges, all components of $\hat{k}$ are either knots with two bridges or unknotted. If k is a link with two bridges, $\hat{k}$ consists of unknotted curves; any pair of them is a link with two bridges. One of the $s + 1$ components covering a knot k has branching index 1 (call it $\hat{k}_0$), the others have a branching index equal to 2. If k is a link, then there is just one link $\hat{k}_0 \cup \hat{k}_1$ in $\hat{k}$ consisting of two components which have branching index 1. If k is a knot, then $\hat{k}_0$ can easily be determined. Let $k = (\alpha, \beta)$, $\alpha'\alpha'' = \alpha$; then $\hat{k}_0$ is the knot (α'', β). This is a consequence of the following commutative diagram:



z_1 and z_2 are cyclic coverings along k and $\hat{k}_0$ with branching index 2. $L(\alpha, \beta)$ denotes a lens space. $q: L(\alpha'', \beta) \to L(\alpha, \beta)$ is the regular α'-sheeted covering of the lens space $L(\alpha, \beta)$. Since $\hat{k}_0$ has a 2-bridge presentation, it is determined by the type of its cyclic covering $L(\alpha'', \beta)$. As an example, take $k = 6_1 = (9, 5)$ and $\alpha' = \alpha'' = 3$. Thus $\hat{k}_0 = (3, -1)$ is a trefoil knot [**3**]. For $\alpha' = \alpha$, $\hat{k}_0$ is unknotted, a fact mentioned in [**6**; **7**] but not proved correctly there.

3. Kinoshita [**5**] determined some links $\hat{k} \subset \hat{S}^3$ explicitly. He investigated 3-sheeted coverings of the 2-bridge knots 3_1 and 6_1. Since his method seems to hinge on the number 3, I shall determine the link $\hat{k}$ in the 5-sheeted irregular covering of $k = 4_1 = (5, 3)$ (see Figure 1). $\hat{k}$ consists of three unknotted curves (Figure 3). The component $\hat{k}_0$ is clearly distinguished. Since some tedious calculations are implied, I only give a sketch of the method used. First, find a defining braid z for a plat 4_1. The corresponding braid automorphism ζ can be described by two sets of special generators (see [**2**]) for $\pi_1(Q_0)$, where Q_0 is the

https://doi.org/10.4153/CJM-1971-009-2 Published online by Cambridge University Press

2-sphere $Q = \{1\} \times S^2$ punctured in the $2m$ points in which it is intersected by k. A geometric study of the covering p: $\hat{Q}_0 \rightarrow Q_0$ yields all necessary information to determine the automorphism $\hat{\zeta}$. Figure 4 shows Q_0 with cuts from a point P to the points A, B, C, D of the knot. Figure 5 shows $\hat{Q}_0$, the $A_i, B_i, \ldots$ covering $A, B, \ldots$. The index zero always represents the component $\hat{k}_0$ with branching index 1. A set of generators for $\pi_1(\hat{Q}_0)$ is indicated in Figure 5. They can be expressed by a set of generators of $\pi_1(Q_0)$ by geometric means. $\hat{\zeta}$ is the automorphism of $\pi_1(\hat{Q}_0)$ induced by ζ. It is easily seen in Figure 5 how it is expressed in the chosen generators of $\pi_1(\hat{Q}_0)$. (Since p: $\hat{Q}_0 \rightarrow Q_0$ is irregular, the correct choice of the base-point of $\pi_1(\hat{Q}_0)$ is essential.)

The class $[\hat{\zeta}]$ modulo the inner automorphisms determines a class of defining braids $[\hat{z}]$. A representative is most easily obtained by the method of projection used in [**2**].

Remark. An easy calculation shows that a covering satisfying $(*)$ is necessarily irregular. Nevertheless, there are regular coverings of knots with two bridges which are 3-spheres: The universal coverings of the cyclic coverings of order 2 which are lens spaces.

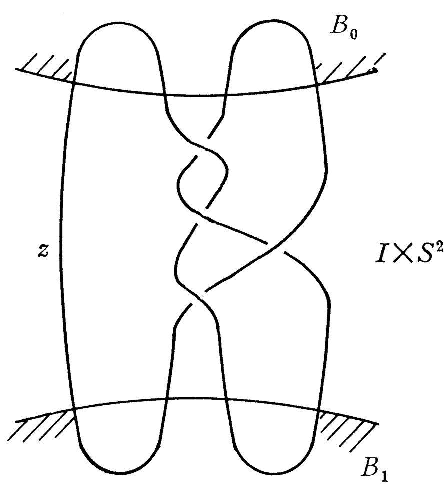


FIGURE 1

For $m = 3, n = 3$ we obtain $g = 1$ from $(**)$ for irregular coverings of knots. $\hat{S}^3$ then possesses a Heegaard diagram of genus 1, the solid tori being $\hat{B}_0 \cup I \times S^2$ and $\hat{B}_1$. Thus $\hat{S}^3$ is homeomorphic to S^3, $S^1 \times S^2$ or a lens space. This throws some light on the fact that only cyclic groups appear in the table contained in [**4**, p. 200].

https://doi.org/10.4153/CJM-1971-009-2 Published online by Cambridge University Press

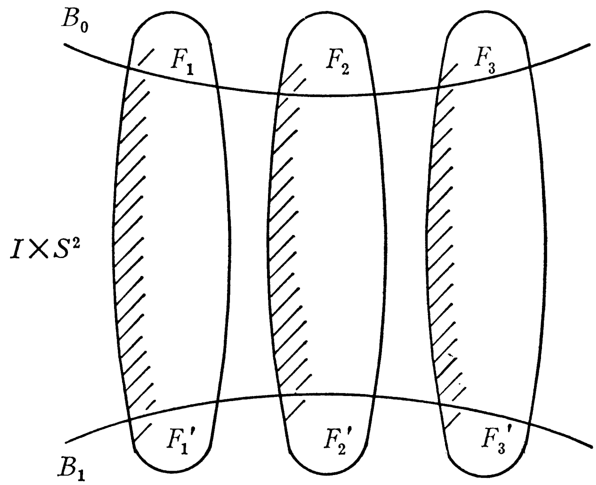


FIGURE 2

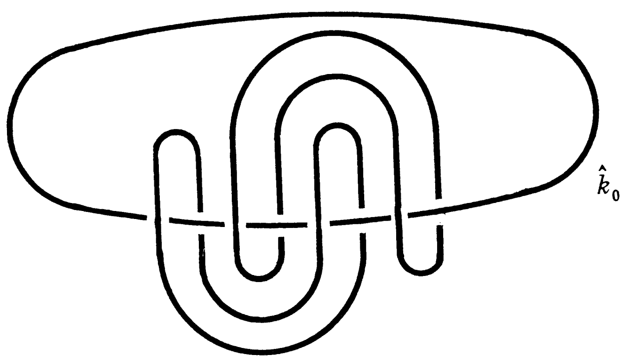


FIGURE 3

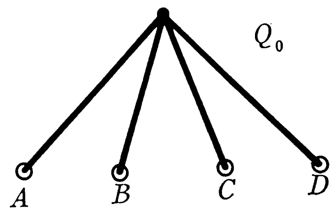


FIGURE 4

https://doi.org/10.4153/CJM-1971-009-2 Published online by Cambridge University Press

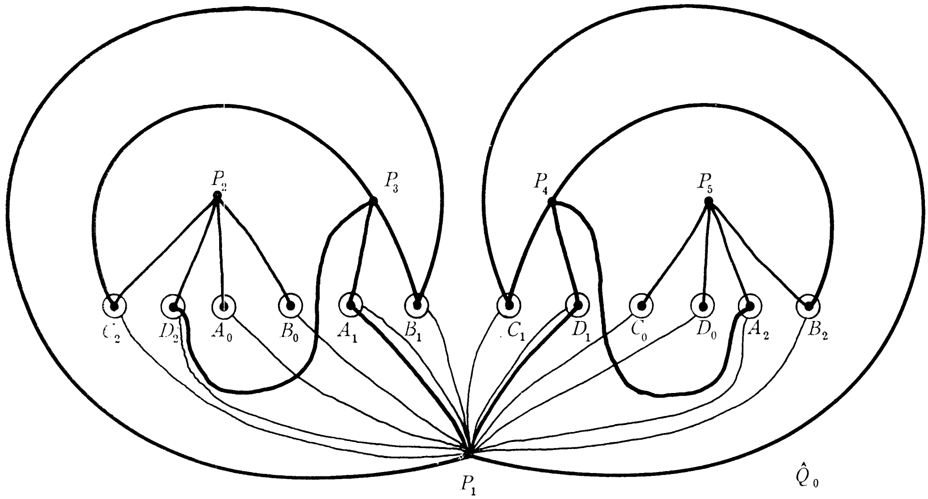


FIGURE 5

REFERENCES

1. E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Hamburg Univ. *4* (1926), 47–72.
2. G. Burde, *Zur Theorie der Zöpfe*, Math. Ann. *151* (1963), 101–107.
3. R. H. Fox, *Construction of simply connected 3-manifolds*, Topology of 3-manifolds and related topics, Proc. The Univ. of Georgia Institute, 1961, pp. 213–216 (Prentice-Hall, Englewood Cliffs, N. J., 1962).
4. ——— *Metacyclic invariants of knots and links*, Can. J. Math. *22* (1970), 193–201.
5. S. Kinoshita, *On irregular branched covering spaces of a kind of knots*, Notices Amer. Math. Soc. *14* (1967), 924.
6. K. Reidemeister, *Knoten und Verkettungen*, Math. Z. *29* (1929), 713–729.
7. ——— *Knotentheorie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 1 (Springer-Verlag, Berlin-Göttingen-Heidelberg, 1932).
8. ——— *Knoten und Geflechte*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II *5* (1960), 105–115.
9. H. Schubert, *Knoten mit zwei Brücken*, Math. Z. *65* (1956), 133–170

Mathematisches Seminar der Johann Wolfgang Goethe-Universität,
Frankfurt, West Germany