Canad. J. Math. Vol. **61** (6), 2009 pp. 1375–1382

# Stable Discrete Series Characters at Singular Elements

Steven Spallone

*Abstract.* Write $\Theta^E$ for the stable discrete series character associated with an irreducible finite-dimensional representation $E$ of a connected real reductive group $G$. Let $M$ be the centralizer of the split component of a maximal torus $T$, and denote by $\Phi_M(\gamma, \Theta^E)$ Arthur's extension of $|D_M^G(\gamma)|^{1/2}\Theta^E(\gamma)$ to $T(\mathbb{R})$. In this paper we give a simple explicit expression for $\Phi_M(\gamma, \Theta^E)$ when $\gamma$ is elliptic in $G$. We do not assume $\gamma$ is regular.

## 1 Introduction

Let $G$ be a connected reductive group over $\mathbb{R}$ and $T$ a maximal torus contained in a Borel subgroup $B$ of $G$. Assume that $G$ has a discrete series of representations. Let $A$ be the split part of $T$ and $M$ the centralizer of $A$ in $G$. It is a Levi subgroup of $G$ containing $T$. Let $E$ be an irreducible finite-dimensional representation of $G(\mathbb{C})$, and consider the packet $\Pi_E$ of discrete series representations $\pi$ of $G(\mathbb{R})$ that have the same infinitesimal and central characters as $E$. Write $\Theta_\pi$ for the character of $\pi$, and put

$$\Theta^E = (-1)^{q(G)} \sum_{\pi \in \Pi_E} \Theta_\pi.$$

Here $q(G)$ is half the dimension of the symmetric space associated with $G$. Note that $\Theta^E(\gamma)$ will not extend to all elements $\gamma \in T(\mathbb{R})$, in particular to $\gamma = 1$. Define the number $D_M^G(\gamma)$ by

$$D_M^G(\gamma) = \det\big(1 - \mathrm{Ad}(\gamma),\ \mathrm{Lie}(G)/\,\mathrm{Lie}(M)\big).$$

Then a result of Arthur and Shelstad [1] states that the function

$$\gamma \mapsto |D_M^G(\gamma)|^{\frac{1}{2}}\,\Theta^E(\gamma),$$

defined on the set of regular elements $T_{\mathrm{reg}}(\mathbb{R})$ extends continuously to $T(\mathbb{R})$. We denote this extension by $\Phi_M(\gamma, \Theta^E)$. The Weyl discriminant $D_M^G$ is a well-understood function, so the extension describes the asymptotic behavior of $\Theta^E$ at singular elements.

Moreover this quantity has appeared in recent harmonic analysis, giving the contribution from the real place to the $L^2$-Lefschetz numbers of Hecke operators in [1,3]. An expression for $\Phi_M(\gamma, \Theta^E)$ as essentially a sum over elements in the Weyl group $W$

Received by the editors March 31, 2007.
AMS subject classification: 22E47.

of $T$ in $G$ appears in the proof of Lemma 4.1 in [3]. Although this expression suffices to prove the lemma, it can be considerably refined when $\gamma$ is in the maximal elliptic subtorus $T_e(\mathbb{R})$ of $T(\mathbb{R})$.

The following theorem is proved in Section 4.

***Theorem*** *If $\gamma \in T_e(\mathbb{R})$, then*

$$\Phi_M(\gamma, \Theta^E) = (-1)^{q(L)} \cdot |W_L| \cdot \sum_{\omega \in W^{LM}} \varepsilon(\omega) \cdot \mathrm{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B}).$$

Here we write $L$ for the centralizer of $T_c$ in $G$, where $T_c$ is the maximal compact subtorus of $T$. Also write $W_L$ and $W_M$ for the Weyl groups of $T$ in $L$ and $M$. The latter are subgroups of $W$ which commute and have trivial intersection. Here

$$W^{LM} = W^L \cap W^M \subseteq W$$

is the set of elements that are simultaneously Kostant representatives for both $L$ and $M$ relative to $B$. We write $\varepsilon$ for the sign character of $W$. Finally, by $V^M_{\omega(\lambda_B+\rho_B)-\rho_B}$ we denote the irreducible finite-dimensional representation of $M$ with highest weight $\omega(\lambda_B + \rho_B) - \rho_B$, where $\lambda_B$ is the $B$-dominant highest weight of $E$. In particular, we derive an extremely simple expression,

$$\Phi_A(1, \Theta^E) = (-1)^{q(G)} \cdot |W|,$$

in the case of a split torus $T = A$.

We now describe the organization of this paper. In Section 2, we spell out the relationship between the root systems of $G$, $L$, and $M$. There are two distinct systems of chambers in $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ obtained from these root systems which are important to understand.

In Section 3, we take the aforementioned lemma a step further to express $\Phi_M(\gamma, \Theta^E)$ explicitly as a linear combination of characters. (Actually, we do the computation for any stable virtual character $\Theta$, as it is no more difficult.) The sum over $W$ simplifies to a sum over Kostant representatives $W^M$.

In Section 4, where we deal specifically with $\Phi_M(\gamma, \Theta^E)$, we distill out the action of $W_L$. A sum over $W^{LM}$ remains. At a key step we use a result of Section 5, the computation of an alternating sum of stable discrete series constants.

In Section 5, we prove the above mentioned result in the context of abstract root systems. It is independent of the rest of the paper.

## 2 $L$-Chambers and $\mathcal{P}$-Chambers

Let $G$ be a connected reductive group over $\mathbb{R}$ and $T$ a maximal torus of $G$. Assume that $G$ has a discrete series, or equivalently, that $G$ has an elliptic maximal torus.

Write $T_c$, respectively $A$, for the maximal compact, resp. split, subtori of $T$ with centralizers $L$, resp. $M$, in $G$. Write $R$ for the root system of $T$ in $G$, and $R_L$, resp. $R_M$, for the set of roots of $T$ in $L$, resp. $M$. Then $R_L$ is the subset of $R$ consisting of real roots, and $R_M$ is the subset of imaginary roots. Write $W_L$ and $W_M$ for the respective

Weyl groups. They are commuting subgroups of $W$ with trivial intersection. Note that $W_L$ fixes each root in $R_M$.

Now $A$ is contained as a split maximal torus in $L_{\mathrm{der}}$, the derived group of $L$, and we may identify $R_L$ with the set of roots of $A$ in $L_{\mathrm{der}}$.

Write $\mathfrak{a}_M$ for $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. For any $\alpha \in R \backslash R_M$ the root hyperplane $H_\alpha$ of

$$X_*(T)_{\mathbb{R}} := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$$

gives a hyperplane in $\mathfrak{a}_M$. Let us consider two kinds of chambers in $\mathfrak{a}_M$ obtained from these. Define $\mathcal{P}$-chambers to be those obtained by deleting from $\mathfrak{a}_M$ all the hyperplanes $H_\alpha$, with $\alpha \in R\backslash R_M$. Define $L$-chambers to be those obtained by deleting all the $H_\alpha$ with $\alpha \in R_L$. The latter are the Weyl chambers for $A$ in $L_{\mathrm{der}}$; therefore $W_L$ acts simply transitively on them.

Observe that $R_L \subset (R\backslash R_M)$. Any additional hyperplanes coming from roots in $R\backslash(R_L \cup R_M)$ divide the $L$-chambers into $\mathcal{P}$-chambers. Thus every $\mathcal{P}$-chamber is contained in a unique $L$-chamber.

Write $\mathcal{P}(M)$ for the set of parabolic subgroups of $G$ admitting $M$ as a Levi component. There is a one-to-one correspondence between $\mathcal{P}(M)$ and the set of $\mathcal{P}$-chambers in $\mathfrak{a}_M$, obtained as follows: for $P = MN \in \mathcal{P}(M)$, the corresponding $\mathcal{P}$-chamber is

$$\mathfrak{a}_P^+ = \{x \in \mathfrak{a}_M : \langle \alpha, x\rangle > 0, \text{ for all } \alpha \in R_N\},$$

where $R_N$ denotes the set of roots of $T$ in $\mathrm{Lie}(N)$.

Recall that the set of $L$-chambers is in bijection with the set of Borel subgroups of $L$ containing $T$, or equivalently the set of positive root systems $R_L^+$ in the root system $R_L$.

Now let $C_P$ be a $\mathcal{P}$-chamber, and let $P = MN$ be the corresponding element of $\mathcal{P}(M)$. It is easy to see that $R_N \cap R_L$ is a positive system in $R_L$, and this corresponds to an $L$-chamber $C_L$. Thus we have defined a map $C_P \mapsto C_L$ from the set of $\mathcal{P}$-chambers to the set of $L$-chambers. It is the obvious one that associates $C_P$ with the unique $L$-chamber containing $C_P$.

## 3 A Linear Combination of Characters

A *stable virtual character* is a finite $\mathbb{Z}$-linear combination $\Theta$ of characters $\Theta_\pi$ such that $\Theta(\gamma) = \Theta(\gamma')$ whenever $\gamma$ and $\gamma'$ are regular, stably conjugate elements of $G(\mathbb{R})$.

In Lemma 4.1 of [3], it was proved that for a stable virtual character $\Theta$ on $G(\mathbb{R})$, the function $\gamma \mapsto |D_M^G(\gamma)|^{1/2}\Theta(\gamma)$ on $T_{\mathrm{reg}}(\mathbb{R})$ extends continuously to $T(\mathbb{R})$. A key ingredient of the proof is the fact that the expression at the bottom of page 497 of [3] is a linear combination of irreducible finite-dimensional representations of $M$. In this section we will compute explicitly the coefficients and the representations involved in the case where the element $a$ appearing in the proof is equal to 1.

We translate the setup of the proof in [3] as follows. We take $\Gamma$ to be the identity component of $T(\mathbb{R})$. The root system $R_\Gamma$ is then simply $R_L$. Fix a Borel $B$ of $G$ containing $T$, and write $R^+$ for the corresponding system of postive roots. Let $R_L^+ = R_L \cap R^+$, and write $C$ for the corresponding $L$-chamber in $\mathfrak{a}_M$. Pick a parabolic subgroup $P = MN$ of $G$ containing $B$. Note that $R_L \cap R_N \subseteq R_L^+$. Since $R_L \cap R_N$ is also a system of positive roots, this implies that actually $R_L \cap R_N = R_L^+$. Thus the $\mathcal{P}$-chamber corresponding to $P$ is contained in $C$.

Although at the end of our computations we will allow $\gamma$ to be nonregular, we now choose $\gamma$ to be a regular element of $\Gamma = T_c(\mathbb{R}) \cdot \exp(\bar{C})$.

The expression is

$$\sum_{B'} m(B') \frac{\Delta_P(\gamma) \cdot \lambda_{B'}(\gamma)}{\Delta_{B'}(\gamma)}. \tag{3.1}$$

The sum runs over Borels containing $T$, which correspond to elements of $W$. Here $\lambda_{B'}$ is the $B'$-dominant highest weight of $E$,

$$\Delta_{B'} = \prod_{\alpha>0} (1-\alpha^{-1}) \quad \text{and} \quad \Delta_P = \prod_{\alpha \in R_N} (1-\alpha^{-1}).$$

Recall the set of Kostant representatives $W^M$ for the Weyl group $W_M$ of $M$, relative to B. It is the set $\{w \in W \mid w^{-1} R_M^+ \subset R^+\}$. If $w \in W$, write $w * B$ for $wBw^{-1}$. We will use the observation that $(\omega * B)_M = B_M$ for $\omega \in W^M$. (Recall that $B_M = B \cap M$.) Indeed, if $\alpha \in R^+ \cap R_M$, then $\omega^{-1}\alpha \in R^+$, which implies that $\alpha \in \omega R^+ \cap R_M$.

Our sum (3.1) breaks up as

$$\sum_{\omega \in W^M} m(\omega * B) \cdot \Delta_P(\gamma) \cdot \sum_{w_M \in W_M} \frac{w_M(\omega\lambda_B)(\gamma)}{\Delta_{w_M \omega * B}(\gamma)}. \tag{3.2}$$

We would prefer the denominator inside the sum to be $\Delta_{w_M * B_M}(\gamma)$. Note that $\Delta_P \cdot \Delta_{B_M} = \Delta_B$, since $R^+$ is the disjoint union of $R_M^+$ and $R_N$. So we consider the quantity

$$\frac{\Delta_P \cdot \Delta_{w_M * B_M}}{\Delta_{w_M \omega * B}} = \frac{\Delta_B \cdot \Delta_{w_M * B_M}}{\Delta_{B_M} \cdot \Delta_{w_M \omega * B}}. \tag{3.3}$$

Observe that if $\mathcal{B}$ is a Borel, $\Delta_{\mathcal{B}} = \delta_{\mathcal{B}} \cdot \rho_{\mathcal{B}}^{-1}$, where $\delta_{\mathcal{B}} = \prod_{\alpha>0}(\alpha^{\frac{1}{2}} - \alpha^{-\frac{1}{2}})$ and $\rho_{\mathcal{B}}$ is the usual half sum of positive roots. Since $\delta_{w*\mathcal{B}} = \varepsilon(w)\delta_{\mathcal{B}}$, we compute that

$$\frac{\Delta_{w*\mathcal{B}}}{\Delta_{\mathcal{B}}} = \varepsilon(w) \cdot (\rho_{\mathcal{B}} - w\rho_{\mathcal{B}}).$$

Thus (3.3) becomes $\varepsilon(\omega)(w_M(\omega\rho_B - \rho_{B_M}) - \rho_B + \rho_{B_M})$.

Next observe that for $w_M \in W_M$, $w_M(\rho_B - \rho_{B_M}) = \rho_B - \rho_{B_M}$. Indeed, the roots of $R^+$ not in $R_M^+$ are in $R_N$, and are thus normalized by $W_M$. So the above expression simplifies to $\varepsilon(\omega) \cdot w_M(\omega\rho_B - \rho_B)$.

We can therefore rewrite (3.2) as

$$\sum_{\omega \in W^M} m(\omega * B) \cdot \varepsilon(\omega) \cdot \sum_{w_M \in W_M} \frac{w_M(\omega(\lambda_B + \rho_B) - \rho_B)(\gamma)}{\Delta_{w_M * B_M}(\gamma)}.$$

Since $\omega$ is a Kostant representative, the weight $\omega(\lambda_B+\rho_B)-\rho_B$ is positive for $B_M$, and we may use the Weyl character formula to rewrite this as

$$\sum_{\omega\in W^M} m(\omega * B)\cdot\varepsilon(\omega)\cdot \operatorname{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B}). \tag{3.4}$$

Here $V^M_{\omega(\lambda_B+\rho_B)-\rho_B}$ denotes the irreducible finite-dimensional representation of $M$ with highest weight $\omega(\lambda_B+\rho_B)-\rho_B$.

## 4 A Formula for $\Phi_M(\gamma,\Theta^E)$

To identify (3.4) with $\Phi_M(\gamma,\Theta^E)$, we replace $m(\omega*B)$ with $n(\gamma,\omega*B)$ (see [3, p. 500]), and multiply it by the factor $\delta_P^{1/2}(\gamma)$:

$$\delta_P^{\frac12}(\gamma)\cdot\sum_{\omega\in W^M} n(\gamma,\omega * B)\cdot\varepsilon(\omega)\cdot \operatorname{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B}). \tag{4.1}$$

Here $\delta_P$ is the modulus character of $P$. (We are still only considering regular $\gamma$.)

Write $A_G$ for the split component of the center of $G$. Let $\lambda_0\in X^*(A_G)$ denote the character by which $A_G$ acts on $E$. It extends to $X^*(T)_{\mathbb{R}}$ in the usual way, and is $W$-invariant.

Let $T_e$ denote the subtorus of $T$ generated by $T_c$ and $A_G$. It is the maximal subtorus of $T$ which is elliptic in $G$.

Write $p_M$ for the projection from $X^*(T)_{\mathbb{R}}$ to $X^*(A)_{\mathbb{R}}$, and note that it is $W_L$-equivariant. The group $W_L$ fixes each root of $M$, thus it acts on $W^M$. For every orbit of this action, there is a unique member $\omega$ so that $p_M(\omega(\lambda_B+\rho_B-\lambda_0))$ is dominant with respect to $C$. One checks dominance using roots in $R_L^+$, and finds that $\omega$ simply needs to satisfy $\omega^{-1}R_L^+\subseteq R^+$. Thus $\omega\in W^L$, the Kostant representatives for $W_L$. We write $W^{LM}=W^L\cap W^M$; there is one element for each orbit of $W_L$ on $W_M$.

If $\lambda\in X^*(T)$ and $w_L\in W_L$, then plainly $w_L\lambda-\lambda\in\mathfrak{a}_M^*$. Write $(\chi_{w_L,\omega,B},\mathbb{C}_{w_L,\omega,B})$ for the one-dimensional representation of $M$, acting through $A$, with weight

$$w_L\omega(\lambda_B+\rho_B)-\omega(\lambda_B+\rho_B).$$

Note that $T_c$ and $A_G$ act trivially on $\mathbb{C}_{w_L,\omega,B}$, thus so does $T_e$. Thus we have

$$V^M_{w_L\omega(\lambda_B+\rho_B)-\rho_B}\cong V^M_{\omega(\lambda_B+\rho_B)-\rho_B}\otimes\mathbb{C}_{w_L,\omega,B}.$$

Our formula (4.1) is now (replacing $\omega\in W^M$ with $w_L\omega$, where $\omega$ is now in $W^{LM}$):

$$\delta_P^{\frac12}(\gamma)\cdot\sum_{\omega\in W^{LM}}\varepsilon(\omega)\cdot\operatorname{tr}(\gamma;V^M_{\omega(\lambda_B+\rho_B)-\rho_B})\cdot\sum_{w_L\in W_L}\varepsilon(w_L)\cdot\chi_{w_L,\omega,B}(\gamma)\cdot n(\gamma,w_L\omega*B). \tag{4.2}$$

Of course, we now wish to simplify the inner sum. Recall [3, p. 500] that

$$n(\gamma,w_L\omega*B)=\bar{c}(x,p_M(w_L\omega\lambda_B+w_L\omega\rho_B-\lambda_0)),$$

https://doi.org/10.4153/CJM-2009-065-x Published online by Cambridge University Press

where $x$ is in the interior of $C$. Here $\bar{c}(x,\lambda)$ is the integer-valued "stable discrete series constant" on $(X_*(A/A_G)_{\mathbb{R}})_{\mathrm{reg}} \times (X^*(A/A_G)_{\mathbb{R}})_{\mathrm{reg}}$, as defined, for instance, on page 493 of [3]. Recall that $\lambda_0 \in X^*(T)_{\mathbb{R}}$ is obtained from the character $\lambda_0 \in X^*(A_G)$ by which $A_G$ acts on $E$, and is thus $W$-invariant.

As $p_M$ commutes with $w_L$, the inner sum of (4.2) is now

$$\sum_{w_L \in W_L} \varepsilon(w_L) \cdot \bar{c}(x, w_L\Lambda) \cdot \chi_{w_L,\omega,B}(\gamma), \tag{4.3}$$

where $\Lambda = p_M(\omega\lambda_B + \omega\rho_B - \lambda_0)$.

We would like to consider the limit of (4.3) as $x$ approaches 0. Recall we can write $\gamma = \gamma_c \cdot \exp(x)$, with $\gamma_c \in T_c(\mathbb{R})$ and $x$ in $\bar{C}$. Also recall that $\gamma$ is still regular (but not for long!). Consider the above formula with $\gamma_c$ fixed and $x$ going to 0 along regular elements of $\bar{C}$. Fix some element $x_0$ in the interior of $C$. The value

$$\bar{c}(x, w_L\Lambda) = \bar{c}(x_0, w_L\Lambda)$$

is unchanged, but $\chi_{w_L,\omega,B}(\gamma)$ approaches $\chi_{w_L,\omega,B}(\gamma_c) = 1$. Thus (4.3) converges to $\sum_{w_L \in W_L} \varepsilon(w_L) \cdot \bar{c}(x_0, w_L\Lambda)$ for some $x_0 \in C$.

But this is simply $(-1)^{q(L)}|W_L|$, by Proposition 5.1(ii) in Section 5 below. Here we use that $\omega \in W^{LM}$. Note that $-1$ is in the Weyl group of the root system by the argument on [3, p. 499]. It is easy to modify this argument to get the same limit as $x$ approaches an element of $X_*(A_G)_{\mathbb{R}}$. Finally note that $\delta_P$ is a positive character and therefore trivial on the compact group $T_c(\mathbb{R})$. It is thus trivial on $T_e(\mathbb{R})$.

Now consider irregular $\gamma$. We take the limit in (4.2) and obtain our theorem.

***Theorem 4.1*** *If $\gamma \in T_e(\mathbb{R})$, then*

$$\Phi_M(\gamma, \Theta^E) = (-1)^{q(L)} \cdot |W_L| \cdot \sum_{\omega \in W^{LM}} \varepsilon(\omega) \cdot \mathrm{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B}). \tag{4.4}$$

Recall that $W^{LM}$ is the intersection of the Kostant representatives $W^L$ and $W^M$ and depends on the choice of $B$ containing $T$.

We now evaluate (4.4) for $\Phi_M$ on the extreme cases for $T$. If $T = A$ is split, then $M = A$, $L = G$, $W^{LM}$ is trivial, but so is $T_c$. We conclude that for $z \in A_G(\mathbb{R})$,

$$\Phi_A(z, \Theta^E) = (-1)^{q(G)} \cdot |W| \cdot \lambda_0(z).$$

If $T$ is elliptic, then $M = G$, $L = T$, $W^{LM}$ is again trivial, and so for $\gamma \in T$,

$$\Phi_G(\gamma, \Theta^E) = \mathrm{tr}(\gamma; E).$$

Note that this agrees with the results of [3, Theorems 5.1, 5.2], since

$$\mathrm{tr}(\gamma^{-1}; E^*) = \mathrm{tr}(\gamma; E).$$

## 5 The Sum of the Stable Discrete Series Constants

Let $(X, X^*, R, \check{R})$ be a root system. Write $W$ for the Weyl group of the root system, and $\varepsilon$ for its sign character. Assume that $R$ generates the real vector space $X$ and that $-1 \in W$. Write $q(R)$ for $(|R^+| + \dim(X))/2$, as in [2]. Let $x_0$ be a regular element of $X$, and $\lambda$ a regular element of $X^*$. Write $C_0$ for the chamber of $X$ containing $x_0$, and $C_0^\vee$ for its dual chamber in $X^*$. Recall the stable discrete series constants $\bar{c}_R(x_0, \lambda)$ from [3, §3].

***Proposition 5.1*** *We have the following formulas for sums of discrete series constants:*

(i) *For all such $\lambda$, $\sum_{w \in W} \bar{c}_R(wx_0, \lambda) = |W|$.*

(ii) *For $\lambda = \lambda_0 \in C_0^\vee$, we have $\sum_{w \in W} \varepsilon(w) \cdot \bar{c}_R(wx_0, \lambda_0) = (-1)^{q(R)}|W|$.*

*The same formulas hold if we sum over the $W$-orbit of $\lambda$ rather than that of $x_0$.*

***Remark*** We make a few comments before beginning the proof. The proof begins by using the "inductive" property of the discrete series constants [3, p. 493, (4)] to change the sum over chambers into a sum over certain facets of $X$. Following [4], we define "panel" to be a facet of codimension 1. Thus a panel is the common face of two chambers, and spans a root hyperplane of $X$.

The hyperplanes $Y$ have their own chambers, and we examine the relationship between the panels and these smaller chambers. Not every panel is equal to such a chamber, as in the case of $B_3$ when $Y$ is the root hyperplane of a long root. The panels in $Y$ give a $B_2$ system, but the chambers of $R_Y$ give an $A_1 \times A_1$ system.

Finally, induction on the rank of the root system gives the calculation.

**Proof** The second formula follows from the first by applying Theorem 3.2(2) of [3, p. 494].

We induce on $r = \dim X$. The proposition is clear when $r = 0$.

We associate these discrete series constants with the various chambers and panels of $X$, and introduce some appropriate notation. Write $c(\mathcal{C})$ for $\bar{c}_R(x, \lambda)$, when $x$ is in the interior of a chamber $\mathcal{C}$.

Suppose $P$ is a panel in $X$, $y$ is in the interior of $P$, and $\bar{P} := \operatorname{span}(P) = Y$. Then write $c(P) = \bar{c}_{R_Y}(y, \lambda_Y)$ (see [3, p. 493]). Thus if $P$ is the common face of distinct chambers $\mathcal{C}$ and $\mathcal{C}'$, then $2c(P) = c(\mathcal{C}) + c(\mathcal{C}')$.

Each chamber has $r$ faces, and it follows that

$$r \cdot \sum_{\mathcal{C}} c(\mathcal{C}) = 2 \sum_P c(P), \tag{5.1}$$

where we are summing over all chambers and then all panels. We show the right-hand side of (5.1) is equal to $r \cdot |W|$ to prove the proposition. Now every panel is on some root hyperplane $X_\alpha = X_{-\alpha}$, so we have

$$2 \sum_P c(P) = \sum_{\alpha \in R} \sum_{\bar{P} = X_\alpha} c(P).$$

We now work with the inner sum. There is a root system on $X_\alpha$ whose set of coroots is $\check{R} \cap X_\alpha$, which defines chambers $\mathcal{C}_\alpha$ in $X_\alpha$ and constants $c_\alpha(\mathcal{C}_\alpha)$. Write $W_\alpha$

https://doi.org/10.4153/CJM-2009-065-x Published online by Cambridge University Press

for the Weyl group of $X_\alpha$. We have

$$\sum_{\bar{P}=X_\alpha} c(P) = \sum_{\mathcal{C}_\alpha} \sum_{P\subset\mathcal{C}_\alpha} c_\alpha(\mathcal{C}_\alpha) = \sum_{\mathcal{C}_\alpha} \sum_{W_\alpha\backslash\{P\subset X_\alpha\}} c_\alpha(\mathcal{C}_\alpha) = \sum_{W_\alpha\backslash\{P\subset X_\alpha\}} \sum_{\mathcal{C}_\alpha} c_\alpha(\mathcal{C}_\alpha).$$

For the first equality, note that every panel $P$ with $\bar{P} = X_\alpha$ is contained in some chamber $\mathcal{C}_\alpha$. The second equality follows because $W_\alpha$ acts transitively on the chambers $C_\alpha$.

Write $n(\alpha)$ for the order of $W_\alpha\backslash\{P \subset X_\alpha\}$. It is equal to the number of panels of $X$ in a given chamber $\mathcal{C}_\alpha$. Then by induction the above is merely $n(\alpha) \cdot |W_\alpha|$, which is exactly the number of panels in $X_\alpha$. It follows that (5.1) is simply equal to twice the total number of panels in $X$.

Since $W$ has $r$ orbits on the set of panels in $X$, and the stabilizer in $W$ of any panel has order 2, we conclude that the total number of panels is half of $r \cdot |W|$, as desired. ∎

**Acknowledgements** I am indebted to my advisor Robert Kottwitz for suggesting the problem and many useful comments. I also thank Christian Kaiser for some helpful conversations. This project was carried out during a stay at the Max-Planck-Institut für Mathematik in Bonn, and I am grateful to the Institut for its support and hospitality.

## References

[1] J. Arthur, *The $L^2$-Lefschetz numbers of Hecke operators.* Invent. Math. **97**(1989), no. 2, 257–290.

[2] N. Bourbaki, *Lie Groups and Lie Algebras.* Chapters 4-6, Springer-Verlag, Berlin, 2002.

[3] M. Goresky, R. Kottwitz, and R. MacPherson, *Discrete series characters and the Lefschetz formula for Hecke operators.* Duke Math. J. **89**(1997), no. 3, 477–554.

[4] R. Herb, *Characters of averaged discrete series on semisimple real Lie groups.* Pacific J. Math. **80**(1979), no. 1, 169–177.

*Purdue University, West Lafayette, IN*
*e-mail*: sspallon@math.purdue.edu