GAP FORMULAE FOR THE WEIERSTRASS TRANSFORMS

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A gap formula for a transform

(1) $$f(x) = \int_{-\infty}^{\infty} k(x-y)\varphi(y)dy$$

is an operator J on f(x)

(2) $$J[f;x] = \varphi(x+) - \varphi(x-) .$$

Such operators are known for Laplace transform [1; p. 91] and [4; pp. 296-299], Stieltjes transform [4; pp. 351-353], a class of convolution transforms [2] and others. Gap formulae for the Weierstrass transform

(3) $$f(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \exp[-(x-y)^2/4] \varphi(y)dy$$

and the Weierstrass-Stieltjes transform

(4) $$f(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \exp[-(x-y)^2/4] d\alpha(y)$$

will be proved in the following two theorems.

THEOREM 1. *Let $\alpha(t)$ be of bounded variation in any finite interval and let the integral (4) relating f(x) to $\alpha(y)$ converge in* $a < x < b$, *then for* d *satisfying* $a < d < b$ *and* $-\infty < x < \infty$

(5) $$\lim_{t\to 1-} -i(1-t)^{1/2} \int_{d-i\infty}^{d+i\infty} \exp[(s-x)^2/4t]f(s)ds = \alpha(x+) - \alpha(x-).$$

THEOREM 2. *Suppose*: (i) $\varphi(y) \in L_1(A, B)$ *for any finite* A *and* B.

(ii) *The integral* (3) *relating* f(x) *to* $\varphi(y)$ *converges* for $a < x < b$.

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(iii) <u>There exist two numbers</u> $\varphi(x+0)$ <u>and</u> $\varphi(x-0)$ <u>such that</u>

$$\int_0^h [\varphi(x \pm u) - \varphi(x \pm 0)]du = o(h) \qquad h \downarrow 0.$$

<u>Then for</u> d <u>satisfying</u> $a < d < b$ <u>we have for</u> $-\infty < x < \infty$

$$(6) \qquad \lim_{t \to 1-} \frac{1}{2i} (1-t)^{1/2} \int_{d-i\infty}^{d+i\infty} (s-x)\exp[(s-x)^2/4]f(s)ds$$

$$= \varphi(x+o) - \varphi(x-o) .$$

<u>Proof of Theorem 1.</u> Theorem 7.3 of [3, pp. 189-190] implies

$$\frac{1}{2\pi i}\sqrt{\frac{\pi}{t}} \int_{d-i\infty}^{d+i\infty} \exp[(s-x)^2/4t]f(s)ds = \int_{-\infty}^{\infty} k_1(x-u, 1-t)\alpha(u)du$$

where $k_1(x, t) = \frac{\partial}{\partial x} [(4\pi t)^{1/2} \exp(-x^2/4t)]$. Writing

$$(4\pi)^{1/2}(1-t)^{1/2} \left\{ \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x} + \int_{\delta}^{x+\delta} + \int_{x+\delta}^{x} \right\} k_1(x-u, 1-t)\alpha(u)du$$

$$\equiv I_1(t) + I_2(t) + I_3(t) + I_4(t) .$$

we have by simple integration for every $\delta > o$

$$(7) \qquad (4\pi)^{1/2}(1-t)^{1/2} \int_{x-\delta}^{x} k_1(x-u, 1-t)du = -1 + o(1) \quad t \to 1-$$

and

$$(8) \qquad (4\pi)^{1/2}(1-t)^{1/2} \int_{x}^{x+\delta} k_1(x-u, 1-t)du = 1 + o(1) \qquad t \to 1- .$$

We can obviously choose δ so that $|\alpha(u) - \alpha(x-)| < \epsilon$ for $x-\delta < u < x$ and $|\alpha(u) - \alpha(x+)| < \epsilon$ for $x < u < x+\delta$ and therefore

$$|I_2(t)+\alpha(x-)| = |(2\pi)^{1/2}(1-t)^{1/2} \int_{x-\delta}^{x} k_1(x-u, 1-t)[\alpha(u)-\alpha(x-)]du| + o(1)$$

$$\leq \epsilon \cdot 1 + o(1), \quad t \to 1- .$$

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Similarly $|I_3(t)-\alpha(x+)| \leq \epsilon \,.\, 1 + o(1) \quad t \to 1- \,.$

Using Lemma 2.1c of [3, Ch. VI] for some finite ξ and η $a < \xi < \eta < b$

$$\alpha(u) = o(\exp((u-\eta)^2/4)), \quad u \to \infty, \quad \text{and}$$

$$\alpha(u) = o(\exp((u-\xi)^2/4)), \quad u \to \infty,$$

and since $\alpha(u)$ is, locally, of bounded variation

(9) $$|\alpha(u)| \leq \begin{cases} K \exp((u-\eta)^2/4), & u > x, \\ K \exp((u-\xi)^2/4), & u < x. \end{cases}$$

Integration yields

$$|I_1(t)| \leq K \int_{-\infty}^{x-\delta} \frac{(x-u)}{(1-t)} \exp(-(x-u)^2/4(1-t) \exp((u-\xi)^2/4)du = o(1) \quad t \to 1-$$

and similarly also $I_4(t) = o(1) \quad t \to 1- \,.$

Proof of Theorem 2. Define $\alpha(u) = \int_0^u \varphi(v)dv$. For $a < d < b$

$$I \equiv -i(1-t)^{1/2} \int_{d-i\infty}^{d+i\infty} \frac{1}{2t} (s-x)\exp[(s-x)^2/4t]f(s)ds =$$

$$= -i(1-t)^{1/2} \int_{d-i\infty}^{d+i\infty} \frac{1}{2t} (s-x)\exp[(s-x)^2/4t]ds \int_{-\infty}^{\infty} k_1(s-u, 1)\alpha(u)du$$

$$= -i(1-t)^{1/2} \int_{-\infty}^{\infty} \alpha(u)du \int_{d-i\infty}^{d+i\infty} \frac{1}{2t} (s-x)\exp((s-x)^2/4t)k_1(s-u, 1)ds \,.$$

The interchange in order of integration is justified by Fubini's theorem since $\alpha(u)$ satisfies (9) and therefore

$$\int_{-\infty}^{\infty} ((d-x)^2+y^2)^{1/2}\exp[((d-x)^2-y^2)/4t]dy \int_{-\infty}^{\infty} (y^2+(d-u)^2)^{1/2}\exp[(y^2-(d-u)^2/4] \,.$$

$$\cdot\ |\alpha(u)|du \,,$$

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which dominates our integral, converges.

By the method employed in [3, p. 176-177] we calculate

$$J = \frac{\pi^{1/2}}{2\pi i t^{1/2}} \int_{d-i\infty}^{d+i\infty} \frac{1}{2t} (s-x)\exp[(s-x)^2/4t]k_1(s-u, 1)ds$$

$$= \frac{-i}{2\pi} \left(\frac{\pi}{t}\right)^{1/2} \int_{-\infty}^{\infty} \frac{1}{2t} \quad (y-i(d-x))\exp[-(y-i(d-x))^2/4t]k_1(iy+d-u, 1)dy$$

$$= \frac{1}{2\pi} \left(\frac{\pi}{t}\right)^{1/2} \int_{-\infty}^{\infty} \exp[-(y-i(d-x))^2/4t]k_2(iy+d-u, 1)dy$$

where $k_2(s, t) = \frac{\partial^2}{\partial s^2} k(s, t) = \frac{\partial^2}{\partial s^2} ((4\pi t)^{-1/2} \exp(-s^2/4t))$. Since by [3, p. 177]

$$\int_{-\infty}^{\infty} k(x-y, t_1)\, k(y-v, t_2-t_1)dy = k(x-v, t_2)$$

we have

$$\text{(10)} \qquad \int_{-\infty}^{\infty} k(x-y, t_1)k_2(y-v, t_2-t_1)dy = k_2(x-v, t_2) .$$

Combining (10) with Corollary 2.2 of [3, p. 176] where $\varphi(y) = k_2(y-v, 1-t)$, $t_2 = 1$ and $t_1 = t$ we obtain $J = k_2(x-u, 1-t)$ and therefore

$$I = (4\pi)^{1/2}(1-t)^{1/2} \left\{ \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x} + \int_{x}^{x+\delta} + \int_{x+\delta}^{\infty} \right\} \alpha(u)k_2(x-u, 1-t)du$$

$$= I_1(t) + I_2(t) + I_3(t) + I_4(t) .$$

By the method employed in the proof of theorem 1, $\lim_{t\to 1-} I_1(t) = \lim_{t\to 1-} I_4(t) = 0$

$$I_2(t) = (4\pi)^{1/2}(1-t)^{1/2} \int_{x-\delta}^{x} \varphi(u)k_1(x-u, 1-t)du + o(1) \quad t \to 1- \quad .$$

Using (7) we have

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$$|I_2(t)+\varphi(x-0)| = |(4\pi)^{1/2}(1-t)^{1/2}\int_{x-\delta}^{x} k_1(x-u, 1-t)[\varphi(u)-\varphi(x-o)]du|$$

$$+ o(1) = |(4\pi)^{1/2}(1-t)^{1/2}\int_{x-\delta}^{x} k_2(x-u, 1-t)\beta(u)du| + o(1)$$

$$\leq \epsilon(4\pi)^{1/2}(1-t)^{1/2}\int_{x-\delta}^{x} k_2(x-u, 1-t)|s-u|du + o(1) \leq \epsilon M + o(1)$$

$$t \to 1- .$$

Similarly $|I_3(t)-\varphi(x+o)| \leq \epsilon M + o(1)$ $t \to t-$, which concludes the proof of our theorem.

Example. Let $\alpha(t) = 0$ for $t < 0$ and $\alpha(t) = 1$ for $t > 0$ then $f(x) = \frac{1}{\sqrt{4\pi}} \exp(-x^2/4)$. Since the integral (4) converges always

$$\lim_{t\to 1-} \frac{-i}{\sqrt{4\pi}} (1-t)^{1/2} \int_{-i\infty}^{i\infty} \exp[(s-x)^2 \cdot \frac{1}{4t} - s^2 \frac{1}{4}] ds$$

$$= \lim_{t\to 1-} \frac{-i}{\sqrt{4\pi}} (1-t)^{1/2} \left\{ \int_{-i\infty}^{i\infty} \exp[s^2 \frac{(1-t)}{4t} - \frac{2sx}{4t} + \frac{x^2}{4t(1-t)}] ds \right\}$$

$$. \exp(-x^2/4t(1-t)) . \exp(x^2/4t)$$

$$= \lim_{t\to 1-} t^{1/2} \exp(-x^2/4t(1-t)) \exp(x^2/4t) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases} .$$

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