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On a classification of irreducible admissible modulo p representations of a p-adic split reductive group

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Abstract

We give a classification of irreducible admissible modulo p representations of a split p-adic reductive group in terms of supersingular representations. This is a generalization of a theorem of Herzig.

1. Introduction

Let p be a prime number and F a finite extension of $\mathbb{Q}_p$. In this paper, we consider modulo p representations of (the group of F-valued points of) a split connected reductive group G over F. The study of such representations was started by Barthel–Livné [BL94, BL95] when $G = \mathrm{GL}_2(F)$. They defined the notion of *supersingular representations* and gave a classification of non-supersingular irreducible representations. In particular, they proved that a representation is supersingular if and only if it is supercuspidal. Here, a representation is called *supercuspidal* if and only if it does not appear as a subquotient of a parabolic induction from an irreducible representation of a proper parabolic subgroup. By this theorem, to classify irreducible representations of $\mathrm{GL}_2(F)$, it is sufficient to classify irreducible supersingular representations. When $G = \mathrm{GL}_2(\mathbb{Q}_p)$, irreducible supersingular representations are classified by Breuil [Bre03]. However, when $F \neq \mathbb{Q}_p$ a classification seems more complicated [BP12].

Herzig [Her11a] gave a definition of a supersingular representation for any split G using the modulo p Satake transform [Her11b]. He also gave a classification of irreducible admissible representations in terms of supersingular representations when $G = \mathrm{GL}_n(F)$. This is a generalization of a theorem of Barthel–Livné. In this paper, we generalize his classification to any split G.

Now we state our main theorem. Let $\overline{\kappa}$ be an algebraic closure of the residue field of F. *All representations in this paper are smooth representations over* $\overline{\kappa} \simeq \overline{\mathbb{F}}_p$. Fix a reductive $\mathcal{O}$-form of G and denote it by the same letter G. Let K be the group of $\mathcal{O}$-valued points of G. We also fix a Borel subgroup B and a split maximal torus $T \subset B$ of G. Then we can define the notion of supersingular representations with respect to (K, T, B). (See Herzig's paper [Her11a, Definition 4.7] or Definition 5.1 in this paper.) Let Π be the set of simple roots. Each subset $\Theta \subset \Pi$ corresponds to the standard parabolic subgroup P_Θ. Let $P_\Theta = M_\Theta N_\Theta$ be the Levi decomposition such that $T \subset M_\Theta$ and N_Θ is the unipotent radical of P_Θ. Consider the set $\mathcal{P}$ of all $\Lambda = (\Pi_1, \Pi_2, \sigma_1)$ such that:

- Π_1 and Π_2 are subsets of Π;

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- σ_1 is an irreducible admissible representation of M_{Π_1} which is supersingular with respect to $(M_{\Pi_1} \cap K, T, M_{\Pi_1} \cap B)$;
- if we let ω_{σ_1} be the central character of σ_1 and put $\Pi_{\sigma_1} = \{\alpha \in \Pi \mid \langle \alpha, \check{\Pi}_1 \rangle = 0,\ \omega_{\sigma_1} \circ \check{\alpha} = \mathbf{1}_{\mathrm{GL}_1(F)}\}$ then $\Pi_2 \subset \Pi_{\sigma_1}$.

Then the main theorem says that there exists a bijection between $\mathcal{P}$ and the set of isomorphism classes of irreducible admissible representations of G.

To state the theorem more precisely, we define the representation $I(\Lambda)$ for $\Lambda = (\Pi_1, \Pi_2, \sigma_1) \in \mathcal{P}$. Let $P_\Lambda = M_\Lambda N_\Lambda$ be the Levi decomposition of the standard parabolic subgroup corresponding to $\Pi_1 \cup \Pi_{\sigma_1}$. First we construct the representation σ_Λ of M_Λ. We can prove that σ_1 can be extended uniquely to M_Λ such that $[M_{\Pi_{\sigma_1}}(F), M_{\Pi_{\sigma_1}}(F)]$ acts on it trivially (Lemma 3.2). We denote the extended representation by the same letter σ_1. Let Q be the parabolic subgroup of M_Λ corresponding to $\Pi_1 \cup \Pi_2$. Then Q defines the special representation of M_Λ [Gro]. We denote it by $\sigma_{\Lambda,2}$. From the definition of the special representation, the restriction of $\sigma_{\Lambda,2}$ to $M_{\Pi_{\sigma_1}}$ is the special representation of $M_{\Pi_{\sigma_1}}$ with respect to the standard parabolic subgroup corresponding to Π_2. Now we define $\sigma_\Lambda = \sigma_1 \otimes \sigma_{\Lambda,2}$.

In the case of GL_n, the construction is as follows. The Levi subgroup M_Λ is given by a product $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. The extension of σ_1 to M_Λ is a tensor product $\tau'_1 \boxtimes \cdots \boxtimes \tau'_r$. For each i, define a representation τ_i of GL_{n_i} as follows. If $\mathrm{GL}_{n_i} \subset M_{\Pi_1}$, then τ'_i is a supersingular representation and put $\tau_i = \tau'_i$. If $\mathrm{GL}_{n_i} \not\subset M_{\Pi_1}$, then τ'_i is a character. In this case, the intersection of the roots of GL_{n_i} and Π_2 gives a parabolic subgroup Q_i of GL_{n_i}. Put $\tau_i = \tau'_i \otimes \mathrm{Sp}_{Q_i}$; here Sp_{Q_i} is the special representation corresponding to Q_i. Then σ_Λ is given by $\sigma_\Lambda = \tau_1 \boxtimes \cdots \boxtimes \tau_r$. Each τ_i is a supersingular representation or a special representation twisted by a character (cf. [Her11a, Theorem 1.1]).

Put $I(\Lambda) = \mathrm{Ind}_{P_\Lambda}^G(\sigma_\Lambda)$. The following is the main theorem of this paper.

Theorem 1.1 (Theorem 5.11). *For $\Lambda \in \mathcal{P}$, $I(\Lambda)$ is irreducible and the correspondence $\Lambda \mapsto I(\Lambda)$ gives a bijection between $\mathcal{P}$ and the set of isomorphism classes of irreducible admissible representations of G.*

Using this theorem, we get the relation between supersingular representations and supercuspidal representations. Recall that a representation is called *supersingular* if it is supersingular with respect to any 3-tuple (K, T, B) chosen as before.

Theorem 1.2 (Corollary 5.13). *For an irreducible admissible representation π of G, the following conditions are equivalent.*

(i) *The representation π is supersingular with respect to the fixed (K, T, B).*

(ii) *The representation π is supersingular.*

(iii) *The representation π is supercuspidal.*

These theorems are proved by Barthel–Livné [BL94, BL95] ($G = \mathrm{GL}_2$) and Herzig [Her11a] ($G = \mathrm{GL}_n$). (In these cases, the equivalence of (i) and (ii) in Theorem 1.2 is almost clear since there is only one hyperspecial maximal compact subgroup of G up to conjugate. See Herzig's argument [Her11a, § 4].)

We also give a criterion of the irreducibility of a principal series representation.

Theorem 1.3. *Let $\nu\colon T \to \overline{\kappa}^\times$ be a character. Then $\mathrm{Ind}_B^G \nu$ is irreducible if and only if $\nu \circ \check{\alpha} \neq \mathbf{1}_{\mathrm{GL}_1(F)}$ for all $\alpha \in \Pi$.*

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This is proved by Barthel–Livné when $G=\mathrm{GL}_2$ [BL94, BL95] and Ollivier [Oll06] when $G=\mathrm{GL}_n$. In fact, we can describe the composition factors of $\mathrm{Ind}_P^G(\sigma)$ where σ is an irreducible admissible supersingular representation of the Levi subgroup of a parabolic subgroup P (Lemma 5.8 and Remark 5.9). When $G=\mathrm{GL}_n$, such description is given by Herzig [Her11a, Theorem 8.7].

Now we give an outline of the proof. Using a z-extension, we may assume that the derived group of G is simply connected. Let $\text{c-Ind}_K^G(V)$ be the compact induction from an irreducible K-representation V and $\mathcal{H}_G(V)$ the endomorphism ring of $\text{c-Ind}_K^G(V)$. Let X_* be the group of cocharacters of T and $X_{*,+}=\{\lambda\in X_*\mid\langle\lambda,\check{\Pi}\rangle\subset\mathbb{Z}_{\geqslant 0}\}$. Then by the Satake transform, we have $\mathcal{H}_G(V)\simeq\overline{\kappa}[X_{*,+}]$ [Her11b, Corollary 1.3]. In particular, $\mathcal{H}_G(V)$ is commutative. Therefore, for each irreducible admissible representation π of G, there exist an irreducible representation V of K and a character χ of $\mathcal{H}_G(V)$ such that π is a quotient of $\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\chi$. To prove the main theorem, we reveal the relation between $\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\chi$ and a parabolic induction.

The first comparison is given by Herzig [Her11a, Theorem 3.1]. He proved the following. Let $P=MN$ be a standard parabolic subgroup and its Levi decomposition and Π_M the set of simple roots of M. By the partial Satake transform, we have an injective homomorphism $\mathcal{H}_G(V)\hookrightarrow\mathcal{H}_M(V^{\overline{N}(\mathcal{O})})$. Fix a character χ of $\mathcal{H}_G(V)$. Let $P=MN$ be a standard parabolic subgroup such that χ factors through $\mathcal{H}_G(V)\to\mathcal{H}_M(V^{\overline{N}(\mathcal{O})})$. Let ν be a lowest weight of V and put $\Pi_V=\{\alpha\in\Pi\mid\langle\nu,\check{\alpha}\rangle=0\}$. Herzig proved that if $\Pi_V\subset\Pi_M$ then we have

$$\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\chi\simeq\mathrm{Ind}_P^G(\text{c-Ind}_{M\cap K}^M(V^{\overline{N}(\mathcal{O})})\otimes_{\mathcal{H}_M(V^{\overline{N}(\mathcal{O})})}\chi). \tag{1.1}$$

(He proved this theorem for any split G.)

Unfortunately, in the above theorem, the condition $\Pi_V\subset\Pi_M$ is needed. For example, if V is the trivial representation, the above theorem does not hold. However, we can prove the following 'changing the weight theorem'. Let V' be another irreducible K-representation and ν' its lowest weight. Assume that there exists a simple root α such that $\alpha\notin\Pi_M$, $\alpha\in\Pi_V$ and $\nu'=\nu-(q-1)\omega_\alpha$ where ω_α is a fundamental weight corresponding to α. Moreover, assume that $\langle\check{\alpha},\Pi_M\rangle\neq 0$ or $\chi(\check{\alpha})\neq 1$. Then we have

$$\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\chi\simeq\text{c-Ind}_K^G(V')\otimes_{\mathcal{H}_G(V)}\chi$$

(Theorem 4.1). In this theorem, $\Pi_{V'}=\Pi_V\setminus\{\alpha\}\subsetneqq\Pi_V$. Therefore, at least if χ is generic, then (1.1) holds. Herzig proved this theorem under some assumptions (which are enough for $G=\mathrm{GL}_n$). We prove it for any split G in this paper.

Finally, we must treat the case when neither theorem can be applied. An argument using a tensor product deduces us to the case of $P=B$. To use such arguments, we need to express the Satake parameters of σ_Λ by those of σ_1 and $\sigma_{\Lambda,2}$. Such calculation is given in § 3. If $G=\mathrm{GL}_n$, this calculation is almost obvious since any Levi subgroup of GL_n is a product of smaller groups GL_m.

Assume that $P=B$. In this case, Herzig studied the structure of the left-hand side of (1.1) by a (mysterious) calculation of the affine Hecke algebra when $G=\mathrm{GL}_n$. Our method is different from his, and ours gives more information on the structure of the left-hand side. In fact, we prove that both sides of (1.1) have a finite length and the same composition factors (Proposition 4.7). To prove it, we prove that $\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\mathcal{H}_T(V^{\overline{U}(\mathcal{O})})$ is free as a $\mathcal{H}_T(V^{\overline{U}(\mathcal{O})})$-module (Proposition 4.22). By the theorem of changing the weight, for a generic χ, $\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\chi$ only depends on $V^{\overline{U}(\mathcal{O})}$ and χ. Using the freeness, it follows that the composition factors of $\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\chi$ only depend on $V^{\overline{U}(\mathcal{O})}$ and χ. Such an argument can

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be found in the paper of Barthel–Livné [BL95] when $G = \mathrm{GL}_2$. They proved the freeness (see Remark 4.23) by the detailed study of a compact induction. We prove the freeness by embedding $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \mathcal{H}_T(V^{\overline{U}(\mathcal{O})})$ to a principal series and considering the filtration coming from the Bruhat decomposition (Lemma 4.21).

Such comparisons are given in § 4. Using these comparisons, the main theorem is proved in § 5.

2. Preliminaries

2.1 Notation

In this paper, we use the following notation. Let p be a prime number, F a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its ring of integers, $\varpi \in \mathcal{O}$ a uniformizer, $\kappa = \mathcal{O}/(\varpi)$ the residue field and $q = \#\kappa$. Let G be a connected split reductive group over $\mathcal{O}$. Fix a Borel subgroup $B \subset G$ and a split maximal torus $T \subset B$. Let U be the unipotent radical of B. Then $B = TU$ is a Levi decomposition of B. Let $\overline{B} = T\overline{U}$ be a Levi decomposition of the opposite group of B. We also denote the group of F-valued points of G by the same letter G. The only confusion coming from using the same letter is the notation '$[G, G]$'. In this paper, $[G, G]$ means the derived group of G as an algebraic group. In general, $[G(F), G(F)] \subset [G, G](F)$ and it is not equal. If $[G, G]$ is simply connected, then $[G, G](F) = [G(F), G(F)]$.

We use similar notation for other groups (for example, $B = B(F)$). Set $K = G(\mathcal{O})$. For any algebraic group H, let Z° be the connected component of H containing the unit element and Z_H the center of H. We also use the notation Z_H for the center of any group H. For closed subgroups $H_1, H_2 \subset H$, we define a closed subgroup $Z_{H_1}(H_2)$ of H_1 by $Z_{H_1}(H_2) = \{h_1 \in H_1 \mid h_1 h_2 = h_2 h_1 \text{ for all } h_2 \in H_2\}$. For a group Γ, $\mathbf{1}_\Gamma$ is the trivial representation of Γ. For a representation V of Γ, V^Γ is the space of invariants and V_Γ is the space of coinvariants.

Let $(X^*, \Delta, X_*, \check{\Delta})$ be the root datum of (G, T). Then B determines the set of positive roots $\Delta^+ \subset \Delta$ and the set of simple roots $\Pi \subset \Delta^+$. Let W be its Weyl group. Let $\mathrm{red}\colon K = G(\mathcal{O}) \to G(\kappa)$ be the canonical morphism. The set of dominant (respectively anti-dominant) elements in X^* is denoted by X^*_+ (respectively X^*_-). We also use notation $X_{*,+}$ and $X_{*,-}$. For $\lambda, \mu \in X_*$, we denote $\mu \leqslant \lambda$ if $\lambda - \mu \in \mathbb{Z}_{\geqslant 0}\check{\Pi}$.

Let P be a standard parabolic subgroup. It has a Levi decomposition $P = MN$. In this paper, we only consider the decomposition such that $T \subset M$. The opposite parabolic subgroup of P is denoted by $\overline{P} = M\overline{N}$. We denote the Levi decomposition of the standard parabolic subgroup corresponding to $\Theta \subset \Pi$ by $P_\Theta = M_\Theta N_\Theta$. The subset of Π corresponding to P is denoted by Π_P or Π_M. Put $\Delta_M = \Delta \cap \mathbb{Z}\Pi_M$ and $\Delta_M^+ = \Delta^+ \cap \Delta_M$. Let W_M be the Weyl group of Δ_M. For dominant $\nu \in X^*$, let $P_\nu = M_\nu N_\nu$ be the standard parabolic subgroup corresponding to $\Pi_\nu = \{\alpha \in \Pi \mid \langle \nu, \check{\alpha} \rangle = 0\}$. Put $W_\nu = \mathrm{Stab}_W(\nu)$, $\Delta_\nu = \{\alpha \in \Delta \mid \langle \nu, \check{\alpha} \rangle = 0\}$ and $\Delta_\nu^+ = \Delta^+ \cap \Delta_\nu$. We use similar notation for dominant $\lambda \in X_*$.

For a subset $A \subset X^*$ and $A' \subset X_*$, $\langle A, A' \rangle = 0$ means $\langle \nu, \lambda \rangle = 0$ for all $\nu \in A$ and $\lambda \in A'$. Notice that this condition is automatically satisfied if A or A' is empty. We write $\langle A, \lambda \rangle = 0$ (respectively $\langle \nu, A' \rangle = 0$) instead of $\langle A, \{\lambda\} \rangle = 0$ (respectively $\langle \{\nu\}, A' \rangle = 0$).

A z-extension of G (over F) is a surjective homomorphism (as algebraic groups) $\widetilde{G} \to G \times_{\mathcal{O}} F$ over F such that the derived group of $\widetilde{G}$ is simply connected and the kernel is a split torus which is central in $G \times_{\mathcal{O}} F$. Since the Galois cohomology of a split torus is trivial, the homomorphism $\widetilde{G} = \widetilde{G}(F) \to G(F) = G$ is also surjective. It is known that a z-extension exists.

LEMMA 2.1. *Let $\widetilde{G} \to G$ be a z-extension. Then there exists a hyperspecial maximal compact subgroup $\widetilde{K}$ of $\widetilde{G}$ such that the following conditions hold.*

(i) *The homomorphism $\widetilde{G} \to G$ induces a surjective homomorphism $\widetilde{K} \to K$.*

(ii) *The induced homomorphism $\widetilde{K} \to K$ induces a surjective homomorphism $\widetilde{G}(\kappa) \to G(\kappa)$. (Here, we denote the $\mathcal{O}$-form of $\widetilde{G}$ corresponding to $\widetilde{K}$ by the same letter $\widetilde{G}$.)*

(iii) *The derived group of $\widetilde{G} \times_{\mathcal{O}} \kappa$ is simply connected.*

Proof. Let $G_{\mathrm{ad}} = \widetilde{G}_{\mathrm{ad}}$ be the adjoint group of G, $\mathcal{B}$ its building and $x \in \mathcal{B}$ the hyperspecial point corresponding to K. The point x defines the hyperspecial maximal compact subgroup $\widetilde{K}$ of $\widetilde{G}$. Then (i) follows from [HR08, Proof of Proposition 3]. Since $\mathrm{Ker}(K \to G(\kappa))$ is the maximal normal pro-p subgroup of K, $\widetilde{K} \to K$ induces $\widetilde{G}(\kappa) \to G(\kappa)$. By (i), this homomorphism is surjective. Since $\widetilde{G} \times_{\mathcal{O}} F$ and $\widetilde{G} \times_{\mathcal{O}} \kappa$ have the same root data, (iii) follows. □

LEMMA 2.2. *The subgroup $[G(F), G(F)]$ is closed in $G(F)$ (with respect to the p-adic topology).*

Proof. Let $1 \to Z \to \widetilde{G} \xrightarrow{r} G \to 1$ be a z-extension. By the surjectivity of $\widetilde{G}(F) \to G(F)$, we have $[G(F), G(F)] = r([\widetilde{G}(F), \widetilde{G}(F)])$. Since $[\widetilde{G}, \widetilde{G}]$ is simply connected, we have $[\widetilde{G}(F), \widetilde{G}(F)] = [\widetilde{G}, \widetilde{G}](F)$. The map $[\widetilde{G}, \widetilde{G}](F) \to [G, G](F)$ is an open map [BZ76, A.3. Lemma]. Therefore $[G(F), G(F)]$ is open in $[G, G](F)$. Hence $[G(F), G(F)]$ is closed in $[G, G](F)$. Since $[G, G](F)$ is a closed subgroup of $G(F)$, $[G(F), G(F)]$ is closed in $G(F)$. □

2.2 Satake transform and irreducible representations of K

Let $\overline{\kappa}$ be an algebraic closure of κ. Recall that *all representations in this paper are smooth representations over $\overline{\kappa}$.* For a finite-dimensional representation V of K, let $\text{c-Ind}_K^G V$ be a representation defined by

$$\text{c-Ind}_K^G V = \{f \colon G \to V \mid f(xk) = k^{-1} f(x)(x \in G, k \in K),\ \mathrm{supp}\, f \text{ is compact}\}.$$

The action of $g \in G$ is given by $(gf)(x) = f(g^{-1}x)$. For $x \in G$ and $v \in V$, let $[x, v] \in \text{c-Ind}_K^G(V)$ be the element defined by $\mathrm{supp}([x, v]) = xK$ and $[x, v](x) = v$. Then $g[x, v] = [gx, v]$ and $[xk, v] = [x, kv]$ for $g \in G$ and $k \in K$. For finite-dimensional representations V_1, V_2 of K, $\mathrm{Hom}_G(\text{c-Ind}_K^G V_1, \text{c-Ind}_K^G V_2)$ is identified with

$$\mathcal{H}_G(V_1, V_2) = \left\{ \varphi \colon G \to \mathrm{Hom}_{\overline{\kappa}}(V_1, V_2) \;\middle|\; \begin{array}{l} \varphi(k_2 x k_1) = k_2 \varphi(x) k_1 \ (k_1, k_2 \in K, x \in G), \\ \mathrm{supp}\, \varphi \text{ is compact} \end{array} \right\}.$$

The operator corresponding to $\varphi \in \mathcal{H}_G(V_1, V_2)$ is given by $f \mapsto \varphi * f$ where

$$(\varphi * f)(x) = \sum_{y \in G/K} \varphi(y) f(xy).$$

We denote $\mathcal{H}_G(V, V)$ by $\mathcal{H}_G(V)$. Let π be a representation of G. Then by the Frobenius reciprocity law, we have $\mathrm{Hom}_K(V, \pi) \simeq \mathrm{Hom}_G(\text{c-Ind}_K^G(V), \pi)$. Hence $\mathrm{Hom}_K(V, \pi)$ is a right $\mathcal{H}_G(V)$-module. We denote the action of $\varphi \in \mathcal{H}_G(V)$ on $\psi \in \mathrm{Hom}_K(V, \pi)$ by $\psi * \varphi$.

When V is irreducible, the structure of $\mathcal{H}_G(V)$ is given by the *Satake transform* [Her11b]. Namely, the Satake transform $S_G \colon \mathcal{H}_G(V) \to \mathcal{H}_T(V^{\overline{U}(\kappa)})$ defined by

$$S_G(\varphi)(t) = \sum_{u \in \overline{U}/\overline{U}(\mathcal{O})} \varphi(ut)|_{V^{\overline{U}(\kappa)}}$$

is injective and its image is $\{\varphi \in \mathcal{H}_T(V^{\overline{U}(\kappa)}) \mid \mathrm{supp}\, \varphi \subset T_+\}$ where $T_+ = \{t \in T \mid \alpha(t) \in \mathcal{O}$ $(\alpha \in \Delta^+)\}$.

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Remark 2.3. The convention about positive and negative are interchanged comparing to Herzig's papers [Her11a, Her11b].

Herzig [Her11a] defined another homomorphism ${}'S_G\colon \mathcal{H}_G(V) \to \mathcal{H}_T(V_{U(\kappa)})$ and, under the identification $V^{\overline{U}(\kappa)} \xrightarrow{\sim} V_{U(\kappa)}$, he proved $S_G = {}'S_G$ if the derived group of G is simply connected [Her11a, Corollary 2.19].

Lemma 2.4. *For any G, $S_G = {}'S_G$.*

Proof. Let $\widetilde{G} \to G$ be a z-extension and Z the kernel of $\widetilde{G} \to G$. Take a hyperspecial maximal compact subgroup $\widetilde{K} \subset \widetilde{G}$ as in Lemma 2.1. Using the surjective homomorphism $\widetilde{K} \to K$, we regard V as an irreducible representation of $\widetilde{K}$. Define $\mathcal{H}_{\widetilde{G}}(V) \to \mathcal{H}_G(V)$ by $\varphi \mapsto (g \mapsto \sum_{z \in Z/(Z\cap\widetilde{K})} \varphi(\widetilde{g}z))$; here $\widetilde{g} \in \widetilde{G}$ is a lift of $g \in G$. (Notice that $Z \cap \widetilde{K}$ acts on V trivially.) The same formula defines a homomorphism $\mathcal{H}_{\widetilde{T}}(V^{\overline{U}(\kappa)}) \to \mathcal{H}_T(V^{\overline{U}(\kappa)})$, here $\widetilde{T}$ is the inverse image of T. Then we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{H}_{\widetilde{G}}(V) & \xrightarrow{S_{\widetilde{G}}} & \mathcal{H}_{\widetilde{T}}(V^{\overline{U}(\kappa)}) \\ \downarrow & & \downarrow \\ \mathcal{H}_G(V) & \xrightarrow[S_G]{} & \mathcal{H}_T(V^{\overline{U}(\kappa)}) \end{array}$$

We have a similar diagram for ${}'S_{\widetilde{G}}$ and ${}'S_G$. Since $\mathcal{H}_{\widetilde{G}}(V) \to \mathcal{H}_G(V)$ is surjective, $S_{\widetilde{G}} = {}'S_{\widetilde{G}}$ implies $S_G = {}'S_G$. ∎

Using this lemma, we identify S_G with ${}'S_G$ and we always denote it by S_G.

A homomorphism $X_* \times T(\mathcal{O}) \to T$ defined by $(\lambda, t_0) \mapsto \lambda(\varpi)t_0$ is an isomorphism and it induces $X_{*,+} \times T(\mathcal{O}) \simeq T_+$. Hence S_G gives an isomorphism $\mathcal{H}_G(V) \simeq \overline{\kappa}[X_{*,+}]$. For $\lambda \in X_{*,+}$, there exists $T_\lambda \in \mathcal{H}_G(V)$ such that $\operatorname{supp} T_\lambda = K\lambda(\varpi)K$ and $T_\lambda(\lambda(\varpi))$ is given by $V \twoheadrightarrow V_{N_\lambda(\kappa)} \simeq V^{\overline{N_\lambda}(\kappa)} \hookrightarrow V$. Then $\{T_\lambda \mid \lambda \in X_{*,+}\}$ gives a basis of $\mathcal{H}_G(V)$. When we want to emphasize the group G, we write T_λ^G instead of T_λ. For $\lambda \in X_*$, let $\tau_\lambda \in \overline{\kappa}[X_*]$ be an element corresponding to λ. (As an element of $\mathcal{H}_T(V^{\overline{U}(\kappa)})$, the support of τ_λ is $T(\mathcal{O})\lambda(\varpi)$ and $\tau_\lambda(\lambda(\varpi)) = \mathrm{id}$.) Then $\{\tau_\lambda \mid \lambda \in X_{*,+}\}$ gives a basis of $\overline{\kappa}[X_{*,+}]$. The relation between $S_G(T_\lambda)$ and τ_λ is given by Herzig [Her11a, Proposition 5.1]. An algebra homomorphism $\overline{\kappa}[X_{*,+}] \to \overline{\kappa}$ is parameterized by (M, χ_M) where M is the Levi subgroup of a standard parabolic subgroup and χ_M is a group homomorphism $X_{M,*,0} \to \overline{\kappa}^\times$ where $X_{M,*,0} = \{\lambda \in X_* \mid \langle \lambda, \Pi_M \rangle = 0\}$ [Her11a, Proposition 4.1]. Therefore, an algebra homomorphism $\mathcal{H}_G(V) \to \overline{\kappa}$ is parameterized by the same pair.

Remark 2.5. Since the isomorphism $\mathcal{H}_T(V^{\overline{U}(\kappa)}) \simeq \overline{\kappa}[X_*]$ depends on a choice of a uniformizer ϖ, the above parameterization is not natural. A more natural way is given by Herzig [Her11b, Corollary 1.5]. In this paper, we fix a uniformizer and identify $\mathcal{H}_G(V)$ with $\overline{\kappa}[X_{*,+}]$. (It is only for a simplification of notation.)

Let $P = MN$ be the Levi decomposition of a standard parabolic subgroup. Then the partial Satake transform $S_G^M\colon \mathcal{H}_G(V) \to \mathcal{H}_M(V^{\overline{N}(\kappa)})$ is injective and it satisfies $S_M \circ S_G^M = S_G$ [Her11a, § 2.3]. We also have ${}'S_G^M$. By Lemma 2.4, we have $S_G^M = {}'S_G^M$ under the identification $V^{\overline{N}(\kappa)} \simeq V_{N(\kappa)}$. Assume that $\chi\colon \mathcal{H}_G(V) \to \overline{\kappa}$ is parameterized by (M, χ_M). Then M is characterized by the following property: χ factors through $S_G^{M'}$ if and only if $M' \supset M$. We also have the following: $\chi_M(\lambda) = \chi(\tau_\lambda)^{-1}$ for all $\lambda \in X_{M,*,0} \cap X_{*,+}$.

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Let V_1, V_2 be irreducible representations of K. For each $\lambda \in X_{*,+}$, there exists $\varphi \in \mathcal{H}(V_1, V_2)\backslash\{0\}$ whose support is $K\lambda(\varpi)K$ if and only if $(V_1)_{N_\lambda(\kappa)} \simeq (V_2)_{N_\lambda(\kappa)}$ as $M_\lambda(\kappa)$-representations. Moreover, such φ is unique up to a constant multiple. The homomorphism $\varphi(\lambda(\varpi))$ is given by $V_1 \twoheadrightarrow (V_1)_{N_\lambda(\kappa)} \simeq V_2^{\overline{N_\lambda}(\kappa)} \hookrightarrow V_2$. (See the proof of [Her11a, Proposition 6.3].)

All irreducible representations of K factor through $K \to G(\kappa)$. If the derived group of G is simply connected, such representation is parameterized by its lowest weight. If $\nu \in X^*$ satisfies $-q < \langle \nu, \check{\alpha} \rangle \leqslant 0$ for all $\alpha \in \Pi$ then the restriction of the irreducible representation of $G(\overline{\kappa})$ with lowest weight ν to $G(\kappa)$ is irreducible and they give all irreducible representations of $G(\kappa)$. When V is the restriction of an irreducible representation with lowest weight ν, we call ν a lowest weight of V. (For $\nu_0 \in X^*$ such that $\langle \nu_0, \check{\Pi} \rangle = 0$, the restriction of the irreducible representations with lowest weight ν and $\nu + (q-1)\nu_0$ are isomorphic to each other. Hence ν is not determined by V uniquely.)

3. Satake parameters

3.1 Definition and some lemmas

We start with the following definition.

Definition 3.1. Let π be a representation of G. An algebra homomorphism $\chi\colon \overline{\kappa}[X_{*,+}] \to \overline{\kappa}$ is called a *Satake parameter of* π if there exist an irreducible K-representation V and $\psi \in \operatorname{Hom}_K(V, \pi)\backslash\{0\}$ such that for all $\varphi \in \mathcal{H}_G(V)$, $\psi * \varphi = \chi(S_G(\varphi))\psi$.

Let $\mathcal{S}(\pi, V)$ be the set of Satake parameters appearing in $\operatorname{Hom}_K(V, \pi)$. We denote the set of Satake parameters of π by $\mathcal{S}(\pi)$. Then we have $\mathcal{S}(\pi) = \bigcup_V \mathcal{S}(\pi, V)$. If π is admissible, then $\mathcal{S}(\pi) \neq \emptyset$. We give some propositions about Satake parameters. Before proving some properties of Satake parameters, we give some fundamental facts about a structure of G.

Lemma 3.2. *Let $\Pi = \Pi_1 \cup \Pi_2$ be a partition of Π such that $\langle \Pi_1, \check{\Pi}_2 \rangle = 0$ and $P_i = M_i N_i$ the standard parabolic subgroup corresponding to Π_i. Let L_2 be the subgroup of $T \subset M_1$ generated by $\{\check{\alpha}(F^\times) \mid \alpha \in \Pi_2\}$. Then we have $G/[M_2(F), M_2(F)] \simeq M_1/L_2$.*

Notice that L_2 is not the group of F-valued points of an algebraic group in general.

Proof. First we assume that the derived group of G is simply connected. Let $\overline{F}$ be a separable closure of F. In this proof, we write $\mathbf{G} = G(\overline{F})$. (The same notation is used for other groups.) Let $\mathbf{L}_2$ be the subgroup of $\mathbf{T}$ generated by $\{\check{\alpha}(\overline{F}^\times) \mid \alpha \in \Pi_2\}$. Namely, $\mathbf{L}_2$ is the image of $(\overline{F}^\times)^{\Pi_2} \to \mathbf{T}$. Since the derived group of G is simply connected, this map is injective. Therefore, $L_2 = \mathbf{L}_2^{\operatorname{Gal}(\overline{F}/F)}$.

Set $\check{\Pi}_2^\perp = \{\nu \in X^* \mid \langle \nu, \check{\Pi}_2 \rangle = 0\}$. Since $\mathbf{G}/[\mathbf{M}_2, \mathbf{M}_2]$ and $\mathbf{M}_1/\mathbf{L}_2$ have the same root data $(\check{\Pi}_2^\perp, \Delta_{M_1}, X_*/\mathbb{Z}\check{\Pi}_2, \check{\Delta}_{M_1})$, these are isomorphic. Since the derived group of $\mathbf{G}$ is simply connected, so is $[\mathbf{M}_2, \mathbf{M}_2]$. Hence the Galois cohomology $H^1(F, [\mathbf{M}_2, \mathbf{M}_2])$ is trivial. Therefore $(\mathbf{G}/[\mathbf{M}_2, \mathbf{M}_2])^{\operatorname{Gal}(\overline{F}/F)} = G/([M_2, M_2](F))$. Using the fact that $[\mathbf{M}_2, \mathbf{M}_2]$ is simply connected again, $[M_2, M_2](F) = [M_2(F), M_2(F)]$. Since $\mathbf{L}_2$ is a split torus, $H^1(F, \mathbf{L}_2)$ is trivial. Hence $(\mathbf{M}_1/\mathbf{L}_2)^{\operatorname{Gal}(\overline{F}/F)} = M_1/\mathbf{L}_2^{\operatorname{Gal}(\overline{F}/F)} = M_1/L_2$. The lemma follows in this case.

In general, let $r\colon \widetilde{G} \to G$ be a z-extension of G. Define $\widetilde{M}_1$ (respectively $\widetilde{M}_2$, $\widetilde{L}_2$) in the same way as M_1 (respectively M_2, L_2). Then $\widetilde{M}_1$ and $\widetilde{M}_2$ are the inverse images of M_1 and M_2, respectively. In particular, $r([\widetilde{M}_2(F), \widetilde{M}_2(F)]) = [M_2(F), M_2(F)]$. By the definition, $r(\widetilde{L}_2) = L_2$. By the above argument, we have $\widetilde{G}/[\widetilde{M}_2(F), \widetilde{M}_2(F)] \simeq \widetilde{M}_1/\widetilde{L}_2$. Consider $f\colon M_1 \hookrightarrow G \to G/[M_2(F), M_2(F)]$. We prove f is surjective and $\operatorname{Ker}(f) = L_2$.

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Let $g \in G$ and take $\widetilde{g} \in \widetilde{G}$ such that $r(\widetilde{g}) = g$. Then there exist $\widetilde{m}_1 \in \widetilde{M}_1$ and $\widetilde{m}_2 \in [\widetilde{M}_2(F), \widetilde{M}_2(F)]$ such that $\widetilde{g} = \widetilde{m}_1\widetilde{m}_2$. Hence $g = r(\widetilde{g}) = r(\widetilde{m}_1)r(\widetilde{m}_2) \in M_1[M_2(F), M_2(F)]$. Therefore, f is surjective.

Take $m \in M_1 \cap [M_2(F), M_2(F)]$. Take $\widetilde{m}_1 \in \widetilde{M}_1$ and $\widetilde{m}_2 \in [\widetilde{M}_2(F), \widetilde{M}_2(F)]$ such that $m = r(\widetilde{m}_1) = r(\widetilde{m}_2)$. Then $\widetilde{m}_2 \in \widetilde{m}_1 \operatorname{Ker}(r) \subset \widetilde{M}_1 \operatorname{Ker}(r) = \widetilde{M}_1$. Hence $\widetilde{m}_2 \in \widetilde{M}_1 \cap [\widetilde{M}_2(F), \widetilde{M}_2(F)] \subset \widetilde{L}_2$. Therefore, $m = r(\widetilde{m}_2) \in L_2$. Hence $\operatorname{Ker}(f) \subset L_2$. Let $m \in L_2$ and take $\widetilde{m} \in \widetilde{L}_2$ such that $r(\widetilde{m}) = m$. Then $\widetilde{m} \in [\widetilde{M}_2(F), \widetilde{M}_2(F)]$. Hence $m \in r([\widetilde{M}_2(F), \widetilde{M}_2(F)]) = [M_2(F), M_2(F)]$. Hence $L_2 \subset \operatorname{Ker}(f)$. □

Proposition 3.3. *There is a one-to-one correspondence between characters ν_G of G and characters ν_T of T such that $\nu_T \circ \check{\alpha}$ is trivial for all $\alpha \in \Pi$. It is characterized by $\nu_T = \nu_G|_T$.*

Proof. Apply the previous lemma for $\Pi_1 = \emptyset$ and $\Pi_2 = \Pi$. □

Corollary 3.4. *Let ν_K be a character of K. Then there exists a character ν_G of G such that $\nu_K = \nu_G|_K$. Moreover, there is a unique character ν_G of G such that $\nu_K = \nu_G|_K$ and $\nu_G(\lambda(\varpi)) = 1$ for all $\lambda \in X_*$.*

Proof. If the derived group of G is simply connected, it is known that ν_K has a lowest weight ν which satisfies $(\nu \circ \check{\alpha})(\mathcal{O}^\times) = 1$ for all $\alpha \in \Pi$. Therefore, the corollary follows from the above proposition. In general, let $1 \to Z \to \widetilde{G} \to G \to 1$ be a z-extension of G, $\widetilde{K}$ as in Lemma 2.1 and $\widetilde{T}$ the inverse image of T in $\widetilde{G}$. Then there exists a character $\nu_{\widetilde{G}}$ such that $\nu_{\widetilde{G}}|_{\widetilde{K}}$ is a pull-back of ν_K and $\nu_{\widetilde{G}}(\lambda(\varpi)) = 1$ for all $\lambda \in X_*(\widetilde{T})$. Hence $\nu_{\widetilde{G}}|_Z$ is trivial. Therefore, it gives a character ν_G of G and $\nu_G|_K = \nu_K$. □

For a character ν of G, $\varphi \mapsto (g \mapsto \varphi_\nu(g) = \varphi(g)\nu(g))$ gives an isomorphism $\mathcal{H}_G(V) \simeq \mathcal{H}_G(V \otimes \nu|_K)$. The following lemma and propositions are essentially proved in [Her11a].

Lemma 3.5 [Her11a, Lemma 4.6]. *For a standard parabolic subgroup $P = MN$, the homomorphism $\varphi \mapsto \varphi_\nu$ is compatible with the partial Satake transform S_G^M.*

Proof. We have

$$(S_G^M \varphi_\nu)(m) = \sum_{\overline{n} \in \overline{N}/(\overline{N} \cap K)} \nu(m\overline{n})\varphi(m\overline{n}).$$

Since $\overline{N} \subset [G, G]$, we have $\nu(\overline{n}) = 1$. Therefore,

$$\sum_{\overline{n} \in \overline{N}/(\overline{N} \cap K)} \nu(m\overline{n})\varphi(m\overline{n}) = \nu(m) \sum_{\overline{n} \in \overline{N}/(\overline{N} \cap K)} \varphi(m\overline{n}) = \nu(m)(S_G^M \varphi)(m).$$ □

Now we give some properties on Satake parameters. The following proposition is obvious.

Proposition 3.6. *If $\pi' \subset \pi$, then $\mathcal{S}(\pi', V) \subset \mathcal{S}(\pi, V)$.*

The following proposition follows from [Her11a, Lemma 2.14].

Proposition 3.7. *Let $P = MN$ be a parabolic subgroup, σ a representation of M and V an irreducible representation of K. Then we have $\mathcal{S}(\operatorname{Ind}_P^G(\sigma), V) = \mathcal{S}(\sigma, V^{\overline{N}(\kappa)})|_{\overline{\kappa}[X_{*,+}]}$. In particular, we have $\mathcal{S}(\operatorname{Ind}_P^G(\sigma)) = \mathcal{S}(\sigma)|_{\overline{\kappa}[X_{*,+}]}$.*

Let $\chi_1, \chi_2 \colon \overline{\kappa}[X_{*,+}] \to \overline{\kappa}$ be algebra homomorphisms. Define $\chi_1 \otimes \chi_2 \colon \overline{\kappa}[X_{*,+}] \to \overline{\kappa}$ by $(\chi_1 \otimes \chi_2)(\tau_\lambda) = \chi_1(\tau_\lambda)\chi_2(\tau_\lambda)$.

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PROPOSITION 3.8. *Assume χ_i is parameterized by (M_i, χ_{M_i}). Then $\chi_1 \otimes \chi_2$ is parameterized by (M, χ_M) where $\Pi_M = \Pi_{M_1} \cup \Pi_{M_2}$ and $\chi_M = \chi_{M_1}|_{X_{M,*,0}} \chi_{M_2}|_{X_{M,*,0}}$.*

Proof. If $\chi\colon \overline{\kappa}[X_*] \to \overline{\kappa}$ corresponds to (M, χ_M), for $\lambda \in X_{*,+}$, $\lambda(\varpi) \in Z_M$ if and only if $\chi(\tau_\lambda) \neq 0$ [Her11a, Corollary 4.2]. Hence $\Pi_M = \Pi_{M_1} \cup \Pi_{M_2}$. The formula $\chi_M = \chi_{M_1}|_{X_{M,*,0}} \chi_{M_2}|_{X_{M,*,0}}$ follows from [Her11a, Corollary 4.2]. ☐

PROPOSITION 3.9 [Her11a, Lemma 4.6]. *Let ν be a character of G and π a representation of G. Then $\mathcal{S}(\pi \otimes \nu) = \mathcal{S}(\pi) \otimes \chi_\nu$ where $\chi_\nu\colon \overline{\kappa}[X_{*,+}] \to \overline{\kappa}$ is given by $\chi_\nu(\tau_\lambda) = \nu(\lambda(\varpi))^{-1}$.*

Proof. This follows from Lemma 3.5. ☐

PROPOSITION 3.10. *Let ν be a character of G. Then $\mathcal{S}(\nu) = \{\chi_\nu\}$.*

Proof. We have an injective homomorphism $\nu \hookrightarrow \operatorname{Ind}_B^G(\nu|_T)$. Hence we have $\emptyset \neq \mathcal{S}(\nu) \subset \mathcal{S}(\operatorname{Ind}_B^G(\nu|_T)) = \mathcal{S}(\nu|_T)|_{\overline{\kappa}[X_{*,+}]} = \{\chi_\nu\}$. ☐

3.2 Restriction and Satake parameter

Let G_1 be a connected subgroup of G which contains the derived group of G. Put $K_1 = G_1 \cap K$. This is a hyperspecial maximal compact subgroup of G_1. We also denote the $\mathcal{O}$-form corresponding to K_1 by the same letter G_1.

LEMMA 3.11. *The restriction of an irreducible K-representation to K_1 is also irreducible.*

Proof. We may replace K (respectively K_1) with $G(\kappa)$ (respectively $G_1(\kappa)$). Let V be an irreducible representation of $G(\kappa)$, $V_1 \subset V$ a non-zero $G_1(\kappa)$-subrepresentation of V. Since $U(\kappa) \subset G_1(\kappa)$, we have $V_1^{U(\kappa)} \subset V^{U(\kappa)}$. The group $U(\kappa)$ is a p-group, hence $V_1^{U(\kappa)} \neq 0$. Since $\dim V^{U(\kappa)} = 1$, we have $V_1^{U(\kappa)} = V^{U(\kappa)}$. Let $\tau\colon G \to G$ be an anti-involution such that $\tau|_T = \mathrm{id}_T$. Since G_1 is generated by $U, \overline{U}$ and $T \cap G_1$, and we have $\tau(T \cap G_1) = T \cap G_1$, $\tau(U) = \overline{U}$ and $\tau(\overline{U}) = U$, τ preserves G_1. We have a perfect paring $\langle \cdot, \cdot \rangle\colon V \times V \to \overline{\kappa}$ such that $\langle gv, v' \rangle = \langle v, \tau(g)v' \rangle$ for $g \in G$, $v, v' \in V$ and $\langle V^{U(\kappa)}, V^{U(\kappa)} \rangle \neq 0$. (See an argument in [Hum06, p. 18].) Put $V_1' = \{v \in V \mid \langle v, V_1 \rangle = 0\}$. Then this is a $G_1(\kappa)$-subrepresentation. If it is not zero, then, by the above argument, we have $(V_1')^{U(\kappa)} = V^{U(\kappa)}$. This contradicts $\langle V^{U(\kappa)}, V^{U(\kappa)} \rangle \neq 0$. Therefore, $V_1' = 0$. Hence $V = V_1$. ☐

Let $X_{G_1,*}$ be the group of cocharacters of $G_1 \cap T$. Put $X_{G_1,*,+} = X_{*,+} \cap X_{G_1,*}$. Then we have $\mathcal{H}_{G_1}(V) \simeq \overline{\kappa}[X_{G_1,*,+}]$. Since $X_{G_1,*,+} \subset X_{*,+}$, we have an injective homomorphism $\overline{\kappa}[X_{G_1,*,+}] \hookrightarrow \overline{\kappa}[X_{*,+}]$. This induces $\Phi\colon \mathcal{H}_{G_1}(V) \hookrightarrow \mathcal{H}_G(V)$.

LEMMA 3.12. *We have $\operatorname{Im}\Phi = \{\varphi \in \mathcal{H}_G(V) \mid \operatorname{supp}\varphi \subset G_1K\}$ and the isomorphism $\operatorname{Im}\Phi \simeq \mathcal{H}_{G_1}(V)$ is given by $\varphi \mapsto \varphi|_{G_1}$.*

Proof. Put $\mathcal{H}_1 = \{\varphi \in \mathcal{H}_G(V) \mid \operatorname{supp}\varphi \subset G_1K\}$. Then $\mathcal{H}_1$ has a basis $\{T_\lambda^G \mid \lambda \in X_{G_1,*,+}\}$. To prove the first statement of the lemma, it is sufficient to prove that if $\lambda \in X_{G_1,*,+}$ then $S_G(T_\lambda^G) \in \overline{\kappa}[X_{G_1,*,+}]$ and $\{S_G(T_\lambda^G) \mid \lambda \in X_{G_1,*,+}\}$ is a basis of $\overline{\kappa}[X_{G_1,*,+}]$. We have $S_G(T_\lambda^G) \in \tau_\lambda + \sum_{\mu<\lambda} \overline{\kappa}\tau_\mu$. Since $\check{\Pi} \subset X_{G_1,*}$, $\lambda \in X_{G_1,*}$ and $\mu \leqslant \lambda$ imply $\mu \in X_{G_1,*}$. Therefore we get the first statement.

Since U is the unipotent radical of the Borel subgroup $B \cap G_1$ of G_1, we have $S_G(T_\lambda^G) = S_{G_1}(T_\lambda^G|_{G_1})$ for $\lambda \in X_{G_1,*,+}$ by the definition of the Satake transform. We get the second statement. ☐

Lemma 3.13. *Let ω be a character of Z_G, V_1 an irreducible representation of K_1 such that Z_{K_1} acts on it by $\omega|Z_{K_1}$. Then there exists an irreducible representation V of K such that $V|_{K_1} = V_1$ and the center of K acts on it by ω.*

Proof. Using a z-extension and the argument in the proof of Lemma 3.11, we may assume that the derived group of G is simply connected. Let $\nu_1 \in X^*_{G_1}$ be a lowest weight of V_1. There exists $\omega_1 \in X^*_{Z_G}$ such that $\omega|_{Z_G \cap K}$ is given by $Z_G \cap K \xrightarrow{\omega_1} \mathcal{O}^\times \to \kappa^\times$. (The character ω_1 gives a continuous character $Z_G \to F^\times$ and the image of $Z_G \cap K$ is a compact subgroup, hence it is contained in $\mathcal{O}^\times$.) By the assumption, $\nu_1|_{Z_{G_1}}$ and $\omega_1|_{Z_{G_1}}$ give the same character of $Z_{G_1} \cap K$. Therefore $\nu_1|_{Z_{G_1}} - \omega_1|_{Z_{G_1}} = (q-1)\omega_2$ for some $\omega_2 \in X^*_{Z_{G_1}}$. Take $\omega_3 \in X^*_{Z_G}$ such that $\omega_3|_{Z_{G_1}} = \omega_2$. Set $\omega_4 = \omega_1 + (q-1)\omega_3$. Then ω_4 gives the character $\omega|_{Z_G \cap K}$ of $Z_G \cap K$ and $\nu_1|_{Z_{G_1}} = \omega_4|_{Z_{G_1}}$. We have an exact sequence $1 \to Z_{G_1} \to Z_G \times (G_1 \cap T) \to T \to 1$ as algebraic groups. Hence we get an exact sequence $0 \to X^*_G \to X^*_{G_1} \oplus X^*_{Z_G} \to X^*_{Z_{G_1}} \to 0$. Therefore there exists $\nu \in X^*_G$ such that $\nu|_{T \cap G_1} = \nu_1$ and $\nu|_{Z_G} = \omega_4$. Then the irreducible representation V of K with a lowest weight ν satisfies the condition of the lemma. $\square$

Proposition 3.14. *Let π be a representation of G and V an irreducible representation of K. Then we have $\mathcal{S}(\pi, V)|_{\overline{\kappa}[X_{G_1,*,+}]} \subset \mathcal{S}(\pi|_{G_1}, V|_{G_1 \cap K})$. Hence $\mathcal{S}(\pi)|_{\overline{\kappa}[X_{G_1,*,+}]} \subset \mathcal{S}(\pi|_{G_1})$.*

Moreover, if π has a central character, then for each irreducible $(G_1 \cap K)$-representation V_1, we have $\mathcal{S}(\pi|_{G_1}, V_1) = \bigcup_{V|_{G_1 \cap K} = V_1} \mathcal{S}(\pi, V)|_{\overline{\kappa}[X_{G_1,*,+}]}$. Hence $\mathcal{S}(\pi|_{G_1}) = \mathcal{S}(\pi)|_{\overline{\kappa}[X_{G_1,*,+}]}$.

Proof. Let V be an irreducible representation of K. We prove $\mathcal{S}(\pi, V)|_{\overline{\kappa}[X_{G_1,*,+}]} \subset \mathcal{S}(\pi|_{G_1}, V|_{K_1})$. It is sufficient to prove that

$$\mathrm{Hom}_K(V, \pi) \hookrightarrow \mathrm{Hom}_{K_1}(V, \pi)$$

is an $\mathcal{H}_{G_1}(V)$-module homomorphism. Let $\varphi \in \mathcal{H}_{G_1}(V)$ and $\psi \in \mathrm{Hom}_K(V, \pi)$. Then for each $v \in V$,

$$(\psi * \Phi(\varphi))(v) = \sum_{g \in G/K} g\psi(\Phi(\varphi)(g^{-1})v) = \sum_{g \in G_1K/K} g\psi(\Phi(\varphi)(g^{-1})v).$$

The claim follows from $G_1/K_1 \simeq G_1K/K$.

Assume that π has a central character. Let V_1 be an irreducible representation of K_1. By the above lemma, there exists an irreducible representation V of K such that $V|_{K_1} = V_1$ and a central character of V is the same as that of π. Set $K' = K_1 Z_K$. Since K_1 is open in G_1 and Z_K is open in Z_G, K' is open in $G_1(F)Z_G(F)$. Applying [BZ76, A.3. Lemma] to $G_1 \times Z_G \to G$, $G_1(F)Z_G(F)$ is open in $G = G(F)$. Hence K' is open in G. Therefore, K' has a finite index in K. We have

$$\mathrm{Hom}_{K_1}(V, \pi) = \mathrm{Hom}_{K'}(V, \pi) \simeq \mathrm{Hom}_K(\mathrm{Ind}_{K'}^K(V), \pi).$$

Since V has a structure of a representation of K, we have $\mathrm{Ind}_{K'}^K(V) \simeq \mathrm{Ind}_{K'}^K(\mathbf{1}_{K'}) \otimes V$. Therefore we have

$$\Psi\colon\ \mathrm{Hom}_{K_1}(V, \pi) \simeq \mathrm{Hom}_K(\mathrm{Ind}_{K'}^K(\mathbf{1}_{K'}) \otimes V, \pi).$$

Explicitly, this isomorphism is given by

$$\Psi(\psi)(f \otimes v) = \sum_{x \in K/K'} f(x) x\psi(x^{-1}(v)).$$

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Therefore, for $\varphi \in \mathcal{H}_{G_1}(V)$, we have

$$\begin{aligned}\Psi(\psi * \varphi)(f \otimes v) &= \sum_{x \in K/K'} f(x)x \sum_{g \in G_1/K_1} g\psi(\varphi(g^{-1})x^{-1}v) \\ &= \sum_{x \in K/K'} \sum_{g \in G_1/K_1} f(x)(xg)\psi(\Phi(\varphi)((xg)^{-1})v).\end{aligned}$$

Replacing g with $x^{-1}gx$, we have

$$\Psi(\psi * \varphi)(f \otimes v) = \sum_{x \in K/K'} \sum_{g \in G_1/K_1} f(x)gx\psi(x^{-1}\varphi(g^{-1})v) = \sum_{g \in G_1/K_1} g\Psi(\psi)(f \otimes \varphi(g^{-1})v).$$

Since K' is a normal subgroup of K and K/K' is commutative, the representation $\operatorname{Ind}_{K'}^{K}(\mathbf{1}_{K'})$ has a filtration $\{X_i\}$ such that $X_i/X_{i-1} \simeq \nu_i$ for some character ν_i of K. Set $X = \operatorname{Ind}_{K'}^{K}(\mathbf{1}_{K'})$, $Y = \operatorname{Hom}_K(X \otimes V, \pi)$ and $Y_i = \operatorname{Hom}_K(X/X_i \otimes V, \pi)$. Then we see that $\{Y_i\}$ is a filtration of Y and $Y_{i-1}/Y_i \hookrightarrow \operatorname{Hom}_K(\nu_i \otimes V, \pi)$. By the above formula, Y_i is stable under the action of $\varphi \in \mathcal{H}_{G_1}(V)$. Hence φ acts on Y_{i-1}/Y_i. Extend ν_i to a character of G such that ν_i is trivial on G_1. Then we have $\mathcal{H}_G(V) \simeq \mathcal{H}_G(\nu_i \otimes V)$ by $\varphi' \mapsto \varphi'_{\nu_i}$. We have an action of $\Phi(\varphi)_{\nu_i} \in \mathcal{H}_G(\nu_i \otimes V)$ on $\operatorname{Hom}_K(\nu_i \otimes V, \pi)$. We prove that these actions are compatible with $Y_{i-1}/Y_i \hookrightarrow \operatorname{Hom}_K(\nu_i \otimes V, \pi)$.

Since ν_i is trivial on G_1, we have $a \otimes \varphi(g^{-1})v = \Phi(\varphi)_{\nu_i}(g^{-1})(a \otimes v)$ for $g \in G_1$. The function $g \mapsto g\Psi(\psi)(\Phi(\varphi)_{\nu_i}(g^{-1})(a \otimes v))$ is right K-invariant. Therefore,

$$\begin{aligned}\sum_{g \in G_1/K_1} g\Psi(\psi)(a \otimes \varphi(g^{-1})v) &= \sum_{g \in G_1K/K} g\Psi(\psi)(\Phi(\varphi)_{\nu_i}(g^{-1})(a \otimes v)) \\ &= \sum_{g \in G/K} g\Psi(\psi)(\Phi(\varphi)_{\nu_i}(g^{-1})(a \otimes v)) = (\Psi(\psi) * \Phi(\varphi)_{\nu_i})(a \otimes v).\end{aligned}$$

This means that the actions are compatible.

Hence each element of $\mathcal{S}(\pi|_{G_1}, V)$ appears in $\mathcal{S}(\pi, \nu_i \otimes V)|_{\overline{\kappa}[X_{G_1,*,+}]}$ for some i. Since ν_i is trivial on K_1, $(\nu_i \otimes V)|_{K_1} \simeq V|_{K_1} \simeq V_1$. We get $\mathcal{S}(\pi|_{G_1}, V) \subset \bigcup_{V'|_{K_1} = V|_{K_1}} \mathcal{S}(\pi, V')|_{\overline{\kappa}[X_{G_1,*,+}]}$. □

3.3 Satake parameter of tensor product

Consider the setting in Lemma 3.2. Namely, let $\Pi = \Pi_1 \cup \Pi_2$ be a partition of Π such that $\langle \Pi_1, \check{\Pi}_2 \rangle = 0$. Let $P_i = M_i N_i$ be the standard parabolic subgroup corresponding to Π_i. Set $H_2 = Z_{M_2}([M_1, M_1])^\circ$. Put $\Pi_1^\perp = \{\lambda \in X_* \mid \langle \lambda, \Pi_1 \rangle = 0\}$. Then the group of cocharacters of $H_2 \cap T$ is $\Pi_1^\perp$. We also have $[M_2, M_2] \subset H_2 \subset M_2$ (as algebraic groups). Put $X_{H_2,*,+} = X_{*,+} \cap \Pi_1^\perp$. We have $N_2 \subset [M_1, M_1]$.

Fix an irreducible representation V of K and put $V_2 = V^{\overline{N}_2(\kappa)}$. Then V_2 is irreducible as a representation of $M_2 \cap K$. Since $[M_2, M_2] \subset H_2 \subset M_2$ (as algebraic groups), V_2 is also irreducible as a representation of $H_2 \cap K$ (Lemma 3.11). We have $\overline{\kappa}[X_{H_2,*,+}] \hookrightarrow \overline{\kappa}[X_{*,+}]$. Hence we get $\Phi' \colon \mathcal{H}_{H_2}(V_2) \hookrightarrow \mathcal{H}_G(V)$.

Lemma 3.15. *For $m \in M_2$ and $\overline{n} \in \overline{N}_2$, if $m\overline{n} \in KH_2K$, then $\overline{n} \in K$.*

Proof. By the Cartan decompositions, we can choose $\lambda \in X_{H_2,*,+}$, $\lambda_2 \in X_{M_2,*,+}$ and $k_1 \in M_2 \cap K$ such that $m\overline{n} \in K\lambda(\varpi)K$ and $m \in (M_2 \cap K)\lambda_2(\varpi)k_1$. Then we have $\lambda_2(\varpi)(k_1 n k_1^{-1}) \in K\lambda(\varpi)K$. Put $\overline{n}_1 = k_1 \overline{n} k_1^{-1} \in \overline{N}_2$. We prove $\overline{n}_1 \in K$.

By the assumption, we have $\overline{N}_2 \subset M_1$. Therefore, $\lambda_2(\varpi)\overline{n}_1$ is in M_1. Take $\lambda_1 \in X_{M_1,*,+}$ such that $\lambda_2(\varpi)\overline{n}_1 \in (M_1 \cap K)\lambda_1(\varpi)(M_1 \cap K)$. Then $K\lambda_1(\varpi)K \cap K\lambda(\varpi)K \neq \emptyset$. Therefore, $\lambda_1 \in W\lambda$. The Weyl group W preserves each connected component of the root system Δ. Hence W

preserves $\Pi_1^\perp$. Hence $\lambda_1 \in \Pi_1^\perp$. Therefore, $\lambda_1(\varpi)$ commutes with any element of M_1. Hence $\lambda_2(\varpi)\overline{n}_1 \in (M_1 \cap K)\lambda_1(\varpi)(M_1 \cap K) = \lambda_1(\varpi)(M_1 \cap K)$. Therefore, $\lambda_1(\varpi)^{-1}\lambda_2(\varpi)\overline{n}_1 \in K$. We get $\overline{n}_1 \in K$. □

Lemma 3.16. *If $\varphi \in \mathcal{H}_G(V)$ satisfies* $\operatorname{supp}\varphi \subset KH_2K$*, then $S_G^{M_2}(\varphi)(m) = \varphi(m)|_{V_2}$ for $m \in M_2$.*

Proof. By the definition, we have

$$S_G^{M_2}(\varphi)(m) = \sum_{\overline{n} \in \overline{N}_2/\overline{N}_2 \cap K} \varphi(m\overline{n})|_{V_2}.$$

Since $\operatorname{supp}\varphi \subset KH_2K$, this is equal to $\varphi(m)|_{V_2}$ by the above lemma. □

Lemma 3.17. *If $\lambda, \mu \in X_{*,+}$ satisfies $\mu \leqslant \lambda$ and $\lambda \in X_{H_2,*,+}$, then $\lambda - \mu \in \mathbb{Z}_{\geqslant 0}\Pi_2$. In particular, $\mu \in X_{H_2,*,+}$.*

Proof. For each $\alpha \in \Pi$, take $n_\alpha \in \mathbb{Z}_{\geqslant 0}$ such that $\lambda - \mu = \sum_{\alpha \in \Pi} n_\alpha \check{\alpha}$. Then for $\beta \in \Pi_1$, we have $\sum_{\alpha \in \Pi_1} n_\alpha \langle \beta, \check{\alpha}\rangle = -\langle \beta, \mu\rangle \leqslant 0$. Since $(d_\beta\langle\beta,\check{\alpha}\rangle)_{\alpha,\beta\in\Pi_1}$ is symmetric and positive definite for some $d_\alpha > 0$, we have $n_\alpha = 0$ for all $\alpha \in \Pi_1$. □

By the above two lemmas and the argument in the proof of Lemma 3.12, we get the following lemma. (Notice that $\varphi(h)$ induces $V_2 \to V_2$ for $h \in H_2$ since H_2 and N_2 commute with each other.)

Lemma 3.18. *We have* $\operatorname{Im}\Phi' = \{\varphi \in \mathcal{H}_G(V) \mid \operatorname{supp}\varphi \subset KH_2K\}$ *and the isomorphism* $\operatorname{Im}\Phi' \simeq \mathcal{H}_{H_2}(V_2)$ *is given by* $\varphi \mapsto \varphi|_{H_2}$.

By Lemma 3.16, we get $S_G(\varphi) = S_{M_2}(\varphi|_{M_2})$ if $\operatorname{supp}(\varphi) \subset KH_2K$. This means that the map is given by the restriction.

Let π be a representation of G. Consider the following homomorphism

$$\operatorname{Hom}_K(V,\pi) \to \operatorname{Hom}_{M_2\cap K}(V_2,\pi).$$

Since V is generated by V_2 as a K-representation, this is injective. The left-hand side is $\mathcal{H}_G(V) \simeq \overline{\kappa}[X_{*,+}]$-module and the right-hand side is $\mathcal{H}_{M_2}(V_2) \simeq \overline{\kappa}[X_{M_2,*,+}]$-module where $X_{M_2,*,+} = \{\lambda \in X_* \mid \langle\lambda,\alpha\rangle \geqslant 0 \ (\alpha \in \Pi_{M_2})\}$. Therefore, both sides are $\overline{\kappa}[X_{H_2,*,+}]$-modules. We prove that the above embedding is a $\overline{\kappa}[X_{H_2,*,+}]$-modules homomorphism.

Lemma 3.19. *Let π be a representation of G. The homomorphism*

$$\operatorname{Hom}_K(V,\pi) \to \operatorname{Hom}_{M_2\cap K}(V_2,\pi)$$

is a $\overline{\kappa}[X_{H_2,*,+}]$-module homomorphism.

Proof. Let $\varphi \in \mathcal{H}_{H_2}(V_2)$. Take $\psi \in \operatorname{Hom}_K(V,\pi)$ and $v \in V_2$. We have

$$\begin{aligned}(\psi * \Phi'(\varphi))(v) &= \sum_{g \in G/K} g\psi(\Phi'(\varphi)(g^{-1})v) \\ &= \sum_{m \in M_2/(M_2\cap K)} \sum_{\overline{n} \in \overline{N}_2/(\overline{N}_2 \cap K)} m\overline{n}\psi(\Phi'(\varphi)(\overline{n}^{-1}m^{-1})v).\end{aligned}$$

Since $\operatorname{supp}\Phi'(\varphi) \subset KH_2K$, $\Phi'(\varphi)(\overline{n}^{-1}m^{-1}) = 0$ if $\overline{n} \notin \overline{N}_2 \cap K$ by the above lemma. Therefore, we have

$$(\psi * \Phi'(\varphi))(v) = \sum_{m \in M_2/(M_2\cap K)} m\psi(\Phi'(\varphi)(m^{-1})v).$$

Using Lemma 3.16, we obtain the lemma. □

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Let π_1, π_2 be representations of G with the central characters such that $[M_2(F), M_2(F)]$ acts on π_1 trivially and the center of M_1 acts on π_1 by a character. Put $\pi = \pi_1 \otimes \pi_2$.

Remark 3.20. The group H_2 is generated by $H_2 \cap T$ and the one-dimensional unipotent subgroup corresponding to each $\alpha \in \Delta \cap \mathbb{Z}\Pi_2$. Since $H_2 \cap T \subset Z_{M_1}^{\circ}$ and the one-dimensional unipotent subgroup corresponding to $\alpha \in \Delta \cap \mathbb{Z}\Pi_2$ is a subgroup of $[M_2(F), M_2(F)]$, H_2 is generated by $[M_2(F), M_2(F)]$ and $Z_{M_1}^{\circ}$. Therefore, H_2 acts on π_1 by a scalar.

Proposition 3.21. *We have* $\mathcal{S}(\pi)|_{\overline{\kappa}[X_{H_2,*,+}]} \subset \mathcal{S}(\pi_1|_{H_2}) \otimes \mathcal{S}(\pi_2|_{H_2})$.

Proof. We have $\mathcal{S}(\pi)|_{\overline{\kappa}[X_{H_2,*,+}]} \subset \mathcal{S}(\pi|_{M_2})|_{\overline{\kappa}[X_{H_2,*,+}]}$ by the above lemma. By Proposition 3.14, we have $\mathcal{S}(\pi|_{M_2})|_{\overline{\kappa}[X_{H_2,*,+}]} \subset \mathcal{S}(\pi|_{H_2})$. Since H_2 acts on π_1 by a scalar, $\mathcal{S}(\pi|_{H_2}) = \mathcal{S}(\pi_1|_{H_2}) \otimes \mathcal{S}(\pi_2|_{H_2})$ by Lemma 3.9 and Proposition 3.10. □

We give some corollaries of Proposition 3.21 which we will use. We make the following additional assumptions.

– The derived group $[M_1(F), M_1(F)]$ acts on π_2 trivially and the center of M_2 acts on π_2 by a character.
– We have $\#\mathcal{S}(\pi_1|_{M_1}) = \#\mathcal{S}(\pi_2|_{M_2}) = 1$.

Since $\#\mathcal{S}(\pi_1|_{M_1}) = \#\mathcal{S}(\pi_2|_{M_2}) = 1$, there exists a unique parabolic subgroup $P = MN$ such that $\mathcal{S}(\pi_1|_{M_1}) = \{\chi_1 = (M \cap M_1, \chi_{M\cap M_1})\}$ and $\mathcal{S}(\pi_2|_{M_2}) = \{\chi_2 = (M \cap M_2, \chi_{M\cap M_2})\}$ for some $\chi_{M\cap M_1}$ and $\chi_{M\cap M_2}$.

Corollary 3.22. *Any $\chi \in \mathcal{S}(\pi)$ is parameterized by (M, χ_M) for some χ_M.*

Proof. Take M' and $\chi_{M'}$ such that χ is parameterized by $(M', \chi_{M'})$. For each $\alpha \in \Pi$, take $\lambda_\alpha \in X_{*,+}$ such that $\langle \Pi\backslash\{\alpha\}, \lambda_\alpha\rangle = 0$ and $\langle \alpha, \lambda_\alpha\rangle \neq 0$. Then M' corresponds to $\{\alpha \in \Pi \mid \chi(\tau_{\lambda_\alpha}) = 0\}$ [Her11a, Proof of Proposition 2.12]. If $\alpha \in \Pi_2$, then $\lambda_\alpha \in X_{H_2,*,+}$. Therefore, there exist $\chi_1' \in \mathcal{S}(\pi_1|_{H_2})$ and $\chi_2' \in \mathcal{S}(\pi_2|_{H_2})$ such that $\chi(\tau_{\lambda_\alpha}) = \chi_1'(\tau_{\lambda_\alpha})\chi_2'(\tau_{\lambda_\alpha})$ by Proposition 3.21. Since $\pi_1|_{H_2}$ is a direct sum of characters, $\chi_1'(\tau_{\lambda_\alpha}) \neq 0$ by Proposition 3.10. Hence $\chi(\tau_{\lambda_\alpha}) = 0$ if and only if $\chi_2'(\tau_{\lambda_\alpha}) = 0$. By Proposition 3.14, $\mathcal{S}(\pi_2|_{H_2}) = \mathcal{S}(\pi_2|_{M_2})|_{\overline{\kappa}[X_{H_2,*,+}]} = \{\chi_2\}|_{\overline{\kappa}[X_{H_2,*,+}]}$. Therefore, we have $\chi_2'(\tau_{\lambda_\alpha}) = \chi_2(\tau_{\lambda_\alpha})$. It is zero if and only if $\alpha \in \Pi_M \cap \Pi_2$. By the same argument, for $\alpha \in \Pi_1$, $\chi(\tau_{\lambda_\alpha}) = 0$ if and only if $\alpha \in \Pi_M \cap \Pi_1$. Hence $M' = M$. □

Moreover, we assume the following conditions.

– The representation π_1 is an admissible G-representation.
– The representation π_2 is an admissible $[M_2(F), M_2(F)]$-representation.

Lemma 3.23. *Under the above conditions, π is admissible as a representation of G.*

Proof. Let K' be a compact open subgroup. Then we have $\pi^{K'} = (\pi_1 \otimes \pi_2^{[M_2(F),M_2(F)]\cap K'})^{K'}$. Since $\pi_2^{[M_2(F),M_2(F)]\cap K'}$ is finite dimensional, there exists a compact open subgroup $K'' \subset K'$ which acts on $\pi_2^{[M_2(F),M_2(F)]\cap K'}$ trivially. Hence $\pi^{K'} \subset (\pi_1 \otimes \pi_2^{[M_2(F),M_2(F)]\cap K'})^{K''} = \pi_1^{K''} \otimes \pi_2^{[M_2(F),M_2(F)]\cap K'}$. The right-hand side is finite dimensional. □

Corollary 3.24. *If $M = M_1$, then $\mathcal{S}(\pi) = \mathcal{S}(\pi_1) \otimes \mathcal{S}(\pi_2) = \{(M_1, \chi_{M\cap M_1}(\chi_{M\cap M_2}|_{X_{M_1,*,0}}))\}$.*

Proof. Take $\chi \in \mathcal{S}(\pi)$ and let $\chi_M\colon X_{M,*,0} \to \overline{\kappa}^\times$ such that χ is parameterized by (M, χ_M). The character χ_M^{-1} is given by a restriction of χ on $X_{*,+} \cap \Pi_M^\perp = X_{*,+} \cap \Pi_1^\perp = X_{H_2,*,+}$. By Proposition 3.21, we have $\chi|_{\overline{\kappa}[X_{H_2,*,+}]} = (\chi_1 \otimes \chi_2)|_{\overline{\kappa}[X_{H_2,*,+}]}$. Hence, by Proposition 3.8, we have $\chi_M|_{X_{H_2,*}} = (\chi_{M\cap M_1}|_{X_{M,*,0}\cap X_{H_2,*}})(\chi_{M\cap M_2}|_{X_{M,*,0}\cap X_{H_2,*}})$. Since $M = M_1$, $X_{H_2,*} = X_{M,*,0}$.

Therefore, $\chi_M = \chi_{M\cap M_1}(\chi_{M\cap M_2}|_{X_{M_1,*,0}})$. Since π is admissible, $\mathcal{S}(\pi) \neq \emptyset$. So we get the corollary. □

3.4 z-extension and Satake parameters

Let $\widetilde{G} \to G$ be a z-extension and take a hyperspecial maximal compact subgroup $\widetilde{K}$ as in Lemma 2.1. A representation π of G can be regarded as a representation of $\widetilde{G}$. Let $\widetilde{\pi}$ be this representation. Denote the inverse image of T by $\widetilde{T}$ and let $X_{\widetilde{G},*}$ be the group of cocharacters of $\widetilde{T}$. We have a surjective map $X_{\widetilde{G},*} \to X_*$ which induces $X_{\widetilde{G},*,+} \to X_{*,+}$.

Lemma 3.25. *Let $r\colon \overline{\kappa}[X_{\widetilde{G},*,+}] \to \overline{\kappa}[X_{*,+}]$ be the induced homomorphism.*

(i) *We have $\mathcal{S}(\widetilde{\pi}) = \{\chi \circ r \mid \chi \in \mathcal{S}(\pi)\}$.*

(ii) *If $\chi\colon \overline{\kappa}[X_{*,+}] \to \overline{\kappa}$ is parameterized by (M, χ_M), then $\chi \circ r$ is parameterized by $(\widetilde{M}, \chi_{\widetilde{M}})$; here $\widetilde{M}$ is the inverse image of M in $\widetilde{G}$ and $\chi_{\widetilde{M}}$ is the composition $X_{\widetilde{M},*,0} \to X_{M,*,0} \xrightarrow{\chi} \overline{\kappa}^\times$.*

Proof. Let Z be the kernel of $\widetilde{G} \to G$. If an irreducible $\widetilde{K}$-representation V' is a subrepresentation of $\widetilde{\pi}$, then $Z \cap \widetilde{K}$ acts on V' trivially. Therefore, V' comes from an irreducible representation of K. Let $\widetilde{V}$ be an irreducible representation of $\widetilde{K}$ coming from an irreducible representation V of K. To prove (i), it is sufficient to prove that $\operatorname{Hom}_{\widetilde{K}}(\widetilde{V}, \widetilde{\pi}) \simeq \operatorname{Hom}_K(V, \pi)$ as $\overline{\kappa}[X_{\widetilde{G},*,+}]$-modules. (Here, $\overline{\kappa}[X_{\widetilde{G},*,+}]$ acts on $\operatorname{Hom}_K(V, \pi)$ through r.)

As a vector space, $\operatorname{Hom}_{\widetilde{K}}(\widetilde{V}, \widetilde{\pi}) \simeq \operatorname{Hom}_K(V, \pi)$. So it is sufficient to prove that this isomorphism is $\overline{\kappa}[X_{\widetilde{G},*,+}]$-equivariant. Define $r_G\colon \mathcal{H}_{\widetilde{G}}(\widetilde{V}) \to \mathcal{H}_G(V)$ as in the proof of Lemma 2.4. Then it is easy to see that the isomorphism $\operatorname{Hom}_{\widetilde{K}}(\widetilde{V}, \widetilde{\pi}) \simeq \operatorname{Hom}_K(V, \pi)$ is $\mathcal{H}_{\widetilde{G}}(\widetilde{V})$-equivariant; here $\mathcal{H}_{\widetilde{G}}(\widetilde{V})$ acts on $\operatorname{Hom}_K(V, \pi)$ through r_G. Hence by the commutative diagram in Lemma 2.4, it is sufficient to prove that $r = r_T|_{\overline{\kappa}[X_{\widetilde{G},*,+}]}$, where $r_T\colon \overline{\kappa}[X_{\widetilde{G},*}] \simeq \mathcal{H}_{\widetilde{T}}(\widetilde{V}^{\overline{U}(\kappa)}) \to \mathcal{H}_T(V^{\overline{U}(\kappa)}) \simeq \overline{\kappa}[X_*]$ is the homomorphism defined in the proof of Lemma 2.4. This follows from the definition of r and r_T.

Take $(\widetilde{M_1}, \chi'_{\widetilde{M_1}})$ which corresponds to $\chi \circ r$. For $\alpha \in \Pi$, take $\widetilde{\lambda}_\alpha \in X_{\widetilde{G},*,+}$ such that $\langle \widetilde{\lambda}_\alpha, \Pi\backslash\{\alpha\}\rangle = 0$ and $\langle \widetilde{\lambda}_\alpha, \alpha\rangle \neq 0$. Put $\lambda_\alpha = r(\widetilde{\lambda}_\alpha)$. Then $\Pi_{\widetilde{M}} = \Pi_M = \{\alpha \in \Pi \mid \chi(\tau_{\lambda_\alpha}) = 0\} = \{\alpha \in \Pi \mid \chi \circ r(\tau_{\widetilde{\lambda}_\alpha}) = 0\} = \Pi_{\widetilde{M_1}}$. Hence $\widetilde{M_1} = \widetilde{M}$. The homomorphism $\chi'_{\widetilde{M_1}}$ is characterized by $\chi'_{\widetilde{M_1}}|_{X_{\widetilde{M},*,0} \cap X_{\widetilde{G},*,+}} = (\chi \circ r|_{X_{\widetilde{M},*,0} \cap X_{\widetilde{G},*,+}})^{-1}$. The homomorphism $\chi_{\widetilde{M}}$ satisfies the same characterization. Hence $\chi'_{\widetilde{M_1}} = \chi_{\widetilde{M}}$. □

4. A theorem of changing the weight

In this section, we assume that the derived group of G is simply connected. For $\alpha \in \Pi$, we denote a fundamental weight corresponding to α by ω_α.

4.1 Changing the weight

We prove the following theorem, which is a generalization of Herzig's theorem [Her11a, Corollary 6.11].

Theorem 4.1. *Let V_1, V_2 be irreducible representations of K with lowest weight ν_1, ν_2, respectively. Assume that $\langle \nu_1, \check{\alpha}\rangle = 0$ and $\nu_2 = \nu_1 - (q-1)\omega_\alpha$ for some $\alpha \in \Pi$. Let $\chi\colon \overline{\kappa}[X_{*,+}] \to \overline{\kappa}$ be an algebra homomorphism parameterized by (M, χ_M). Assume that $\alpha \notin \Pi_M$. If $\check{\alpha} \notin X_{M,*,0}$*

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or $\chi_M(\check{\alpha}) \neq 1$, *then*

$$\text{c-Ind}_K^G V_1 \otimes_{\mathcal{H}_G(V_1)} \chi \simeq \text{c-Ind}_K^G V_2 \otimes_{\mathcal{H}_G(V_2)} \chi.$$

Let V_1, V_2, ν_1, ν_2 be as above. Fix $\lambda \in X_{*,+}$ such that $\langle \lambda, \Pi \backslash \{\alpha\} \rangle = 0$ and $\langle \lambda, \alpha \rangle \neq 0$. Then there exist non-zero $\varphi_{21} \in \mathcal{H}_G(V_1, V_2)$ and $\varphi_{12} \in \mathcal{H}_G(V_2, V_1)$ whose support is $K\lambda(\varpi)K$. By the proof of [Her11a, Corollary 6.11], Theorem 4.1 follows from the following lemma.

LEMMA 4.2. *We have* $S_G(\varphi_{12} * \varphi_{21}) \in \overline{\kappa}^\times(\tau_{2\lambda} - \tau_{2\lambda - \check{\alpha}})$.

This lemma follows from the following two lemmas by [Her11a, Proposition 5.1]. These also answer Herzig's question [Her11a, Question 6.9].

LEMMA 4.3. *The composition* $\varphi_{12} * \varphi_{21}$ *is non-zero and its support is* $K\lambda(\varpi)^2K$.

LEMMA 4.4. *For* $\mu \in X_{*,+}$, *if* $\mu \leqslant 2\lambda$ *then* $\mu = 2\lambda$ *or* $\mu \leqslant 2\lambda - \check{\alpha}$.

First, we prove Lemma 4.3. For each $w \in W \simeq N_K(T(\mathcal{O}))/T(\mathcal{O})$, we fix a representative of w and denote it by the same letter w.

LEMMA 4.5. *Let* $P = MN$ *be a standard parabolic subgroup. Then we have*

$$G(\mathcal{O}) = \coprod_{w \in W/W_M} w(w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))P(\mathcal{O}).$$

Proof. Since $(w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))(w^{-1}\overline{I}w \cap P(\mathcal{O})) = w^{-1}\overline{I}w$, it is sufficient to prove $G(\mathcal{O}) = \coprod_{w \in W/W_M} \overline{I}wP(\mathcal{O})$. By the Bruhat decomposition $G(\kappa) = \coprod_{w \in W/W_M} \overline{B}(\kappa)wP(\kappa)$, for $g \in G(\mathcal{O})$, there exists $w \in W$ and $p \in P(\mathcal{O})$ such that $(\text{red}(wp))^{-1}\text{red}(g) \in \overline{B}$. Hence $(wp)^{-1}g \in \overline{I}$. Therefore, $g \in \overline{I}wp$. Hence $G(\mathcal{O}) = \bigcup_{w \in W} \overline{I}wP(\mathcal{O})$. Assume that $\overline{I}w_1P(\mathcal{O}) \cap \overline{I}w_2P(\mathcal{O}) \neq \emptyset$ for $w_1, w_2 \in W$. Applying red, we have $\overline{B}(\kappa)w_1P(\kappa) \cap \overline{B}(\kappa)w_2P(\kappa) \neq \emptyset$. Therefore, by the Bruhat decomposition of $G(\kappa)$, we have $w_1 \in w_2W_M$. □

To prove Lemma 4.3, we use the following lemma. We use the argument in the proof of [Her11a, Proposition 6.7].

LEMMA 4.6. *Let* V, V' *be irreducible representations of* K *with lowest weight* ν, ν', *and lowest weight vector* $v \in V, v' \in V'$, *respectively. Assume that for* $\mu \in X_{*,+}$, $V^{\overline{N_\mu}(\kappa)} \simeq (V')^{\overline{N_\mu}(\kappa)}$ *as* $M_\mu(\kappa)$*-representations. Let* $\varphi \in \mathcal{H}_G(V, V')$ *be such that* $\operatorname{supp} \varphi = K\mu(\varpi)K$ *and* $\varphi(\mu(\varpi))v = v'$. *Put* $\overline{I} = \text{red}^{-1}(\overline{B}(\kappa))$ *and* $t = \mu(\varpi)$. *Then we have*

$$\varphi * [1, v] = \sum_{w \in W_{-\nu}/(W_{-\nu} \cap W_\mu)} \sum_{a \in (w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))/t^{-1}\overline{N}(\mathcal{O})t} [wat^{-1}, v'].$$

Proof. We have

$$(\varphi * [1, v])(x) = \sum_{y \in G/K} \varphi(y)[1, v](xy) = \sum_{y \in KtK/K} \varphi(y)[1, v](xy).$$

If this is not zero, then $xy \in K$ for some $y \in KtK$. Hence $x \in Kt^{-1}K$. Namely, $\operatorname{supp}(\varphi * [1, v]) \subset Kt^{-1}K$. The value at $x = kt^{-1}$ for $k \in K$ is

$$(\varphi * [1, v])(kt^{-1}) = \sum_{y \in KtK/K} \varphi(y)[1, v](kt^{-1}y) = \varphi(t)[1, v](k) = \varphi(t)k^{-1}v.$$

Therefore, we have

$$\varphi * [1, v] = \sum_{k \in K/(K \cap t^{-1}Kt)} [kt^{-1}, \varphi(t)k^{-1}v].$$

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Put $P = P_\mu$. We have $K \cap t^{-1}Kt \supset P(\mathcal{O})$ and $\mathrm{red}(K \cap t^{-1}Kt) = P(\kappa)$. Therefore, we have a surjective map $G(\mathcal{O})/P(\mathcal{O}) \twoheadrightarrow K/(K \cap t^{-1}Kt)$. For each $w \in W \simeq N_K(T(\mathcal{O}))/T(\mathcal{O})$, we fix a representative of w and denote it by the same letter w. Then, by the above lemma, we have

$$G(\mathcal{O}) = \coprod_{w \in W/W_\mu} w(w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))P(\mathcal{O}).$$

Hence $\varphi * [1, v]$ is a sum of a form $[wat^{-1}, \varphi(t)a^{-1}w^{-1}v]$ for $a \in w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O})$ and $w \in W/W_\mu$. We prove that $\varphi(t)a^{-1}w^{-1}v \neq 0$ implies $w \in W_{-\nu}W_\mu$. Since $\mathrm{red}(a) \in w^{-1}\overline{B}(\kappa)w \cap \overline{N}(\kappa) \subset w^{-1}\overline{U}(\kappa)w$, we have $a^{-1}w^{-1}v = w^{-1}v$. The homomorphism $\varphi(t)$ is given by $V \twoheadrightarrow (V)_{N_\mu(\kappa)} \simeq (V')^{\overline{N_\mu}(\kappa)} \hookrightarrow V'$. Hence if $\varphi(t)w^{-1}v \neq 0$, then $w^{-1}v \in V^{\overline{N_\mu}(\kappa)}$. Since $\{g \in G(\kappa) \mid gv \in \overline{\kappa}v\} = \overline{P_{-\nu}}(\kappa)$, we have $\overline{P_{-\nu}}(\kappa) \supset w\overline{N_\mu}(\kappa)w^{-1}$. Then $\Delta^-_{-\nu} \cup \Delta^+ \supset w(\Delta^+ \setminus \Delta^+_\mu)$. Hence, $(\Delta^- \setminus \Delta^-_{-\nu}) \cap w(\Delta^+ \setminus \Delta^+_\mu) = \emptyset$. Take $w' \in W_{-\nu}wW_\mu$ such that w' is shortest in $W_{-\nu}wW_\mu$ [Bou02, ch. IV, Exercises, § 1 (3)]. Then $(\Delta^- \setminus \Delta^-_{-\nu}) \cap w'(\Delta^+ \setminus \Delta^+_\mu) = \emptyset$. By the condition of w', $\Delta^- \cap w'(\Delta^+ \setminus \Delta^+_\mu) = \Delta^- \cap w'\Delta^+$ and $(\Delta^- \setminus \Delta^-_{-\nu}) \cap w'\Delta^+ = \Delta^- \cap w'\Delta^+$. Therefore, we have $\Delta^- \cap w'\Delta^+ = \emptyset$. Hence $w' = 1$. We have $w \in W_{-\nu}W_\mu/W_\mu = W_{-\nu}/(W_{-\nu} \cap W_\mu)$. Hence we may assume $w \in W_{-\nu}$. Therefore, $\varphi(t)w^{-1}v = \varphi(t)v = v'$. Hence,

$$\varphi * [1, v] = \sum_{w \in W_{-\nu}/(W_{-\nu} \cap W_\mu)} \sum_{a \in (w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))/(w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}) \cap t^{-1}Kt)} [wat^{-1}, v'].$$

Since $\langle \alpha, \mu \rangle < 0$ for all weights α of $\overline{N}$, $t = \mu(\pi)$ satisfies $t\overline{N}(\mathcal{O})t^{-1} \supset \overline{N}(\mathcal{O})$. Hence $t\overline{N}(\mathcal{O})t^{-1} \cap K = \overline{N}(\mathcal{O})$. Equivalently, we have $\overline{N}(\mathcal{O}) \cap t^{-1}Kt = t^{-1}\overline{N}(\mathcal{O})t$. We also have that $\mathrm{red}(t^{-1}\overline{N}(\mathcal{O})t)$ is trivial. Hence $t^{-1}\overline{N}(\mathcal{O})t \subset w^{-1}\overline{I}w$. Therefore, $w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}) \cap t^{-1}Kt = t^{-1}\overline{N}(\mathcal{O})t$. Hence we have

$$\varphi * [1, v] = \sum_{w \in W_{-\nu}/(W_{-\nu} \cap W_\mu)} \sum_{a \in (w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))/t^{-1}\overline{N}(\mathcal{O})t} [wat^{-1}, v']. \qquad \square$$

Proof of Lemma 4.3. Put $t = \lambda(\varpi)$. Let $v_1 \in V_1, v_2 \in V_2$ be lowest weight vectors. We may assume $\varphi_{21}(t)v_1 = v_2$ and $\varphi_{12}(t)v_2 = v_1$. By Lemma 4.6, we have

$$\varphi_{21} * [1, v_1] = \sum_{w \in W_{-\nu_1}/(W_{-\nu_1} \cap W_\lambda)} \sum_{a \in (w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))/t^{-1}\overline{N}(\mathcal{O})t} [wat^{-1}, v_2].$$

By the assumption, $W_{-\nu_2} \cap W_\lambda = W_{-\nu_2}$. Hence we have

$$\varphi_{12} * [1, v_2] = \sum_{b \in \overline{N}(\mathcal{O})/t^{-1}\overline{N}(\mathcal{O})t} [bt^{-1}, v_1]$$

by Lemma 4.6. Therefore, we have

$$\begin{aligned}\varphi_{12} * \varphi_{21} * [1, v_1] &= \varphi_{12} * \Bigg(\sum_{w \in W_{-\nu_1}/(W_\lambda \cap W_{-\nu_1})} \sum_{a \in (w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))/t^{-1}\overline{N}(\mathcal{O})t} [wat^{-1}, v_2]\Bigg) \\ &= \sum_{w \in W_{-\nu_1}/(W_\lambda \cap W_{-\nu_1})} \sum_{a \in (w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))/t^{-1}\overline{N}(\mathcal{O})t} wat^{-1}\varphi_{12} * [1, v_2] \\ &= \sum_{w \in W_{-\nu_1}/(W_\lambda \cap W_{-\nu_1})} \sum_{a \in (w^{-1}\overline{I}w \cap \overline{N}(\mathcal{O}))/t^{-1}\overline{N}(\mathcal{O})t} \sum_{b \in \overline{N}(\mathcal{O})/t^{-1}\overline{N}(\mathcal{O})t} [wat^{-1}bt^{-1}, v_1]\end{aligned}$$

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$$= \sum_{w\in W_{-\nu_1}/(W_\lambda\cap W_{-\nu_1})}\sum_{a\in(w^{-1}\overline{I}w\cap\overline{N}(\mathcal{O}))/t^{-1}\overline{N}(\mathcal{O})t}\sum_{b\in t^{-1}\overline{N}(\mathcal{O})t/t^{-2}\overline{N}(\mathcal{O})t^2}[wabt^{-2},v_1]$$
$$= \sum_{w\in W_{-\nu_1}/(W_\lambda\cap W_{-\nu_1})}\sum_{c\in(w^{-1}\overline{I}w\cap\overline{N}(\mathcal{O}))/t^{-2}\overline{N}(\mathcal{O})t^2}[wct^{-2},v_1].$$

Let $\varphi\in\mathcal{H}_G(V_1)$, whose support is $K\lambda(\varpi)^2K$, and $\varphi(\lambda(\varpi)^2)v_1=v_1$. By Lemma 4.6, the right-hand side of the above equation is $\varphi*[1,v_1]$. (Notice that $W_\lambda=W_{2\lambda}$.) Since $[1,v_1]$ generates $\text{c-Ind}_K^G(V_1)$, we obtain the lemma. □

Finally, we prove Lemma 4.4.

Proof of Lemma 4.4. Assume that $\mu\leqslant 2\lambda$ and $\mu\not\leqslant 2\lambda-\check{\alpha}$. Since $\mu\leqslant 2\lambda$, there exists $n_\beta\in\mathbb{Z}_{\geqslant 0}$ such that $2\lambda-\mu=\sum_{\beta\in\Pi}n_\beta\check{\beta}$. Then for $\gamma\in\Pi\backslash\{\alpha\}$, we have $\sum_\beta n_\beta\langle\gamma,\check{\beta}\rangle=\langle\gamma,2\lambda-\mu\rangle=-\langle\gamma,\mu\rangle\leqslant 0$. By the assumption, $n_\alpha=0$. Then $\sum_{\beta\neq\alpha}n_\beta\langle\gamma,\check{\beta}\rangle\leqslant 0$. Since $(d_\gamma\langle\gamma,\check{\beta}\rangle)_{\beta,\gamma\in\Pi\backslash\{\alpha\}}$ is symmetric and positive definite for some $d_\gamma>0$, we have $n_\beta=0$ for all $\beta\in\Pi\backslash\{\alpha\}$. Hence $\mu=2\lambda$. □

4.2 Comparison of composition factors

We prove the following proposition in this section.

Proposition 4.7. *Let $\chi\colon\overline{\kappa}[X_{*,+}]\to\overline{\kappa}$ be an algebra homomorphism and V an irreducible representation of K. Assume that χ can be extended to $\overline{\kappa}[X_*]\to\overline{\kappa}$. Then $\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\chi$ has a finite length and its composition factors depend only on χ and the $T(\kappa)$-representation $V^{\overline{U}(\kappa)}$.*

When $G=\mathrm{GL}_2$, this proposition is proved by Barthel–Livné [BL95, Theorem 20].

Before proving this proposition, we give an application. For a parabolic subgroup $P\subset G$, let Sp_P be the special representation [Gro]. If we want to emphasize G, we write $\mathrm{Sp}_{P,G}$. We have the following corollary.

Corollary 4.8. *Let V be an irreducible K-representation such that $V^{\overline{U}(\kappa)}$ is the trivial representation and $\chi\colon\overline{\kappa}[X_*]\to\overline{\kappa}$ an algebra homomorphism parameterized by $(T,\mathbf{1}_{X_{T,*,0}}=\mathbf{1}_{X_*})$. Then the composition factors of $\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\chi$ are $\{\mathrm{Sp}_P\mid P\subset G\}$.*

Proof. Let V_1 be the irreducible K-representation with lowest weight $-\sum_{\alpha\in\Pi}(q-1)\omega_\alpha$. Then we have $V^{\overline{U}(\kappa)}\simeq V_1^{\overline{U}(\kappa)}\simeq\mathbf{1}_{T(\kappa)}$. By Proposition 4.7, we have that $\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\chi$ and $\text{c-Ind}_K^G(V_1)\otimes_{\mathcal{H}_G(V_1)}\chi$ have the same composition factors. By Herzig's theorem [Her11a, Theorem 3.1], we have

$$\text{c-Ind}_K^G(V_1)\otimes_{\mathcal{H}_G(V_1)}\chi\simeq\mathrm{Ind}_B^G(\text{c-Ind}_{T\cap K}^T(\mathbf{1}_{T\cap K})\otimes_{\mathcal{H}_T(\mathbf{1}_{T\cap K})}\chi)=\mathrm{Ind}_B^G(\mathbf{1}_T).$$

Hence the corollary follows from [Her11a, Corollary 7.3]. □

This corollary implies the following proposition. This proposition is proved by Herzig when $G=\mathrm{GL}_n$ [Her11a, Proposition 9.1] in a different way. Let $\mathrm{Ord}_{\overline{P}}(\pi)$ be the ordinary part of π defined by Emerton [Eme10].

Proposition 4.9. *Let π be an admissible representation of G which contains the trivial representation of K. Assume that there exists $\chi\in\mathcal{S}(\pi,\mathbf{1}_K)$ which is parameterized by $(T,\mathbf{1}_{X_{T,*,0}}=\mathbf{1}_{X_*})$. Then π contains the trivial representation, or $\mathrm{Ord}_{\overline{P}}(\pi)\neq 0$ for some proper parabolic subgroup P.*

Proof. From the assumption, we have a non-zero homomorphism $\text{c-Ind}_K^G(\mathbf{1}_K)\otimes_{\mathcal{H}_G(\mathbf{1}_K)}\chi\to\pi$. Hence π contains an irreducible subquotient of $\text{c-Ind}_K^G(\mathbf{1}_K)\otimes_{\mathcal{H}_G(\mathbf{1}_K)}\chi$ as a subrepresentation. By Corollary 4.8, such subquotient is Sp_P for a parabolic subgroup P. If $P=G$, then $\mathbf{1}_G=\mathrm{Sp}_G\subset\pi$. If $P\neq G$, then $0\neq\mathrm{Ord}_{\overline{P}}(\mathrm{Sp}_P)\hookrightarrow\mathrm{Ord}_{\overline{P}}(\pi)$. □

Remark 4.10. If π is irreducible, then $\pi\simeq\mathrm{Sp}_P$. Since π contains the trivial K-representation, π is trivial by [Her11a, Proposition 7.4].

In the rest of this section, we prove Proposition 4.7. We use the following theorem due to Herzig [Her11a, Theorem 3.1].

Theorem 4.11. *Let V be an irreducible representation of K with lowest weight ν, $P=MN$ a standard parabolic subgroup. Assume that $\mathrm{Stab}_W(\nu)\subset W_M$. Then we have*

$$\text{c-Ind}_K^G(V)\otimes_{\mathcal{H}_G(V)}\mathcal{H}_M(V^{\overline{N}(\kappa)})\simeq\mathrm{Ind}_P^G(\text{c-Ind}_{M\cap K}^M V^{\overline{N}(\kappa)})$$

as G-representations and $\mathcal{H}_M(V^{\overline{N}(\kappa)})$-modules.

Remark 4.12. In fact, the theorem of Herzig is weaker than this theorem. However, his proof can be applicable for this theorem. See a paper of Henniart and Vigneras [HV12], in which this theorem is proved for a more general G.

For a parabolic subgroup $P=MN$, let V_P be the irreducible representation of K with lowest weight $-\sum_{\alpha\in\Pi\setminus\Pi_M}(q-1)\omega_\alpha$. Put $\pi_P=\mathrm{Ind}_K^G(V_P)\otimes_{\mathcal{H}_G(V_P)}\overline{\kappa}[X_*]$. Then we have $\pi_P\simeq\mathrm{Ind}_P^G(\text{c-Ind}_{M\cap K}^M(\mathbf{1}_{M\cap K})\otimes_{\mathcal{H}_M(\mathbf{1}_{M(\kappa)})}\overline{\kappa}[X_*])$ by Theorem 4.11. (Notice that $(V_P)^{\overline{N}(\kappa)}$ is the trivial representation.) In particular, we have $\pi_B\simeq\mathrm{Ind}_B^G(\overline{\kappa}[X_*])$. Here, T acts on $\overline{\kappa}[X_*]$ by $T\to T/T(\mathcal{O})\simeq X_*\to\mathrm{End}(\overline{\kappa}[X_*])$. (The last map is given by the multiplication.)

Lemma 4.13. *For parabolic subgroups $P\subset P'$, there exist $\Phi_{P,P'}\colon\pi_{P'}\to\pi_P$ and $\Phi_{P',P}\colon\pi_P\to\pi_{P'}$ which have the following properties:*

(i) *$\Phi_{P,P'}$ and $\Phi_{P',P}$ are G- and $\overline{\kappa}[X_*]$-equivariant;*
(ii) *$\Phi_{P,P}=\mathrm{id}$;*
(iii) *for $P_1\subset P_2\subset P_3$, $\Phi_{P_1,P_2}\circ\Phi_{P_2,P_3}=\Phi_{P_1,P_3}$ and $\Phi_{P_3,P_2}\circ\Phi_{P_2,P_1}=\Phi_{P_3,P_1}$;*
(iv) *for $P\subset P'$, compositions $\Phi_{P,P'}\circ\Phi_{P',P}$ and $\Phi_{P',P}\circ\Phi_{P,P'}$ are given by $\prod_{\alpha\in\Pi_{P'}\setminus\Pi_P}(\tau_{\check{\alpha}}-1)$.*

Proof. For each $\alpha\in\Pi$, fix $\lambda_\alpha\in X_{*,+}$ such that $\langle\lambda_\alpha,\Pi\setminus\{\alpha\}\rangle=0$ and $\langle\lambda_\alpha,\alpha\rangle\neq0$. We also fix a lowest weight vector v_P of V_P.

Let $P_1\subset P_2$ be parabolic subgroups such that $\#\Pi_{P_2}=\#\Pi_{P_1}+1$ and $\Pi_{P_2}=\Pi_{P_1}\cup\{\alpha\}$. Take $\varphi_{P_2,P_1}\in\mathcal{H}_G(V_{P_1},V_{P_2})$ and $\varphi_{P_1,P_2}\in\mathcal{H}_G(V_{P_2},V_{P_1})$ such that their support is $K\lambda_\alpha(\varpi)K$ and their values at $\lambda_\alpha(\varpi)$ send the lowest weight vector to the lowest weight vector (as in § 4.1). The elements φ_{P_2,P_1} and φ_{P_1,P_2} give homomorphisms $\pi_{P_1}\to\pi_{P_2}$ and $\pi_{P_2}\to\pi_{P_1}$. Let Φ_{P_1,P_2} (respectively Φ_{P_2,P_1}) be a homomorphism given by φ_{P_1,P_2} (respectively $-\tau_{\check{\alpha}-2\lambda_\alpha}\varphi_{P_2,P_1}$). By Lemma 4.2, these homomorphisms satisfy condition (iv). For general $P'\subset P$, take a chain of parabolic subgroups $P'=P_1\subset\cdots\subset P_r=P$ such that $\#\Pi_{P_{i+1}}=\#\Pi_{P_i}+1$. Define $\Phi_{P',P}=\Phi_{P_1,P_2}\circ\cdots\circ\Phi_{P_{r-1},P_r}$ and $\Phi_{P,P'}=\Phi_{P_r,P_{r-1}}\circ\cdots\circ\Phi_{P_2,P_1}$. Then by [Her11a, Proposition 6.3], condition (iv) is satisfied.

It is sufficient to prove that $\Phi_{P',P}$ and $\Phi_{P,P'}$ are independent of the choice of a chain. To prove this, we may assume that the length of the chain is 2. So let P,P',P_1,P_2 be parabolic subgroups and $\alpha,\beta\in\Pi$ such that $\alpha\neq\beta$, $\alpha,\beta\notin\Pi_P$, $\Pi_{P_1}=\Pi_P\cup\{\alpha\}$, $\Pi_{P_2}=\Pi_P\cup\{\beta\}$ and $\Pi_{P'}=\Pi_P\cup\{\alpha,\beta\}$. Put $t_\alpha=\lambda_\alpha(\varpi)$ and $t_\beta=\lambda_\beta(\varpi)$. Then by Lemma 4.6, we have

$$\begin{aligned}(\Phi_{P',P_1}\circ\Phi_{P_1,P})([1,v_P]) &= \sum_{a\in\overline{N}(\mathcal{O})/t_\alpha^{-1}\overline{N}(\mathcal{O})t_\alpha}\Phi_{P',P_1}([at_\alpha^{-1},v_{P_1}])\\ &= \sum_{a\in\overline{N}(\mathcal{O})/t_\alpha^{-1}\overline{N}(\mathcal{O})t_\alpha}\ \sum_{b\in\overline{N}(\mathcal{O})/t_\beta^{-1}\overline{N}(\mathcal{O})t_\beta}[at_\alpha^{-1}bt_\beta^{-1},v_{P'}]\\ &= \sum_{c\in\overline{N}(\mathcal{O})/(t_\alpha t_\beta)^{-1}\overline{N}(\mathcal{O})(t_\alpha t_\beta)}[c(t_\alpha t_\beta)^{-1},v_{P'}].\end{aligned}$$

Hence we have $(\Phi_{P',P_1}\circ\Phi_{P_1,P})([1,v_P])=(\Phi_{P',P_2}\circ\Phi_{P_2,P})([1,v_P])$. Therefore, we have $\Phi_{P',P_1}\circ\Phi_{P_1,P}=\Phi_{P',P_2}\circ\Phi_{P_2,P}$.

Since $\Phi_{P',P_1}\circ\Phi_{P_1,P}$ satisfies condition (iv),

$$(\tau_{\check\alpha}-1)(\tau_{\check\beta}-1)(\Phi_{P,P_2}\circ\Phi_{P_2,P'})=(\Phi_{P,P_2}\circ\Phi_{P_2,P'})\circ(\Phi_{P',P_1}\circ\Phi_{P_1,P}\circ\Phi_{P,P_1}\circ\Phi_{P_1,P'}).$$

By $\Phi_{P',P_1}\circ\Phi_{P_1,P}=\Phi_{P',P_2}\circ\Phi_{P_2,P}$, the right-hand side is equal to

$$(\Phi_{P,P_2}\circ\Phi_{P_2,P'}\circ\Phi_{P',P_2}\circ\Phi_{P_2,P})\circ(\Phi_{P,P_1}\circ\Phi_{P_1,P'}).$$

Using condition (iv) for $\Phi_{P,P_2}\circ\Phi_{P_2,P'}$, this is equal to

$$(\tau_{\check\alpha}-1)(\tau_{\check\beta}-1)(\Phi_{P,P_1}\circ\Phi_{P_1,P'}).$$

Since π_P is a torsion-free $\overline{\kappa}[X_*]$-module [Her11a, Corollary 6.5], we have $\Phi_{P,P_2}\circ\Phi_{P_2,P'}=\Phi_{P,P_1}\circ\Phi_{P_1,P'}$. We get the lemma. □

We fix such homomorphisms. Since π_P is a torsion-free $\overline{\kappa}[X_*]$-module [Her11a, Corollary 6.5], condition (iv) implies $\Phi_{P,P'}$ and $\Phi_{P',P}$ are injective.

LEMMA 4.14. *We have* $\pi_P^K\simeq\overline{\kappa}[X_*]$.

Proof. We have $\pi_P\simeq\operatorname{Ind}_{P\cap K}^K(\text{c-Ind}_{M\cap K}^M(\mathbf{1}_{M\cap K})\otimes_{\mathcal{H}_M(\mathbf{1}_{M\cap K})}\overline{\kappa}[X_*])$ by the Iwasawa decomposition $G=KP$. Therefore, we have

$$\begin{aligned}\pi_P^K &= \operatorname{Hom}_K(\mathbf{1}_K,\operatorname{Ind}_{P\cap K}^K(\text{c-Ind}_{M\cap K}^M(\mathbf{1}_{M\cap K})\otimes_{\mathcal{H}_M(\mathbf{1}_{M\cap K})}\overline{\kappa}[X_*]))\\ &\simeq \operatorname{Hom}_{M\cap K}(\mathbf{1}_{M\cap K},\text{c-Ind}_{M\cap K}^M(\mathbf{1}_{M\cap K})\otimes_{\mathcal{H}_M(\mathbf{1}_{M\cap K})}\overline{\kappa}[X_*])\\ &\simeq \operatorname{End}_M(\text{c-Ind}_{M\cap K}^M(\mathbf{1}_{M\cap K}))\otimes_{\mathcal{H}_M(\mathbf{1}_{M\cap K})}\overline{\kappa}[X_*]\simeq\overline{\kappa}[X_*].\end{aligned}$$ □

Remark 4.15. The homomorphism $\operatorname{Ind}_B^G(\overline{\kappa}[X_*])\ni f\mapsto f(1)\in\overline{\kappa}[X_*]$ gives an isomorphism $\pi_B^K\simeq\overline{\kappa}[X_*]$.

Set $f_0=[1,1]\otimes 1\in\text{c-Ind}_K^G(\mathbf{1}_K)\otimes_{\mathcal{H}_G(\mathbf{1}_K)}\overline{\kappa}[X_*]=\pi_G$. Then π_G^K is generated by f_0 as a $\overline{\kappa}[X_*]$-module. We also have that π_G is generated by $\pi_G^K=\overline{\kappa}[X_*]f_0$ as a G-module. We can prove the following lemma using an argument of Vigneras [Vig08]. This lemma also follows from [Eme10, Proposition 4.3.4, Theorem 4.4.6].

LEMMA 4.16. *Let $P=MN$ be a parabolic subgroup and σ_1,σ_2 representations of M. Then we have* $\operatorname{Hom}_M(\sigma_1,\sigma_2)\simeq\operatorname{Hom}_G(\operatorname{Ind}_P^G(\sigma_1),\operatorname{Ind}_P^G(\sigma_2))$.

Proof. Set $W(M)=\{w\in W\mid w(\Pi_M)\subset\Delta^+\}$. Then this is a set of complete representatives of W/W_M [Bou02, ch. IV, Exercises, §1 (3)]. Hence we have the Bruhat decomposition

$G/P=\coprod_{w\in W(M)} UwP/P$. For $w\in W(M)$, set

$$\pi'_w=\left\{f\colon UwP\to\sigma_1\,\middle|\,\begin{array}{l} f \text{ is a locally constant function, supp } f \text{ is compact modulo } P,\\ f(gp)=p^{-1}f(g) \text{ for } g\in UwP,\ p\in P\end{array}\right\}.$$

This is a representation of U and it is sufficient to prove that $(\pi'_w)_N=0$ if $w\neq 1$. We have $UwP/P\simeq U\cap w\overline{N}w^{-1}$. Since $w\in W(M)$, $U\cap w\overline{N}w^{-1}=U\cap w\overline{U}w^{-1}$. Therefore, as a representation of $U\cap w\overline{U}w^{-1}$, $\pi'_w\simeq\pi_{w^{-1}}\otimes\sigma_1$ where $\pi_{w^{-1}}$ is the representation defined in [Vig08, Definition 1]. If $w\neq 1$, then $w^{-1}\notin W_M$. Hence there exists $\alpha\in\Delta^+\setminus\Delta^+_M$ such that $w^{-1}(\alpha)<0$. Let $U_\alpha\subset G$ be the one-dimensional subgroup corresponding to α. Then $U_\alpha\subset N$ and as a representation of U_α, we have $\pi'_w\simeq\pi_{w^{-1}}\otimes\sigma_1$. Hence $(\pi'_w)_{U_\alpha}=(\pi_{w^{-1}})_{U_\alpha}\otimes\sigma_1$. By [Vig08, Proposition 2], $(\pi_{w^{-1}})_{U_\alpha}=0$. Hence $(\pi'_w)_{U_\alpha}=0$. Since $U_\alpha\subset N$, we have $(\pi'_w)_N=0$. Now the lemma follows from the argument in the proof of [Vig08, Théorème 8]. □

Lemma 4.17. *The element $\tau_{\check\alpha}-1\in\overline{\kappa}[X_*]$ is irreducible.*

Proof. Take $d\in\mathbb{Z}_{>0}$ and $\lambda\in X_*$ such that $\langle\alpha, X_*\rangle=d\mathbb{Z}$ and $\langle\alpha,\lambda\rangle=d$. Then we have $X_*=\mathbb{Z}\lambda\oplus\operatorname{Ker}\alpha$. Let $a,b\in\overline{\kappa}[X_*]$ such that $\tau_{\check\alpha}-1=ab$. Put $t=\tau_\lambda$. Then we have $a=\sum_n a_nt^n$ and $b_n=\sum_n b_nt^n$ where $a_n,b_n\in\overline{\kappa}[\operatorname{Ker}\alpha]$. Put $k_a=\max\{n\mid a_n\neq 0\}$, $l_a=\min\{n\mid a_n\neq 0\}$, $k_b=\max\{n\mid b_n\neq 0\}$, $l_b=\min\{n\mid b_n\neq 0\}$. We may assume $k_a-l_a\leqslant k_b-l_b$. Take $c\in\mathbb{Z}$ and $\lambda_0\in\operatorname{Ker}\alpha$ such that $\check\alpha=c\lambda+\lambda_0$. Then $c=1$ or 2 and we have $ab=\tau_{\check\alpha}-1=t^c\tau_{\lambda_0}-1$. Therefore, $k_a+k_b=c$ and $a_{k_a}b_{k_b}=\tau_{\lambda_0}\in\overline{\kappa}[\operatorname{Ker}\alpha]^\times$. Replacing (a,b) with (au^{-1},bu) for $u=t^{k_a-1}a_{k_a}\in\overline{\kappa}[X_*]^\times$, we may assume $k_a=1$ and $a_{k_a}=1$. Hence $k_b=c-1$. We prove $a\in\overline{\kappa}[X_*]^\times$. If $k_a=l_a$, then $a=t\in\overline{\kappa}[X_*]^\times$. Hence we may assume $k_a\neq l_a$. By $ab=\tau_{\check\alpha}-1=t^c\tau_{\lambda_0}-1$, we have $l_a+l_b=0$. Therefore, (c,k_a,l_a,k_b,l_b) satisfies the following conditions:

$$c=1 \text{ or } 2,\quad k_a=1,\quad k_b=c-1,\quad l_a<k_a,\quad k_a-l_a\leqslant k_b-l_b,\quad l_a+l_b=0.$$

From $k_a=1$, $k_b=c-1$ and $k_a-l_a\leqslant k_b-l_b$, we have $1-l_a\leqslant c-1-l_b$. Since $l_a+l_b=0$, we have $1-l_a\leqslant c-1+l_a$. Therefore, $l_a\geqslant 1-c/2$. We also have $1=k_a>l_a$. Hence $l_a\leqslant 0$. From this, $0\geqslant 1-c/2$. Hence $c=2$. Therefore $0\leqslant l_a\leqslant 1-c/2=0$. Hence $l_a=0$ and $l_b=-l_a=0$. We get $(c,k_a,l_a,k_b,l_b)=(2,1,0,1,0)$.

Now we have $a=t+a_0$ and $b=b_1t+b_0$. Since $ab=\tau_{\lambda_0}t^2-1$, we have

$$b_1=\tau_{\lambda_0},\quad a_0b_1+b_0=0\quad\text{and}\quad a_0b_0=-1.$$

By the last equation, $b_0\in\overline{\kappa}[X_*]^\times$. Hence $b_0\in\overline{\kappa}^\times\tau_\mu$ for some $\mu\in X_*$. We have $\tau_{\lambda_0}=b_1=-b_0a_0^{-1}=b_0^2$. Therefore, $\lambda_0=2\mu$. Hence $\check\alpha=2(\lambda+\mu)\in 2X_*$. This is a contradiction since we assume that the derived group of G is simply connected. □

Lemma 4.18. *The image of f_0 under $\Phi_{B,G}$ is a basis of π_B^K.*

Proof. It is sufficient to prove that $\Phi_{B,G}(\pi_G^K)=\pi_B^K$. We prove $\Phi_{B,G}(\pi_G^K)\supset\prod_{\beta\in\Pi\setminus\{\alpha\}}(\tau_{\check\beta}-1)\pi_B^K$ for all $\alpha\in\Pi$. Then for each $\alpha\in\Pi$, there exists $a_\alpha\in\overline{\kappa}[X_*]$ such that $a_\alpha\Phi_{B,G}(f_0)=\prod_{\beta\in\Pi\setminus\{\alpha\}}(\tau_{\check\beta}-1)f'_0$ where f'_0 is a basis of π_B^K. Since $(\tau_{\check\alpha}-1)$ are distinct irreducible elements and $\overline{\kappa}[X_*]$ is a unique factorization domain, we have $\Phi_{B,G}(f_0)\in\overline{\kappa}[X_*]^\times f'_0$. Hence the lemma is proved.

So it is sufficient to prove $\Phi_{B,G}(\pi_G^K)\supset\prod_{\beta\in\Pi\setminus\{\alpha\}}(\tau_{\check\beta}-1)\pi_B^K$ for all $\alpha\in\Pi$. Fix $\alpha\in\Pi$ and let P be the parabolic subgroup corresponding to $\{\alpha\}$. Since $\Phi_{P,G}(\pi_G^K)\supset\Phi_{P,G}(\Phi_{G,P}(\pi_P^K))=\prod_{\beta\in\Pi\setminus\{\alpha\}}(\tau_{\check\beta}-1)\pi_P^K$, it is sufficient to prove $\Phi_{B,P}(\pi_P^K)=\pi_B^K$. By Lemma 4.16, $\Phi_{B,P}$ is given by a certain homomorphism $\Phi\colon \text{c-Ind}_{M\cap K}^M(\mathbf{1}_{M\cap K})\otimes_{\mathcal{H}_M(\mathbf{1}_{M\cap K})}\overline{\kappa}[X_*]\to\operatorname{Ind}_{M\cap B}^M(\mathbf{1}_{M\cap B})$. We also have that $\Phi_{P,B}$ is induced by some $\Phi'\colon \operatorname{Ind}_{M\cap B}^M(\mathbf{1}_{M\cap B})\to\text{c-Ind}_{M\cap K}^M(\mathbf{1}_{M\cap K})\otimes_{\mathcal{H}_M(\mathbf{1}_{M\cap K})}\overline{\kappa}[X_*]$. It is sufficient to prove that Φ induces surjective homomorphism between the spaces of K-invariants. Since $\Phi_{P,G}\circ\Phi_{G,P}=(\tau_{\check\alpha}-1)$ (respectively $\Phi_{G,P}\circ\Phi_{P,G}=(\tau_{\check\alpha}-1)$), $\Phi'\circ\Phi$ (respectively $\Phi\circ\Phi'$) is induced by $(\tau_{\check\alpha}-1)$. Hence $\Phi'\circ\Phi=(\tau_{\check\alpha}-1)$ and $\Phi\circ\Phi'=(\tau_{\check\alpha}-1)$ by Lemma 4.16. Namely, Φ' and Φ satisfy the conditions of Lemma 4.13 for M. Therefore, it is sufficient to prove the lemma for $G=M$. We assume that the semisimple rank of G is one.

Now we assume that the semisimple rank of G is one. Let $\Pi=\{\alpha\}$. Take $a,b\in\overline{\kappa}[X_*]$ such that $\Phi_{B,G}(\pi_G^K)=a\pi_B^K$, $\Phi_{G,B}(\pi_B^K)=b\pi_G^K$ and $ab=\tau_{\check\alpha}-1$. Assume $\Phi_{B,G}(\pi_G^K)\neq\pi_B^K$. It is equivalent to $a\notin\overline{\kappa}[X_*]^\times$. By the above lemma, $b\in\overline{\kappa}[X_*]^\times$. Hence $\Phi_{G,B}(\pi_B^K)=\pi_G^K$. Since π_G is generated by π_G^K, $\Phi_{G,B}$ is surjective. Therefore, $\Phi_{G,B}$ is isomorphic. Let $\chi\colon\overline{\kappa}[X_*]\to\overline{\kappa}$ be a homomorphism defined by $\chi(\tau_\lambda)=1$ for all $\lambda\in X_*$. Then we have $\pi_B\otimes_{\overline{\kappa}[X_*]}\chi=\operatorname{Ind}_B^G(\mathbf{1}_T)$. Hence we have $\operatorname{Ind}_B^G(\mathbf{1}_T)\simeq\pi_G\otimes_{\overline{\kappa}[X_*]}\chi$. Consider a homomorphism $\text{c-Ind}_K^G(\mathbf{1}_K)\to\mathbf{1}_G$ defined by $f\mapsto\sum_{g\in G/K}f(g)$. This gives a homomorphism $\pi_G\otimes_{\overline{\kappa}[X_*]}\chi\to\mathbf{1}_G$ and the induced homomorphism $(\pi_G\otimes_{\overline{\kappa}[X_*]}\chi)^K\to(\mathbf{1}_G)^K=\mathbf{1}_G$ is surjective since an image of $[1,1]\in(\text{c-Ind}_K^G(\mathbf{1}_K))^K$ is non-zero. Consider the following maps: $\mathbf{1}_G\hookrightarrow\operatorname{Ind}_B^G(\mathbf{1}_T)\simeq\pi_G\otimes_{\overline{\kappa}[X_*]}\chi\to\mathbf{1}_G$. Take K-invariants. Then we have that $\mathbf{1}_G=(\mathbf{1}_G)^K\hookrightarrow(\operatorname{Ind}_B^G(\mathbf{1}_T))^K$ is isomorphic (by the Iwasawa decomposition) and $(\pi_G\otimes_{\overline{\kappa}[X_*]})^K\chi\to(\mathbf{1}_G)^K=\mathbf{1}_G$ is surjective. Hence the composition $\mathbf{1}_G\to\mathbf{1}_G$ is surjective. Since both sides are one-dimensional, it is isomorphic. Hence $\mathbf{1}_G$ is a direct summand of $\operatorname{Ind}_B^G(\mathbf{1}_T)$. Therefore, $\operatorname{End}_G(\operatorname{Ind}_B^G(\mathbf{1}_T))$ has a non-trivial idempotent. However, by Lemma 3.19, $\operatorname{End}_G(\operatorname{Ind}_B^G(\mathbf{1}_T))\simeq\operatorname{End}_T(\mathbf{1}_T)\simeq\overline{\kappa}$. This is a contradiction. □

By this lemma, $\operatorname{Im}\Phi_{B,G}$ is a subrepresentation of π_B generated by π_B^K. For each $w\in W\simeq N_K(T(\mathcal{O}))/T(\mathcal{O})$, we fix a representative of w and denote it by the same letter w. For a subset $A\subset W$ of W, let $X_{G,A}\subset\pi_B=\operatorname{Ind}_B^G\overline{\kappa}[X_*]$ be a B-stable subspace defined by $X_{G,A}=\{f\in\pi_B\mid\operatorname{supp}f\subset\bigcup_{w'\in A}Bw'B/B\}$. For $w\in W$, put $X_{G,>w}=X_{G,\{w'\in W\mid w'>w\}}$ and $X_{G,\geqslant w}=X_{G,\{w'\in W\mid w'\geqslant w\}}$. Set $X_A=X_{G,A}$, $X_{\geqslant w}=X_{G,\geqslant w}$ and $X_{>w}=X_{G,>w}$ for $A\subset W$, $w\in W$. Set $Y=\Phi_{B,G}(\pi_G)$, $Y_A=Y\cap X_A$. For a parabolic subgroup $P=MN$, put $W(M)=\{w\in W\mid w(\Pi_M)\subset\Delta^+\}$. Then $W(M)\times W_M\to W$ is bijective [Bou02, ch. IV, Exercises, § 1 (3)].

Let $A\subset W$ be a subset such that $\bigcup_{w\in A}BwB$ is open. (In other words, if $w\in A$ and $w'>w$ then $w'\in A$.) Let $w\in A$ be a minimal element and set $A'=A\backslash\{w\}$.

Lemma 4.19. *Let $I\subset\overline{\kappa}[X_*]$ be a principal ideal. For $f\in\pi_B$, $f\in X_A+I\pi_B$ if and only if $f(x)\in I$ for all $x\in BvB$ and $v\in W\backslash A$. In particular, if $I_1,I_2\subset\overline{\kappa}[X_*]$ are principal ideals then $(X_A+I_1\pi_B)\cap(X_A+I_2\pi_B)=X_A+(I_1\cap I_2)\pi_B$.*

Proof. It is obvious that if $f\in X_A+I\pi_B$ then $f(x)\in I$ for all $x\in BvB$ and $v\in W\backslash A$. Assume that $f(x)\in I$ for all $x\in BvB$ and $v\in W\backslash A$. Let $a\in I$ be a generator of I. Since $\overline{\kappa}[X_*]$ is an integral domain, there exists a locally constant function $f_0\colon\bigcup_{v\in W\backslash A}BvB\to\overline{\kappa}[X_*]$ such that $f(x)=af_0(x)$. Since $\bigcup_{v\in W\backslash A}BvB$ is closed, there exists $f_1\in\pi_B$ such that $f_1|_{\bigcup_{v\in W\backslash A}BvB}=f_0$. Then $f=(f-af_1)+af_1$ and $f-af_1\in X_A$, $af_1\in I\pi_B$.

Since $\overline{\kappa}[X_*]$ is a unique factorization domain, if I_1, I_2 are principal ideals, then $I_1\cap I_2$ is also a principal ideal. Hence the second statement follows from the first statement. □

Lemma 4.20. *Let $P=MN$ be a parabolic subgroup, $w,v_0\in W(M)$ and $v_1\in W_M$. Then $v_0v_1\geqslant w$ if and only if $v_0\geqslant w$.*

Proof. Put $v=v_0v_1$. Let ℓ be the length function of W. Then $\ell(v)=\ell(v_0)+\ell(v_1)$ [Bou02, ch. IV, Exercises, § 1 (3)]. Hence $v\geqslant v_0$. Therefore, $v_0\geqslant w$ implies $v\geqslant w$.

We prove $v\geqslant w$ implies $v_0\geqslant w$ by induction on $\ell(v_1)$. If $\ell(v_1)=0$, then $v_1=1$. Hence there is nothing to prove. Assume that $\ell(v_1)>0$ and take $\alpha\in\Pi_M$ such that $v_1s_\alpha<v_1$ where $s_\alpha\in W_M$

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is the reflection corresponding to α. Put $s = s_\alpha$. Then $\ell(v_0v_1s) = \ell(v_0) + \ell(v_1s) = \ell(v_0) + \ell(v_1) - 1 = \ell(v_0v_1) - 1$. Hence $vs < v$. By the definition of $W(M)$, we have $ws > w$. Hence we get $vs \geqslant w$ [Deo77, Theorem 1.1 (II, ii)]. Therefore, $v_0(v_1s) \geqslant w$. Since $\ell(v_1s) < \ell(v_1)$, we have $v_0 \geqslant w$ by inductive hypothesis. □

Lemma 4.21. *We have* $Y_A/Y_{A'} = \prod_{\alpha\in\Pi, ws_\alpha<w}(\tau_{\check\alpha} - 1)(X_A/X_{A'})$.

Proof. Set $\Theta = \{\alpha \in \Pi \mid ws_\alpha < w\}$ and put $I = \prod_{\alpha\in\Theta}(\tau_{\check\alpha} - 1)\overline{\kappa}[X_*]$. First we prove $Y_A/Y_{A'} \subset I(X_A/X_{A'})$; namely, we prove $Y_A \subset I\pi_B + X_{A'}$. Let $I_\alpha = (\tau_{\check\alpha} - 1)\overline{\kappa}[X_*]$. By Lemma 4.19, it is sufficient to prove $Y_A \subset I_\alpha\pi_B + X_{A'}$ for all $\alpha \in \Theta$. Let $P_\alpha = M_\alpha N_\alpha$ be the parabolic subgroup corresponding to $\{\alpha\}$. Recall that T acts on $\overline{\kappa}[X_*]$ and $\pi_B = \mathrm{Ind}_B^G(\overline{\kappa}[X_*])$. This action induces the action of T on $\overline{\kappa}[X_*]/I_\alpha$. The image of $\check\alpha$ acts on $\overline{\kappa}[X_*]/I_\alpha$ trivially. Therefore, the action of T on $\overline{\kappa}[X_*]/I_\alpha$ is extended to the action of M_α such that $[M_\alpha(F), M_\alpha(F)]$ acts on it trivially by Lemma 3.2. We have $\mathrm{Ind}_{P_\alpha}^G(\overline{\kappa}[X_*]/I_\alpha) \subset \mathrm{Ind}_B^G(\overline{\kappa}[X_*]/I_\alpha) = \pi_B/I_\alpha\pi_B$.

Let $f \in (\pi_B/I_\alpha\pi_B)^K = (\mathrm{Ind}_B^G(\overline{\kappa}[X_*]/I_\alpha))^K$. We prove $f \in \mathrm{Ind}_{P_\alpha}^G(\overline{\kappa}[X_*]/I_\alpha)$; namely, $f(gp) = p^{-1}f(g)$ for $g \in G$ and $p \in P_\alpha$. Let $g_0 \in G$ and $p_0 \in P_\alpha$. By the Iwasawa decomposition $G = KP_\alpha$, there exist $k_0 \in K$ and $p_0' \in P_\alpha$ such that $g_0 = k_0p_0'$. Since $P_\alpha = M_\alpha N_\alpha = [M_\alpha(F), M_\alpha(F)]TN_\alpha = ([M_\alpha(F), M_\alpha(F)] \cap K)([M_\alpha(F), M_\alpha(F)] \cap B)TN_\alpha = ([M_\alpha(F), M_\alpha(F)] \cap K)B$, there exist $k_0' \in [M_\alpha(F), M_\alpha(F)] \cap K$ and $b_0 \in B$ such that $p_0'p_0 = k_0'b_0$. Hence $f(g_0p_0) = f(k_0p_0'p_0) = f(k_0k_0'b_0) = b_0^{-1}f(1)$. Since $k_0' \in [M_\alpha(F), M_\alpha(F)]$, we have $(k_0')^{-1}f(1) = f(1)$. Hence $f(g_0p_0) = (k_0'b_0)^{-1}f(1) = (p_0'p_0)^{-1}f(1)$. Let $g \in G$ and $p \in P_\alpha$. Take $k \in K$ and $p' \in P_\alpha$ such that $g = kp'$. Then applying the above formula for $g_0 = g$, $k_0 = k$, $p_0' = p'$ and $p_0 = p$, we have $f(gp) = (p'p)^{-1}f(1)$. Applying the above formula for $g_0 = 1$, $k_0 = 1$, $p_0' = 1$ and $p_0 = p'$, we get $f(p') = (p')^{-1}f(1)$. Hence $f(gp) = p^{-1}f(p') = p^{-1}f(kp') = p^{-1}f(g)$. So $f \in \mathrm{Ind}_{P_\alpha}^G(\overline{\kappa}[X_*]/I_\alpha)$. Hence the image of $\Phi_{G,B}(f_0)$ under $\pi_B \to \pi_B/I_\alpha\pi_B$ is in $\mathrm{Ind}_{P_\alpha}^G(\overline{\kappa}[X_*]/I_\alpha)$. (Recall that $f_0 = [1, 1] \otimes 1 \in \pi_G^K$.) Since π_G is generated by f_0, the image of Y is contained in $\mathrm{Ind}_{P_\alpha}^G(\overline{\kappa}[X_*]/I_\alpha)$.

For $f \in \pi_B$, let $\overline{f}$ be the image of f under the canonical projection $\pi_B \to \pi_B/I_\alpha\pi_B = \mathrm{Ind}_B^G(\overline{\kappa}[X_*]/I_\alpha)$. Let $f \in Y_A$. Then $\mathrm{supp}\,\overline{f} \subset \bigcup_{w'\in A} Bw'B/B$. Since $\overline{f} \in \mathrm{Ind}_{P_\alpha}^G(\overline{\kappa}[X_*]/I_\alpha)$, its support is right P_α-invariant. Hence if $\mathrm{supp}\,\overline{f} \cap BwB/B \neq \emptyset$, $\mathrm{supp}\,\overline{f} \cap Bws_\alpha B/B \neq \emptyset$. By the definition of Θ, $ws_\alpha < w$. This contradicts $\mathrm{supp}\,\overline{f} \subset \bigcup_{w'\in A} Bw'B/B$ and the minimality of w. So we have $\mathrm{supp}\,\overline{f} \subset \bigcup_{w'\in A'} Bw'B/B$. Hence $f \in X_{A'} + I_\alpha\pi_B$.

We prove $Y_A/Y_{A'} \supset I(X_A/X_{A'})$. Let $P = MN$ be a parabolic subgroup corresponding to $\Pi\backslash\Theta$. First we prove that $\Phi_{B,P}(\pi_P) \cap X_A \to X_A/X_{A'}$ is surjective. Since $X_A/X_{A'} \simeq X_{\geqslant w}/X_{>w}$ and $X_A \supset X_{\geqslant w}$, we may assume $A = \{w' \in W \mid w' \geqslant w\}$. For each parabolic subgroup $P_1 = M_1N_1 \subset P$, put $\pi_{M,P_1} = \mathrm{Ind}_{M\cap P_1}^M(\text{c-}\mathrm{Ind}_{M_1\cap K}^{M_1} \mathbf{1}_{M_1\cap K} \otimes_{\mathcal{H}_{M_1}(\mathbf{1}_{M_1\cap K})} \overline{\kappa}[X_*])$. Then $\pi_{P_1} = \mathrm{Ind}_P^G(\pi_{M,P_1})$. By Lemma 4.16, for each $P_1 \subset P_2 \subset P$, Φ_{P_1,P_2} and Φ_{P_2,P_1} are induced by some $\Phi^M_{P_1,P_2}\colon \pi_{M,P_2} \to \pi_{M,P_1}$ and $\Phi^M_{P_2,P_1}\colon \pi_{M,P_1} \to \pi_{M,P_2}$. Such homomorphisms satisfy the conditions of Lemma 4.13. Therefore, $\Phi^M_{P_1,P_2}$ induces a bijection $\pi_{M,P_2}^{M\cap K} \simeq \pi_{M,P_1}^{M\cap K}$ by Lemma 4.18. Put $\Phi = \Phi^M_{B,P}$. Then $\Phi_{B,P}(\pi_P) = \mathrm{Ind}_P^G(\Phi(\pi_{M,P}))$.

Let $f \in \Phi_{B,P}(\pi_P)$. By the definition of $X_{\geqslant w}$, $f \in X_{\geqslant w}$ if and only if $\mathrm{supp}\,f \subset \bigcup_{v\geqslant w} BvB$. For $v \in W$, take $v_0 \in W(M)$ and $v_1 \in W_M$ such that $v = v_0v_1$. Since $w \in W(M)$, $v \geqslant w$ if and only if $v_0 \geqslant w$ by the above lemma. Hence $\bigcup_{v\geqslant w} BvB = \bigcup_{v\geqslant w, v\in W(M)} BvW_MB = \bigcup_{v\geqslant w, v\in W(M)} BvP$. Therefore, $\Phi_{B,P}(\pi_P) \cap X_{\geqslant w} = \{f \in \mathrm{Ind}_P^G(\Phi(\pi_{M,P})) \mid \mathrm{supp}\,f \subset \bigcup_{v\geqslant w, v\in W(M)} BvP/P\}$. Let $Z_{\geqslant w}$ be this space. Put $Z_{>w} = \{f \in \mathrm{Ind}_P^G(\Phi(\pi_{M,P})) \mid \mathrm{supp}\,f \subset \bigcup_{v>w, v\in W(M)} BvP/P\}$. Then the homomorphism $Z_{\geqslant w} = \Phi_{B,P}(\pi_{M,P}) \cap X_{\geqslant w} \to X_{\geqslant w}/X_{>w}$ induces $Z_{\geqslant w}/Z_{>w} \to X_{\geqslant w}/X_{>w}$. By the Bruhat decomposition $G/P = \bigcup_{v\in W(M)} BvP/P$, the space $Z_{\geqslant w}/Z_{>w}$ is isomorphic to the space of locally constant compact support $\Phi(\pi_{M,P})$-valued functions on $BwP/P \simeq BwB/B$. The space $X_{\geqslant w}/X_{>w}$ is isomorphic to the space of locally constant compact support $\overline{\kappa}[X_*]$-valued functions on BwB/B. The homomorphism $Z_{\geqslant w}/Z_{>w} \to X_{\geqslant w}/X_{>w}$ is induced by $\Phi(\pi_{M,P}) \hookrightarrow \pi_{M,B} \to \pi_{M,B}/X_{M,>1} \simeq \overline{\kappa}[X_*]$. By Remark 4.15, $\pi_{M,B}^{M\cap K} \hookrightarrow \pi_{M,B} \to \pi_{M,B}/X_{M,>1} \simeq \overline{\kappa}[X_*]$ is isomorphic. Since Φ induces $\pi_{M,P}^{M\cap K} \simeq \pi_{M,B}^{M\cap K}$, $\Phi(\pi_{M,P}) \hookrightarrow \pi_{M,B} \to \pi_{M,B}/X_{M,>1} \simeq \overline{\kappa}[X_*]$ is surjective. Therefore $\Phi_{B,P}(\pi_P) \cap X_{\geqslant w} \to X_{\geqslant w}/X_{>w}$ is surjective.

By the above argument, we get $(\Phi_{B,P}(\pi_P) \cap X_A) + X_{A'} = X_A$. Hence we get $I\Phi_{B,P}(\pi_P) = \Phi_{B,P}(I\pi_P) = \Phi_{B,P}(\Phi_{P,G}(\Phi_{G,P}(\pi_P))) = \Phi_{B,G}(\Phi_{G,P}(\pi_P)) \subset \Phi_{B,G}(\pi_G) = Y$, $IX_A \subset Y \cap X_A + IX_{A'} \subset Y_A + X_{A'}$. This gives us the lemma. □

From this lemma, we obtain the following proposition.

PROPOSITION 4.22. *Let V be an irreducible representation of K. The module* $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \overline{\kappa}[X_*]$ *is free as a* $\overline{\kappa}[X_*]$*-module.*

Remark 4.23. Barthel–Livné proved that, as an $\mathrm{End}_G(\text{c-Ind}_{KZ_G}^G(V))$-module, $\text{c-Ind}_{KZ_G}^G(V)$ is free if $G = \mathrm{GL}_2$ [BL94, Theorem 19].

Proof. Let ν be a lowest weight of V. By Theorem 4.11, we have $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \overline{\kappa}[X_*] \simeq \mathrm{Ind}_{P_{-\nu}}^G(\text{c-Ind}_{M_{-\nu}\cap K}^{M_{-\nu}}(V^{\overline{N_{-\nu}}(\kappa)}) \otimes_{\mathcal{H}_{M_{-\nu}}(V^{\overline{N_{-\nu}}(\kappa)})} \overline{\kappa}[X_*])$. Therefore, it is sufficient to prove that $\text{c-Ind}_{M_{-\nu}\cap K}^{M_{-\nu}}(V^{\overline{N_{-\nu}}(\kappa)}) \otimes_{\mathcal{H}_{M_{-\nu}}(V^{\overline{N_{-\nu}}(\kappa)})} \overline{\kappa}[X_*]$ is free. Hence we may assume $P_{-\nu} = G$. Therefore, V is a character of K. By Corollary 3.4, there exists a character ν_G of G such that $\nu_G|_K \simeq V$. Then $\varphi \mapsto \varphi_{\nu_G^{-1}}$ gives an isomorphism $\mathcal{H}_G(V) \simeq \mathcal{H}_G(\mathbf{1}_K)$ (see § 3.1). By this isomorphism, we can identify $\mathcal{H}_G(V)$ and $\mathcal{H}_G(\mathbf{1}_K)$. Under this identification, we have $\text{c-Ind}_K^G(V) \otimes \nu_G^{-1} \simeq \text{c-Ind}_K^G(\mathbf{1}_K)$. Hence we may assume $V = \mathbf{1}_K$. Therefore, $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \overline{\kappa}[X_*] = \pi_G \simeq Y$. Since $X_A/X_{A'}$ is free [Vig08, Lemma 3], $Y_A/Y_{A'}$ is free by Lemma 4.21. Hence Y is free. □

Proof of Proposition 4.7. We prove the proposition by induction on $\#\Pi_{-\nu}$. Namely, we prove the following by induction on n: if ν satisfies $\#\Pi_{-\nu} \leqslant n$ then the module $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi$ has a finite length and its composition factors depend only on χ and the $T(\kappa)$-representation $V^{\overline{U}(\kappa)}$.

If $\Pi_{-\nu} = \emptyset$, then $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi$ is isomorphic to a principal series representation [Her11a, Theorem 3.1]. Hence the proposition follows.

Assume $\Pi_{-\nu} \neq \emptyset$ and take $\alpha \in \Pi_{-\nu}$. Put $\nu' = \nu - (q-1)\omega_\alpha$ and let V' be the irreducible K-representation with lowest weight ν'. By inductive hypothesis, $\text{c-Ind}_K^G(V') \otimes_{\mathcal{H}_G(V')} \chi$ has a finite length. Define $\chi' \colon \overline{\kappa}[X_*] \to \overline{\kappa}[t, t^{-1}]$ by $\chi'(\tau_\lambda) = \chi(\tau_\lambda) t^{\langle \omega_\alpha, \lambda \rangle}$ for $\lambda \in X_*$. (Here, t is an indeterminant.) Then χ factors through χ'. Put $\pi = \text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi'$ and $\pi' = \text{c-Ind}_K^G(V') \otimes_{\mathcal{H}_G(V')} \chi'$. These are free $\overline{\kappa}[t, t^{-1}]$-modules by Proposition 4.22. Take $\lambda \in X_*$ such that $\langle \lambda, \Pi \backslash \{\alpha\} \rangle = 0$ and $\langle \lambda, \alpha \rangle \neq 0$. Put $a = \chi(\tau_{\check{\alpha}})$. As in § 4.1, λ gives $\Phi \colon \pi \to \pi'$ and $\Phi' \colon \pi' \to \pi$ such that $\Phi \circ \Phi' = (at - 1)$. Therefore, Φ' is injective and $\mathrm{Im}\,\Phi' \subset (at-1)\pi$. By [CG97, Lemma 2.3.4], $\pi/(t-1)\pi$ has a finite length and $\pi/(t-1)\pi$ and $\pi'/(t-1)\pi'$ have the same composition factors. □

5. Classification theorem

Using results in §§ 3 and 4, we prove the main theorem. Almost all the proof of the theorem is a copy of Herzig's proof.

5.1 Construction of representations

We recall the definition of supersingular representations. Recall that a character $\overline{\kappa}[X_{*,+}] \to \overline{\kappa}$ is parameterized by a pair (M, χ_M) where M is the Levi subgroup of a standard parabolic subgroup of G and $\chi_M \colon X_{M,*,0} \to \overline{\kappa}^{\times}$ is a character of $X_{M,*,0}$ where $X_{M,*,0} = \{\lambda \in X_* \mid \langle \lambda, \Pi_M \rangle = 0\}$. (See § 2.2.)

Definition 5.1 (Herzig [Her11a, Definition 4.7]). Let π be an irreducible admissible representation of G.

(i) The representation π is *supersingular with respect to* (K, T, B) if each $\chi \in \mathcal{S}(\pi)$ corresponds to (G, χ_G) for some $\chi_G \colon X_{G,*,0} \to \overline{\kappa}^{\times}$.

(ii) The representation π is *supersingular* if it is supersingular with respect to all (K, T, B).

It will be proved that π is supersingular if and only if π is supersingular with respect to (K, T, B) for a fixed (K, T, B) (Corollary 5.13).

Now we introduce the set of parameters $\mathcal{P} = \mathcal{P}_G$. It will parameterize the isomorphism classes of irreducible admissible representations. Before giving $\mathcal{P}$, we give one notation. Let M be the Levi subgroup of a standard parabolic subgroup and σ its representation with the central character ω_σ. Then set $\Pi_\sigma = \{\alpha \in \Pi \mid \langle \Pi_M, \check{\alpha} \rangle = 0,\ \omega_\sigma \circ \check{\alpha} = \mathbf{1}_{\mathrm{GL}_1(F)}\}$.

Let $\mathcal{P} = \mathcal{P}_G$ be the set of $\Lambda = (\Pi_1, \Pi_2, \sigma_1)$ such that:

- Π_1 and Π_2 are subsets of Π;
- σ_1 is an irreducible admissible representation of M_{Π_1} with central character ω_{σ_1}, which is supersingular with respect to $(M_{\Pi_1} \cap K, T, M_{\Pi_1} \cap B)$;
- $\Pi_2 \subset \Pi_{\sigma_1}$.

We consider $\Lambda = (\Pi_1, \Pi_2, \sigma_1)$ and $\Lambda' = (\Pi_1', \Pi_2', \sigma_1')$ to be equal to each other if $\Pi_1 = \Pi_1'$, $\Pi_2 = \Pi_2'$ and $\sigma_1 \simeq \sigma_1'$.

For $\Lambda = (\Pi_1, \Pi_2, \sigma_1) \in \mathcal{P}$, we attach the representation $I(\Lambda)$ of G in the following way. Let $P_\Lambda = M_\Lambda N_\Lambda$ be the standard parabolic subgroup corresponding to $\Pi_1 \cup \Pi_{\sigma_1}$. By Lemma 3.2, there exists the unique extension of σ_1 to M_Λ such that $[M_{\Pi_{\sigma_1}}(F), M_{\Pi_{\sigma_1}}(F)]$ acts on it trivially. We denote this representation by the same letter σ_1. By the definition, $\Pi_1 \cup \Pi_2$ is a subset of $\Pi_1 \cup \Pi_{\sigma_1}$. Hence this set defines a standard parabolic subgroup of M_Λ. Let $\sigma_{\Lambda,2}$ be the special representation of M_Λ corresponding to this parabolic subgroup. By the construction, $\sigma_{\Lambda,2}|_{M_{\Pi_{\sigma_1}}}$ is a special representation of $M_{\Pi_{\sigma_1}}$. By the following general lemma, the restriction of $\sigma_{\Lambda,2}$ to $[M_{\Pi_{\sigma_1}}(F), M_{\Pi_{\sigma_1}}(F)]$ is irreducible and admissible [Her11a, Theorem 7.2].

Lemma 5.2. *Let π be a special representation of G. Then the restriction of π to $[G(F), G(F)]$ is irreducible and admissible.*

Proof. By the definition of a special representation, the restriction of π to $[G, G](F)$ is a special representation of $[G, G](F)$. Hence it is irreducible and admissible [Her11a, Theorem 7.2]. If the derived group of G is simply connected, $[G, G](F) = [G(F), G(F)]$. Hence the lemma follows. In general, let $\widetilde{G} \to G$ be a z-extension of G. Then the pull-back $\widetilde{\pi}$ of π to $\widetilde{G}$ is a special representation of $\widetilde{G}$ and by the above argument, the restriction of $\widetilde{\pi}$ to $[\widetilde{G}(F), \widetilde{G}(F)]$ is irreducible and admissible. Since the image of $[\widetilde{G}(F), \widetilde{G}(F)]$ in G is $[G(F), G(F)]$, π is irreducible and admissible as a representation of $[G(F), G(F)]$. □

Put $\sigma_\Lambda = \sigma_1 \otimes \sigma_{\Lambda,2}$ and $I(\Lambda) = I_G(\Lambda) = \mathrm{Ind}_{P_\Lambda}^{G}(\sigma_\Lambda)$. It is easy to check that the tuple $(M_1, M_2, \sigma_1, \sigma_2) = (M_{\Pi_1}, M_{\Pi_\sigma}, \sigma_1, \sigma_{\Lambda,2})$ satisfies the conditions of § 3.3. By Lemma 3.23, σ_Λ is

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admissible. Hence $I(\Lambda)$ is admissible. By the following lemma, σ_Λ is irreducible. (Apply for $H = M_\Lambda$ and $H' = [M_{\Pi_{\sigma_1}}(F), M_{\Pi_{\sigma_1}}(F)]$.)

Lemma 5.3. *Let H be a group, H' a normal subgroup of H and σ_2 a representation of H which is irreducible as a representation of H' and $\operatorname{End}_{H'}(\sigma_2) = \overline{\kappa}$. For a representation σ of H, $\operatorname{Hom}_{H'}(\sigma_2, \sigma)$ has a structure of a representation of H/H' defined by $(h\psi)(v) = h\psi(h^{-1}v)$ for $h \in H$, $\psi \in \operatorname{Hom}_{H'}(\sigma_2, \sigma)$ and $v \in \sigma_2$.*

(i) *The natural homomorphism $\operatorname{Hom}_{H'}(\sigma_2, \sigma) \otimes \sigma_2 \to \sigma$ is injective.*

(ii) *If σ is irreducible, then $\operatorname{Hom}_{H'}(\sigma_2, \sigma)$ is zero or irreducible.*

(iii) *For an irreducible representation σ_1 of H/H', $\sigma_1 \otimes \sigma_2$ is an irreducible H-representation.*

Proof. (i) Assume that the kernel of the homomorphism is non-zero. Take a non-zero finite-dimensional subspace $V \subset \operatorname{Hom}_{H'}(\sigma_2, \sigma)$ such that $V \otimes \sigma_2 \to \sigma$ is not injective. This is an H'-homomorphism. Therefore, there exists a non-zero subspace V_1 of V such that the kernel is $V_1 \otimes \sigma_2$. This means $V_1 = 0$ in $\operatorname{Hom}_{H'}(\sigma_2, \sigma)$. This is a contradiction.

(ii) Assume that σ is irreducible and $\operatorname{Hom}_{H'}(\sigma_2, \sigma) \neq 0$. Then by (i), we have an injective homomorphism $\operatorname{Hom}_{H'}(\sigma_2, \sigma) \otimes \sigma_2 \hookrightarrow \sigma$. Since σ is irreducible, we have $\operatorname{Hom}_{H'}(\sigma_2, \sigma) \otimes \sigma_2 \simeq \sigma$. Therefore, $\operatorname{Hom}_{H'}(\sigma_2, \sigma)$ is irreducible.

(iii) Let $\sigma \subset \sigma_1 \otimes \sigma_2$ be a non-zero subrepresentation. As a representation of H', $\sigma_1 \otimes \sigma_2$ is a direct sum of σ_2. Hence $\operatorname{Hom}_{H'}(\sigma_2, \sigma) \neq 0$. Since $\operatorname{End}_{H'}(\sigma_2) = \overline{\kappa}$, we have $\operatorname{Hom}_{H'}(\sigma_2, \sigma_1 \otimes \sigma_2) \simeq \sigma_1$. This is an isomorphism between H/H'-representations. Therefore, we have $\operatorname{Hom}_{H'}(\sigma_2, \sigma) \subset \sigma_1$. Since σ_1 is irreducible, we have $\operatorname{Hom}_{H'}(\sigma_2, \sigma) = \sigma_1$. Therefore, $\sigma = \sigma_1 \otimes \sigma_2$. □

We have the following calculations of Satake parameters.

– If π is a special representation, then $\mathcal{S}(\pi) = \{(T, \chi_{\mathrm{triv}})\}$ where $\chi_{\mathrm{triv}} \colon X_{T,*,0} = X_* \to \overline{\kappa}^\times$ is given by $\lambda \mapsto 1$ [Her11a, Proposition 7.4].

– If π is supersingular with the central character ω_π, then $\mathcal{S}(\pi) = \{(G, \chi_{\omega_\pi})\}$; here, the homomorphism $\chi_{\omega_\pi} \colon X_{G,*,0} \to \overline{\kappa}^\times$ is defined by $\chi_{\omega_\pi}(\lambda) = \omega_\pi(\lambda(\varpi))$ [Her11a, Definition 4.7].

Applying Proposition 3.7 and Corollary 3.24 for $(M_1, M_2, \pi_1, \pi_2) = (M_{\Pi_1}, M_{\Pi_{\sigma_1}}, \sigma_1, \sigma_{\Lambda,2})$, we have the following lemma.

Lemma 5.4. *We have $\mathcal{S}(I(\Lambda)) = \{(M_{\Pi_1}, \chi_{\omega_{\sigma_1}})\}$; here, $\chi_{\omega_{\sigma_1}} \colon X_{M_{\Pi_1},*,0} \to \overline{\kappa}^\times$ is defined by $\chi_{\omega_{\sigma_1}}(\lambda) = \omega_{\sigma_1}(\lambda(\varpi))$.*

5.2 Irreducibility of the representation

In this subsection, we assume that the derived group of G is simply connected. We prove the irreducibility of $I(\Lambda)$. We need a lemma.

Lemma 5.5. *Let $\Lambda = (\Pi_1, \Pi_2, \sigma_1) \in \mathcal{P}$, V an irreducible representation of K and ν its lowest weight. Assume that $\operatorname{Hom}_K(V, I(\Lambda)) \neq 0$ and $\alpha \in \Pi$ satisfies $\langle \Pi_1, \check{\alpha} \rangle = 0$. Then we have $\omega_{\sigma_1} \circ \check{\alpha}|_{\mathcal{O}^\times} = \nu \circ \check{\alpha}$.*

Before the proof, we give a remark on a result of [Gro]. Let $\overline{I}_1 = \mathrm{red}^{-1}(\overline{U}(\kappa))$ and $\overline{\mathrm{Sp}}_P$ the special representation for the finite group $G(\kappa)$. Then we have a K-homomorphism $\overline{\mathrm{Sp}}_P \hookrightarrow \mathrm{Sp}_P$ and under this embedding, we have $\overline{\mathrm{Sp}}_P^{\overline{B}(\kappa)} = \mathrm{Sp}_P^{\overline{I}} = \mathrm{Sp}_P^{\overline{I}_1}$ [Her11a, (7.5)]. (See also the proof of [Gro, Corollary 4.3].) Since $\overline{\mathrm{Sp}}_P \hookrightarrow \mathrm{Sp}_P$ is a K-homomorphism, we have $\overline{\mathrm{Sp}}_P^{\overline{U}(\kappa)} = \overline{\mathrm{Sp}}_P^{\overline{I}_1} \subset \mathrm{Sp}_P^{\overline{I}_1}$. Obviously, $\overline{\mathrm{Sp}}_P^{\overline{B}(\kappa)} \subset \overline{\mathrm{Sp}}_P^{\overline{U}(\kappa)}$. Hence $\overline{\mathrm{Sp}}_P^{\overline{U}(\kappa)} = \overline{\mathrm{Sp}}_P^{\overline{B}(\kappa)}$. In other words, $T(\kappa)$ acts trivially on $\overline{\mathrm{Sp}}_P^{\overline{U}(\kappa)}$.

Proof. Set $V_1 = V^{\overline{N_\Lambda}(\kappa)}$. Then V_1 is an irreducible representation of $M_\Lambda \cap K$ with a lowest weight ν. Moreover, we have $\mathrm{Hom}_{M_\Lambda\cap K}(V_1, \sigma_\Lambda) \neq 0$.

Let Q be the parabolic subgroup of M_Λ corresponding to $\Pi_1 \cup \Pi_2$. Then we have $\sigma_{\Lambda,2} = \mathrm{Sp}_{Q,M_\Lambda}$. Put $L = [M_{\Pi_{\sigma_1}}, M_{\Pi_{\sigma_1}}]$. This is an algebraic group and, since we assumed that the derived group of G (hence, also of $M_{\Pi_{\sigma_1}}$) is simply connected, we have $L(F) = [M_{\Pi_{\sigma_1}}(F), M_{\Pi_{\sigma_1}}(F)]$. Then $\sigma_{\Lambda,2}|_L = \mathrm{Sp}_{Q\cap L,L}$. Put $\sigma_2 = \sigma_{\Lambda,2}$ and $M_1 = M_{\Pi_1}$.

Fix $\psi \in \mathrm{Hom}_{M_\Lambda\cap K}(V_1, \sigma_\Lambda)\backslash\{0\}$ and consider V_1 as a subspace of σ_Λ. Let $v \in V_1$ be a lowest weight vector. Then we have $v \in \sigma_\Lambda^{\overline{I}_{M_\Lambda,1}}$ where $\overline{I}_{M_\Lambda,1}$ is the inverse image of $(M_\Lambda \cap \overline{U})(\kappa)$ in $M_\Lambda \cap K$. Since L acts on σ_1 trivially, we have $v \in \sigma_\Lambda^{\overline{I}_{M_\Lambda,1}} \subset \sigma_\Lambda^{\overline{I}_{M_\Lambda,1}\cap L} = \sigma_1 \otimes \sigma_2^{\overline{I}_{M_\Lambda,1}\cap L}$. Let $\overline{\sigma_2}$ be the special representation of $M_\Lambda(\kappa)$ with respect to the parabolic subgroup $Q(\kappa)$. Then, by the remark before the proof, we have $\overline{\sigma_2} \hookrightarrow \sigma_2$ and we have $\overline{\sigma_2}^{(\overline{U}\cap L)(\kappa)} = \sigma_2^{\overline{I}_{M_\Lambda,1}\cap L}$. Since $\langle \Pi_{\sigma_1}, \check{\Pi}_1\rangle = 0$, we have $\overline{U} \cap M_\Lambda \simeq (\overline{U} \cap L) \times (\overline{U} \cap [M_1, M_1])$ as algebraic groups. By the construction, $[M_1, M_1](\kappa)$ acts on $\overline{\sigma_2}$ trivially. Hence we have $\overline{\sigma_2}^{(\overline{U}\cap L)(\kappa)} = \overline{\sigma_2}^{(\overline{U}\cap M_\Lambda)(\kappa)}$. By the remark before the proof, $T(\kappa)$ acts on $\overline{\sigma_2}^{(\overline{U}\cap M_\Lambda)(\kappa)}$ trivially. Hence $T(\mathcal{O})$ acts on $\sigma_2^{\overline{I}_{M_\Lambda,1}\cap L}$ trivially.

Take α as in the lemma. Then $\mathrm{Im}\,\check{\alpha} \subset Z_{M_1}$. Hence for $t \in \mathcal{O}^\times$, $\check{\alpha}(t)$ acts on σ_1 by the scalar $\omega_{\sigma_1}(\check{\alpha}(t))$. By the above argument, $\check{\alpha}(t)$ acts on $\sigma_2^{\overline{I}_{M_\Lambda,1}\cap L}$ trivially. Hence it acts on $\sigma_\Lambda^{\overline{I}_{M_\Lambda,1}}$ by the scalar $\omega_{\sigma_1}(\check{\alpha}(t))$. On the other hand, $\check{\alpha}(t)$ acts on v by the scalar $t^{\langle\nu,\check{\alpha}\rangle} = \nu(\check{\alpha}(t))$. This gives the lemma. □

Remark 5.6. If we treat the Satake transform in a natural way (see Remark 2.5), Lemma 5.4 should be $\mathcal{S}(I(\Lambda)) = \{(M_{\Pi_1}, \omega_{\sigma_1})\}$. (We use a notation of Herzig [Her11a, Proposition 4.1].) Hence the above lemma should be a consequence of Lemma 5.4.

Proposition 5.7. *For $\Lambda \in \mathcal{P}$, $I(\Lambda)$ is irreducible.*

Proof. Take $\Lambda = (\Pi_1, \Pi_2, \sigma_1) \in \mathcal{P}$ and put $M_1 = M_{\Pi_1}$ and $M_2 = M_{\Pi_2}$. Let χ be the algebra homomorphism $\overline{\kappa}[X_{*,+}] \to \overline{\kappa}$ corresponding to $(M_1, \chi_{\omega_{\sigma_1}})$. Then $\mathcal{S}(I(\Lambda)) = \{\chi\}$. Let $\pi \subset I(\Lambda)$ be a subrepresentation of $I(\Lambda)$. Take an irreducible K-subrepresentation V of π. Then $\emptyset \neq \mathcal{S}(\pi, V) \subset \mathcal{S}(I(\Lambda)) = \{\chi\}$. Therefore, we have a non-zero homomorphism $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi \to \pi$.

Let ν be a lowest weight of V. We take V such that the set $\{\alpha \in \Pi\backslash\Pi_{M_\Lambda} \mid \langle \nu, \check{\alpha}\rangle = 0\}$ is minimal. We claim that this set is empty. Assume that there exists $\alpha \in \Pi\backslash\Pi_{M_\Lambda}$ such that $\langle \check{\alpha}, \nu\rangle = 0$. Put $\nu' = \nu - (q-1)\omega_\alpha$ and let V' be the irreducible K-representation with lowest weight ν'. Since $\alpha \notin \Pi_{M_\Lambda}$, we have $\alpha \notin \Pi_{\sigma_1}$. By the definition of Π_{σ_1}, we have:

- $\langle \check{\alpha}, \Pi_{M_1}\rangle \neq 0$; or
- $\omega_{\sigma_1}(\check{\alpha}(\varpi)) \neq 1$ or $\omega_{\sigma_1} \circ \check{\alpha}|_{\mathcal{O}^\times}$ is not trivial.

The above lemma shows that if $\langle \check{\alpha}, \Pi_{M_1}\rangle = 0$ then $\omega_{\sigma_1} \circ \check{\alpha}|_{\mathcal{O}^\times}$ is trivial. Therefore we have that $\langle \check{\alpha}, \Pi_{M_1}\rangle \neq 0$ or $\chi_{\omega_{\sigma_1}}(\check{\alpha}) \neq 1$. Hence we have $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi \simeq \text{c-Ind}_K^G(V') \otimes_{\mathcal{H}_G(V')} \chi$ by Theorem 4.1. Therefore, we get a non-zero homomorphism $\text{c-Ind}_K^G(V') \otimes_{\mathcal{H}_G(V')} \chi \to \pi$. Namely, V' is an irreducible K-subrepresentation of π. This contradicts the minimality of $\{\alpha \in \Pi\backslash\Pi_{M_\Lambda} \mid \langle \check{\alpha}, \nu\rangle = 0\}$.

Therefore, we have $\langle \nu, \check{\alpha}\rangle \neq 0$ for $\alpha \in \Pi\backslash\Pi_{M_\Lambda}$. Put $V_1 = V^{\overline{N_\Lambda}(\kappa)}$. Since χ is parameterized by $(M_1, \chi_{\omega_{\sigma_1}})$ and $M_1 \subset M_\Lambda$, χ factors through $S_G^{M_\Lambda}$. By [Her11a, Theorem 3.1], $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi \simeq \mathrm{Ind}_{P_\Lambda}^G(\text{c-Ind}_{M_\Lambda\cap K}^{M_\Lambda}(V_1) \otimes_{\mathcal{H}_{M_\Lambda}(V_1)} \chi)$. Therefore, we have $\mathrm{Ind}_{P_\Lambda}^G(\text{c-Ind}_{M_\Lambda\cap K}^{M_\Lambda}(V_1) \otimes_{\mathcal{H}_{M_\Lambda}(V_1)} \chi) \to \pi \hookrightarrow \mathrm{Ind}_{P_\Lambda}^G \sigma_\Lambda$. By Lemma 4.16, the composition is given by a certain homomorphism

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$\text{c-Ind}_{M_\Lambda \cap K}^{M_\Lambda}(V_1) \otimes_{\mathcal{H}_{M_\Lambda}(V_1)} \chi \to \sigma_\Lambda$. Since σ_Λ is irreducible, this homomorphism is surjective. Therefore, $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi \to \text{Ind}_{P_\Lambda}^G(\sigma_\Lambda)$ is surjective. In particular, $\pi \hookrightarrow \text{Ind}_{P_\Lambda}^G(\sigma_\Lambda)$ is surjective. Hence $\pi = \text{Ind}_{P_\Lambda}^G(\sigma_\Lambda)$. □

5.3 Classification theorem

We will use the following lemma.

LEMMA 5.8. *Let $P = MN$ be a parabolic subgroup, σ an irreducible admissible representation of M which is supersingular with respect to $(M \cap K, T, M \cap B)$ and ω_σ the central character of σ. Then $\text{Ind}_P^G(\sigma)$ has a filtration whose graded pieces are $\{I(\Pi_M, \Pi_2, \sigma) \mid \Pi_2 \subset \Pi_\sigma\}$.*

Proof. Let $P' = M'N'$ be the standard parabolic subgroup corresponding to $\Pi_M \cup \Pi_\sigma$. Then by Lemma 3.2, we can extend σ to M' such that $[M_{\Pi_\sigma}(F), M_{\Pi_\sigma}(F)]$ acts on it trivially. We have $\text{Ind}_{P\cap M'}^{M'}(\sigma) = (\text{Ind}_{P\cap M'}^{M'} \mathbf{1}_M) \otimes \sigma$. So we have $\text{Ind}_P^G(\sigma) = \text{Ind}_{P'}^G((\text{Ind}_{P\cap M'}^{M'} \mathbf{1}_{M'}) \otimes \sigma)$. The definition of the special representations implies that $\text{Ind}_{P\cap M'}^{M'} \mathbf{1}_{M'}$ has a filtration whose graded pieces are $\{\text{Sp}_{Q_2,M'}\}$ where Q_2 is a parabolic subgroup of M' which contains $P \cap M'$. Hence $\text{Ind}_P^G(\sigma)$ has a filtration whose graded pieces are $\{\text{Ind}_{P'}^G(\text{Sp}_{Q_2,M'} \otimes \sigma)\}$. Let $\Pi_2' \subset \Pi_{M'}$ be a subset corresponding to Q_2. Then we have $\text{Ind}_{P'}^G(\text{Sp}_{Q_2,M'} \otimes \sigma) = I(\Pi_M, \Pi_2' \backslash \Pi_M, \sigma)$. □

Remark 5.9. If the derived group of G is simply connected, then $I(\Lambda)$ is irreducible by Proposition 5.7. Hence the above lemma gives the composition factors of $\text{Ind}_P^G(\sigma)$. In particular, it has a finite length. The irreducibility of $I(\Lambda)$ will be proved in § 5.4. Hence the above lemma gives the composition factors of $\text{Ind}_P^G(\sigma)$ for any G.

PROPOSITION 5.10. *Assume that the derived group of G is simply connected. The correspondence $\Lambda \mapsto I(\Lambda)$ gives a bijection between $\mathcal{P}$ and the set of isomorphism classes of irreducible admissible representations.*

Proof. First, we prove that the map is surjective by induction on $\#\Pi$. Let π be an irreducible admissible representation. Let χ be an element of $\mathcal{S}(\pi)$ and assume that it is parameterized by (M_1, χ_{M_1}). We assume that M_1 is minimal. If $M_1 = G$, then π is supersingular. Therefore, we assume that $M_1 \neq G$. Take an irreducible K-representation V such that $\chi \in \mathcal{S}(\pi, V)$. Let ν be a lowest weight of V. We assume that $\Pi_{-\nu}$ is minimal with respect to the condition $\chi \in \mathcal{S}(\pi, V)$.

Assume that there exists $\alpha \in \Pi_{-\nu} \backslash \Pi_{M_1}$ such that $\langle \Pi_{M_1}, \check{\alpha} \rangle \neq 0$ or $\chi_{M_1}(\check{\alpha}) \neq 1$. Set $\nu' = \nu - (q-1)\omega_\alpha$ and let V' be the irreducible K-representation with lowest weight ν'. Then $\Pi_{-\nu'} = \Pi_{-\nu} \backslash \{\alpha\} \subsetneq \Pi_{-\nu}$. By Theorem 4.1, we have $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi \simeq \text{c-Ind}_K^G(V') \otimes_{\mathcal{H}_G(V')} \chi$. Hence $\chi \in \mathcal{S}(\pi, V')$. This contradicts the minimality of $\Pi_{-\nu}$. Therefore, for all $\alpha \in \Pi_{-\nu} \backslash \Pi_{M_1}$, $\langle \Pi_{M_1}, \check{\alpha} \rangle = 0$ and $\chi_{M_1}(\check{\alpha}) = 1$. From the first condition, $\langle \Pi_{-\nu} \backslash \Pi_{M_1}, \check{\Pi}_{M_1} \rangle = 0$.

Let $P = MN$ be a parabolic subgroup corresponding to $\Pi_{-\nu} \cup \Pi_{M_1}$. First assume that $M \neq G$. Put $V_1 = V^{\overline{N}(\kappa)}$. Then we have $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi \simeq \text{Ind}_P^G(\text{c-Ind}_{M\cap K}^M(V_1) \otimes_{\mathcal{H}_M(V_1)} \chi)$ [Her11a, Theorem 3.1]. Recall that we have a surjective homomorphism $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \chi \to \pi$. Hence there exist an irreducible admissible representation σ of M and a surjective homomorphism $\text{Ind}_P^G(\sigma) \to \pi$ [Her11a, Lemma 9.9]. By inductive hypothesis, $\sigma = I_M(\Lambda')$ for some $\Lambda' \in \mathcal{P}_M$. Hence there exists a parabolic subgroup $P_0 = M_0 N_0 \subset P$ and an irreducible admissible representation σ_0 of M_0 which is supersingular with respect to $(M_0 \cap K, T, M_0 \cap B)$ such that σ is a subquotient of $\text{Ind}_{P_0 \cap M}^M \sigma_0$ by Lemma 5.8. Hence π is a subquotient of $\text{Ind}_{P_0}^G(\sigma_0)$. By Lemma 5.8, all composition factors of $\text{Ind}_{P_0}^G(\sigma_0)$ are $I(\Lambda)$ for some $\Lambda \in \mathcal{P}$. Hence $\pi = I(\Lambda)$ for some $\Lambda \in \mathcal{P}$.

Therefore, we may assume that $\Pi_{-\nu} \cup \Pi_{M_1} = \Pi$. Let $P' = M'N'$ be the standard parabolic subgroup corresponding to $\Pi \backslash \Pi_{M_1}$. Then for all $\alpha \in \Pi_{M'}$, $\langle \nu, \check{\alpha} \rangle = 0$, $\langle \alpha, \check{\Pi}_{M_1} \rangle = 0$ and

$\chi_{M_1}(\check{\alpha}) = 1$. Set $L' = [M', M']$. Then the group of coweights $X_{L',*}$ of $L' \cap T$ is $\mathbb{Z}\check{\Pi}_{M'}$ which is a subgroup of $X_* \cap \Pi_{M_1}^{\perp}$. Put $X_{L',*,+} = X_{*,+} \cap \mathbb{Z}\check{\Pi}_{M'}$. By Lemma 3.19 and Proposition 3.14, we have $\mathcal{S}(\pi, V)|_{\overline{\kappa}[X_{L',*,+}]} \subset \mathcal{S}(\pi|_{M'}, V^{\overline{N}'(\kappa)})|_{\overline{\kappa}[X_{L',*,+}]} \subset \mathcal{S}(\pi|_{L'}, V^{\overline{N}'(\kappa)}|_{L'\cap K})$. Since $\langle \nu, \check{\Pi}_{M'} \rangle = 0$, $V^{\overline{N}'(\kappa)}|_{L'\cap K}$ is trivial. Therefore, $\chi|_{\overline{\kappa}[X_{L',*,+}]} \in \mathcal{S}(\pi|_{L'}, \mathbf{1}_{L'\cap K})$. Set $\chi' = \chi|_{\overline{\kappa}[X_{L',*,+}]}$. We have a non-zero homomorphism $\text{c-Ind}_{L'\cap K}^{L'} \mathbf{1}_{L'\cap K} \otimes_{\mathcal{H}_{L'}(\mathbf{1}_{L'\cap K})} \chi' \to \pi$. Since χ is parameterized by (M_1, χ_{M_1}), χ' is parameterized by $(L' \cap T, \chi_{M_1}|_{X_{L',*}})$. Since we have $\chi_{M_1}(\check{\alpha}) = 1$ for all $\alpha \in \Pi_{M'}$, we have $\chi_{M_1}|_{X_{L',*}} = \mathbf{1}_{X_{L',*}}$. Hence χ' is parameterized by $(L' \cap T, \mathbf{1}_{X_{L',*}})$. Therefore, by Proposition 4.7, the set of composition factors of $\text{c-Ind}_{L'\cap K}^{L'} \mathbf{1}_{L'\cap K} \otimes_{\mathcal{H}_{L'}(\mathbf{1}_{L'\cap K})} \chi'$ is $\{\mathrm{Sp}_{Q',L'} \mid Q' \subset L' \text{ is a parabolic subgroup}\}$. Hence there exists a unique parabolic subgroup $P_2 = M_2 N_2$ such that $\Pi_{M_1} \subset \Pi_{M_2}$ and $\mathrm{Sp}_{P_2\cap L',L'} \hookrightarrow \pi$. Let σ_2 be the special representation Sp_{P_2}. Then the restriction of σ_2 to L' is $\mathrm{Sp}_{P_2\cap L',L'}$. Put $\sigma_1 = \mathrm{Hom}_{L'}(\sigma_2, \pi)$. This is non-zero. By Lemma 5.3, σ_1 is an irreducible representation of G and $\sigma_1 \otimes \sigma_2 \xrightarrow{\sim} \pi$.

We prove that σ_1 is admissible. Let K' be an open compact subgroup and take an open compact subgroup K'' such that $\sigma_2^{K''} \neq 0$. Let K''' be an open compact subgroup which is contained in K' and K''. Then we have $\sigma_1^{K'} \otimes \sigma_2^{K''} \subset \sigma_1^{K'''} \otimes \sigma_2^{K''} \subset (\sigma_1 \otimes \sigma_2)^{K'''} = \pi^{K'''}$. Since π is admissible, $\pi^{K'''}$ is finite dimensional. Hence the dimension of $\sigma_1^{K'}$ is finite.

We prove σ_1 is supersingular with respect to $(M_1 \cap K, T, M_1 \cap B)$ as a representation of M_1. Since L' acts on σ_1 trivially, σ_1 is regarded as a representation of G/L'. By Lemma 3.2, $M_1 \to G/L'$ is surjective. Therefore, $\sigma_1|_{M_1}$ is irreducible and admissible. By inductive hypothesis, $\sigma_1|_{M_1} \simeq I_{M_1}(\Lambda')$ for some $\Lambda' \in \mathcal{P}_{M_1}$. In particular, $\#\mathcal{S}(\sigma_1|_{M_1}) = 1$. Since $\chi \in \mathcal{S}(\sigma_1 \otimes \sigma_2)$ is parameterized by (M_1, χ_{M_1}), the element of $\mathcal{S}(\sigma_1|_{M_1})$ is parameterized by (M_1, χ'_{M_1}) for some χ'_{M_1} by Corollary 3.22. Hence σ_1 is supersingular.

We prove that the map is injective. Let $\Lambda' = (\Pi'_1, \Pi'_2, \sigma'_1)$ and assume that $I(\Lambda) \simeq I(\Lambda')$. Then we have $\mathcal{S}(I(\Lambda), V) = \mathcal{S}(I(\Lambda'), V) \neq \emptyset$ for some irreducible representation V of K. By Lemma 5.4, $(M_{\Pi_1}, \chi_{\omega_{\sigma_1}}) = (M_{\Pi'_1}, \chi_{\omega_{\sigma'_1}})$. Hence $\Pi_1 = \Pi'_1$. Let ν be a lowest weight of V. Then by Lemma 5.5, for $\alpha \in \Pi$ such that $\langle \Pi_1, \check{\alpha} \rangle = 0$, $\omega_{\sigma_1} \circ \check{\alpha}|_{\mathcal{O}^\times} = \nu \circ \check{\alpha} = \omega_{\sigma'_1} \circ \check{\alpha}|_{\mathcal{O}^\times}$. On the other hand, we have $\omega_{\sigma_1} \circ \check{\alpha}(\varpi) = \chi_{\omega_{\sigma_1}}(\check{\alpha}) = \omega_{\sigma'_1} \circ \check{\alpha}(\varpi)$. Hence $\omega_{\sigma_1} \circ \check{\alpha} = \omega_{\sigma'_1} \circ \check{\alpha}$. Therefore, we have $\Pi_{\sigma_1} = \Pi_{\sigma'_1}$. Hence $P_\Lambda = P_{\Lambda'}$.

Now we have $\mathrm{Ind}_{P_\Lambda}^G(\sigma_\Lambda) \simeq \mathrm{Ind}_{P_\Lambda}^G(\sigma_{\Lambda'})$. By Lemma 4.16, we have a non-zero homomorphism $\sigma_\Lambda \to \sigma_{\Lambda'}$. Since σ_Λ and $\sigma_{\Lambda'}$ are irreducible, $\sigma_\Lambda \simeq \sigma_{\Lambda'}$. Set $L = [M_{\Pi_{\sigma_1}}(F), M_{\Pi_{\sigma_1}}(F)]$. As a representation of L, σ_Λ is a direct sum of special representations $\mathrm{Sp}_{Q_2,L}$ where Q_2 is the parabolic subgroup of L corresponding to Π_2. Hence we have $\Pi_2 = \Pi'_2$. Therefore, $\sigma_{\Lambda,2} \simeq \sigma_{\Lambda',2}$. Hence we have $\sigma_1 \simeq \mathrm{Hom}_L(\sigma_{2,\Lambda}, \sigma_\Lambda) \simeq \mathrm{Hom}_L(\sigma_{2,\Lambda'}, \sigma_{\Lambda'}) \simeq \sigma'_1$. We get $\Lambda = \Lambda'$. □

5.4 General case and corollaries

Theorem 5.11. *Let G be a connected split reductive algebraic group. Then $I(\Lambda)$ is irreducible for all $\Lambda \in \mathcal{P}$ and $\Lambda \mapsto I(\Lambda)$ gives a bijection between $\mathcal{P}$ and the set of isomorphism classes of irreducible admissible representations.*

Proof. Take a z-extension $1 \to Z \to \widetilde{G} \to G \to 1$ of G. For each parabolic subgroup $P = MN$, let $\widetilde{M}$ be the Levi subgroup of the parabolic subgroup of $\widetilde{G}$ corresponding to Π_M. Then $1 \to Z \to \widetilde{M} \to M \to 1$ is a z-extension of M. For each representation π of G, let $\widetilde{\pi}$ be the pull-back of π to $\widetilde{G}$. Then we have $I_G(\Pi_1, \Pi_2, \sigma_1)^\sim = I_{\widetilde{G}}(\Pi_1, \Pi_2, \widetilde{\sigma_1})$. In general, the representation π of G is supersingular with respect to $(\widetilde{K}, \widetilde{B}, \widetilde{T})$ if and only if its pull-back to $\widetilde{G}$ is supersingular with respect to (K, B, T) by Lemma 3.25; here, $\widetilde{K}$ is as in Lemma 2.1 and $\widetilde{B}$, $\widetilde{T}$ are the inverse images of B, T, respectively. By Proposition 5.7, this is irreducible. Hence $I_G(\Lambda)$ is irreducible for $\Lambda \in \mathcal{P}$.

Obviously, we also have that $I_G(\Pi_1, \Pi_2, \sigma_1) \simeq I_G(\Pi_1', \Pi_2', \sigma_1')$ if and only if $I_{\widetilde{G}}(\Pi_1, \Pi_2, \widetilde{\sigma_1}) \simeq I_{\widetilde{G}}(\Pi_1', \Pi_2', \widetilde{\sigma_1'})$. Hence we have $\Pi_1 = \Pi_1'$, $\Pi_2 = \Pi_2'$ and $\widetilde{\sigma_1} \simeq \widetilde{\sigma_1'}$ by Proposition 5.10. Hence we have $\sigma_1 \simeq \sigma_1'$.

Let π be an irreducible admissible representation of G. Then there exists $\Lambda_0 = (\Pi_1, \Pi_2, \sigma_{1,0}) \in \mathcal{P}_{\widetilde{G}}$ such that $\widetilde{\pi} = I_{\widetilde{G}}(\Lambda_0)$. Since Z is contained in the center of M_{Π_1}, it acts on $\sigma_{1,0}$ by a character. By the construction of $I_{\widetilde{G}}(\Lambda_0)$, Z acts on $I_{\widetilde{G}}(\Lambda_0) \simeq \widetilde{\pi}$ by the same scalar. It is trivial since Z acts on $\widetilde{\pi}$ trivially. Hence Z acts on $\sigma_{1,0}$ trivially; namely, $\sigma_{1,0} \simeq \widetilde{\sigma_1}$ for some representation of G. Hence $\pi = I_G(\Pi_1, \Pi_2, \sigma_1)$. This gives us the theorem. □

We give corollaries of this theorem.

COROLLARY 5.12. *For any irreducible admissible representation π of G, $\#\mathcal{S}(\pi) = 1$.*

Proof. Obvious from Lemma 5.4 and Theorem 5.11. □

COROLLARY 5.13. *Let π be an irreducible admissible representation of G. Then the following conditions are equivalent.*

(i) *The representation π is supersingular.*

(ii) *The representation π is supersingular with respect to (K, T, B).*

(iii) *The representation π is supercuspidal.*

Proof. Take $\Lambda = (\Pi_1, \Pi_2, \sigma_1) \in \mathcal{P}$ such that $\pi = I(\Lambda)$. Then by Lemma 5.4, π is supersingular with respect to (K, T, B) if and only if $\Pi_1 = \Pi$. By Lemma 5.8, π is a subquotient of $\mathrm{Ind}_{P_1}^G(\sigma_1)$. Hence, if π is not supersingular with respect to (K, T, B), then π is not supercuspidal.

Assume that π is a subquotient of $\mathrm{Ind}_{P_0}^G \sigma_0$ for a proper parabolic subgroup $P_0 = M_0 N_0$ and an irreducible admissible representation σ_0. By Lemma 5.8, we may assume σ_0 is supersingular with respect to (K, T, B). By Lemma 5.8, $P_{\Pi_1} = P_0$. Hence π is not supersingular with respect to (K, T, B).

Hence (ii) and (iii) are equivalent. Since the property (iii) is independent of a choice of (K, T, B), (i) and (ii) are equivalent. □

COROLLARY 5.14. *Let $P = MN$ be a parabolic subgroup and σ a finite length admissible representation of M. Then $\mathrm{Ind}_P^G \sigma$ has a finite length.*

Proof. We may assume σ is irreducible. This follows from Lemma 5.8 and Remark 5.9. □

COROLLARY 5.15. *Let $\nu\colon T \to \overline{\kappa}^\times$ be a character. Then $\mathrm{Ind}_B^G(\nu)$ has a length 2^C where $C = \#\{\alpha \in \Pi \mid \nu \circ \check{\alpha} = \mathbf{1}_{\mathrm{GL}_1}\}$. In particular, $\mathrm{Ind}_B^G(\nu)$ is irreducible if and only if $\nu \circ \check{\alpha} \neq \mathbf{1}_{\mathrm{GL}_1}$ for all $\alpha \in \Pi$.*

Proof. Notice that any character of T is supersingular. Hence this follows from Lemma 5.8 and Remark 5.9. □

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