18

Characteristic problems for the conformal field equations

This chapter discusses the basic theory of characteristic problems for the conformal field equations. Characteristic problems have been of great conceptual value in the development of the modern theory of gravitational radiation. Indeed, the seminal works by Bondi et al. (1962) and Sachs (1962b), in which the modern understanding of gravitational waves was established, were carried out in a setting based on a characteristic initial value problem; see also Sachs (1962c) and Newman and Penrose (1962). The connection between characteristic problems and the notion of asymptotic flatness, already present in the seminal work by Penrose (1963), was further elaborated in Penrose (1965, 1980). From a mathematical point of view, the realisation that the characteristic initial value problem for the Einstein field equations leads to a symmetric hyperbolic evolution system for which the machinery of the theory of partial differential equations (PDEs) is available was first established in Friedrich (1981b). In Friedrich (1981a, 1982) these ideas were subsequently extended to a situation in which part of the data is prescribed at null infinity –a so-called ***asymptotic characteristic initial value problem***, the subject of this chapter. These results established the local existence of *analytic solutions* and were later extended to the smooth case by Kánnár (1996b) using the method of reduction to a standard Cauchy problem by Rendall (1990); see Section 12.5.3.

There are two basic types of asymptotic characteristic problem for the conformal Einstein field equations. The first type is the so-called ***standard asymptotic characteristic problem*** – introduced in Friedrich (1981b) – where initial data are prescribed on null infinity and a null hypersurface intersecting null infinity in a two-dimensional surface with the topology of a 2-sphere; see Figure 18.1, left. In the second type – the so-called ***characteristic problem on a cone***, first discussed in Friedrich (1986c) – one prescribes information on a null cone down to its vertex; see Figure 18.1, right. For reasons discussed in Section 12.5, characteristic problems on a cone are more technically involved. Existence results have been obtained in Chruściel and Paetz (2013).

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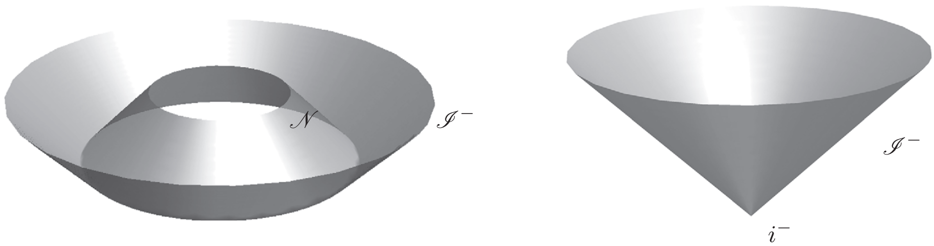

Figure 18.1 Two possible asymptotic characteristic problems for the conformal field equations: on the left, initial data are prescribed on an outgoing null hypersurface $\mathscr{N}$ and null infinity $\mathscr{I}^-$; on the right, data are prescribed on a null cone representing past null infinity $\mathscr{I}^-$. The vertex of the cone corresponds to past timelike infinity, i^-.

The standard and characteristic initial value problems have several structural properties in common. Moreover, the characteristic problem on a cone can be regarded as a limiting case of the standard characteristic problem. In both cases, the Einstein field equations on the initial hypersurfaces split into a set of ***interior (or intrinsic) equations*** and a set of ***transverse equations***. The interior equations split, in turn, into ***constraint equations*** which need to be satisfied only on some subsets of the initial hypersurface (the intersection of the null hypersurfaces or the vertex of the cone) and ***transport equations*** which propagate information along the generators of the null hypersurfaces. The transverse equations dictate the evolution off the initial hypersurfaces. One of the key aspects of the analysis of asymptotic characteristic problems is the identification of ***freely specifiable data*** from which the full data for the evolution equations can be derived. An appealing feature of this type of setting is the natural interpretation of the free data in terms of radiation fields so that a clear-cut connection with the theory of asymptotics as discussed in Chapter 10 can be established.

The discussion in the present chapter is mostly concerned with standard characteristic problems. Certain aspects of the characteristic problem on a cone are briefly considered. The existence results discussed are local in nature. That is, one obtains existence of solutions in a neighbourhood of the intersection of the null hypersurfaces or the vertex of the initial cone. From the perspective of the physical spacetime these local neighbourhoods represent unbounded domains in the asymptotic region.

18.1 Geometric and gauge aspects of the standard characteristic initial value problem

This section provides a discussion of the geometric setting and the gauge fixing procedure for the standard asymptotic characteristic problem. Taking into account the general theory of characteristic problems described in Section 12.5.1 one can consider two possible configurations (see Figure 18.2): (i) that of a

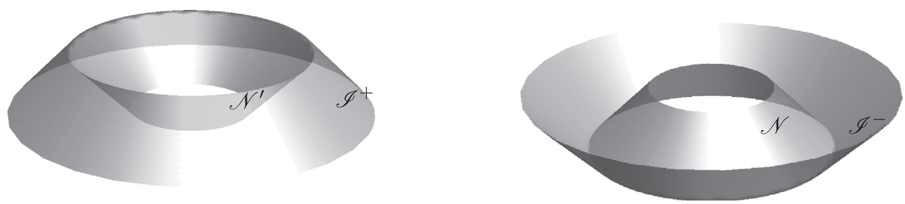

Figure 18.2 The two possible *standard* asymptotic characteristic problems for the conformal Einstein field equations. Case (i) where data are prescribed on a future-oriented (*outgoing*) null hypersurface $\mathscr{N}'$ and future null infinity $\mathscr{I}^+$, and case (ii) where data are prescribed on a past-oriented (*incoming*) null hypersurface $\mathscr{N}$ and past null infinity $\mathscr{I}^-$.

future-oriented (i.e. *outgoing*) null hypersurface intersecting future null infinity or (ii) a past-oriented (i.e. *incoming*) null hypersurface intersecting past null infinity. In order to compare with the characteristic problem on a cone, the present discussion focuses in the latter case. A careful inspection of the setting discussed here leads to the formulation of case (i).

18.1.1 Geometric setting

In what follows, let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ denote a conformal extension of an asymptotically simple spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfying $\boldsymbol{Ric}[\tilde{\boldsymbol{g}}] = 0$ which contains past null infinity $\mathscr{I}^-$. Let $\mathcal{W}$ denote a region of $\mathcal{M}$ with $\mathcal{W} \approx \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{S}^2$ bounded by an *incoming null hypersurface* $\mathscr{N}$ and past null infinity $\mathscr{I}^-$. It will be assumed that both $\mathscr{N}$ and $\mathscr{I}^-$ have the topology of $\mathbb{R}^+ \times \mathbb{S}^2$. Let $\mathscr{Z} \equiv \mathscr{N} \cap \mathscr{I}^-$ with $\mathscr{Z} \approx \mathbb{S}^2$. One has that $\mathcal{W} \subset J^+(\mathscr{Z})$. A schematic representation of the geometric setting can be seen in Figure 18.3.

An adapted coordinate system (x^μ) and an associated null tetrad $\{\boldsymbol{e}_{\boldsymbol{AA}'}\}$will be used to describe the geometry of the region $\mathcal{W}$. Let $\{\boldsymbol{\omega}^{\boldsymbol{AA}'}\}$ denote the associated coframe and require that

$$\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{AA}'}, \boldsymbol{e}_{\boldsymbol{BB}'}) = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A}'\boldsymbol{B}'}. \tag{18.1}$$

On $\mathscr{Z}$ one considers some coordinate system $(x^{\mathcal{A}})$ where $\mathcal{A} = 2, 3$. The complex vectors $\boldsymbol{e}_{\boldsymbol{01}'}$ and $\boldsymbol{e}_{\boldsymbol{10}'} = \overline{\boldsymbol{e}_{\boldsymbol{01}'}}$ of the null tetrad $\{\boldsymbol{e}_{\boldsymbol{AA}'}\}$ will be chosen so that they span the tangent bundle $T(\mathscr{Z})$ – recall that in standard Newman-Penrose notation the vectors $\boldsymbol{e}_{\boldsymbol{01}'}$ and $\boldsymbol{e}_{\boldsymbol{10}'}$ correspond to $\boldsymbol{m}$ and $\overline{\boldsymbol{m}}$.

Now, choose $\boldsymbol{e}_{\boldsymbol{00}'}$ so that, on $\mathscr{I}^-$, it is tangent to the null generators of the conformal boundary – in standard Newman-Penrose notation this vector corresponds to $\boldsymbol{l}$. Let v denote an affine parameter of these generators with the property that $v|_{\mathscr{Z}} = 0$. Thus, one has that $\boldsymbol{e}_{\boldsymbol{00}'} \simeq \boldsymbol{\partial}_v$ where, following the conventions of Chapter 10, the symbol $\simeq$ denotes equality at $\mathscr{I}^-$. The vectors $\boldsymbol{e}_{\boldsymbol{01}'}$ and $\boldsymbol{e}_{\boldsymbol{10}'}$ can be extended to the rest of $\mathscr{I}^-$ by parallel propagation along the null generators. Accordingly, one has

$$\nabla_{\boldsymbol{00}'}\boldsymbol{e}_{\boldsymbol{00}'} \simeq 0, \quad \nabla_{\boldsymbol{00}'}\boldsymbol{e}_{\boldsymbol{01}'} \simeq 0, \quad \nabla_{\boldsymbol{00}'}\boldsymbol{e}_{\boldsymbol{10}'} \simeq 0, \qquad \text{on } \mathscr{I}^-, \tag{18.2}$$

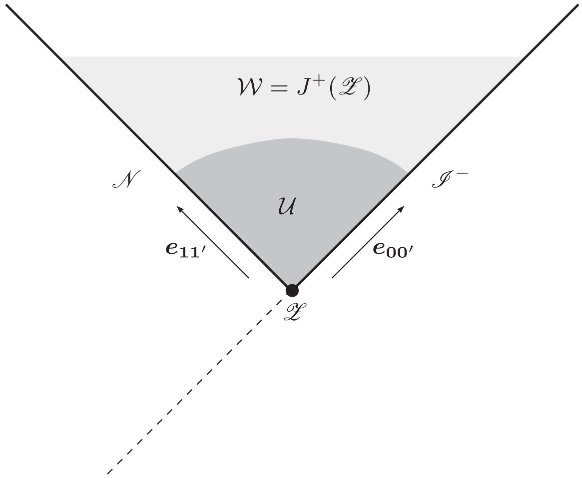

Figure 18.3 Schematic representation of the set up for the standard asymptotic characteristic problem. The existence results are restricted to a neighbourhood $\mathcal{U}$ of $\mathscr{Z}$ in $J^+(\mathscr{Z})$.

where $\nabla_{\mathbf{00'}} \equiv \boldsymbol{e}_{\mathbf{00'}}{}^a \nabla_a$ is the directional derivative in the direction of $\boldsymbol{e}_{\mathbf{00'}}$. Given $v_\bullet \in [0, \infty)$, let $\mathscr{Z}_{v_\bullet} \subset \mathscr{I}^-$ denote the two-dimensional surfaces given by

$$\mathscr{Z}_{v_\bullet} \equiv \{p \in \mathscr{I}^- \mid v(p) = v_\bullet\}.$$

As a result of their parallel propagation, the vectors $\boldsymbol{e}_{\mathbf{01'}}$ and $\boldsymbol{e}_{\mathbf{10'}}$ span $T(\mathscr{Z}_{v_\bullet})$. Having fixed the vectors $\boldsymbol{e}_{\mathbf{00'}}$, $\boldsymbol{e}_{\mathbf{01'}}$ and $\boldsymbol{e}_{\mathbf{10'}}$ on $\mathscr{I}^-$, regarding the conformal boundary as a submanifold of $\mathcal{M}$, and given that the spacetime metric $\boldsymbol{g}$ is assumed to be known, it follows that at every point $p \in \mathscr{I}^-$, there exists a unique future-pointing null vector linearly independent to $\{\boldsymbol{e}_{\mathbf{00'}}, \boldsymbol{e}_{\mathbf{01'}}, \boldsymbol{e}_{\mathbf{10'}}\}$. This vector is used to complete the null frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ on $\mathscr{I}^-$ – accordingly, it will be denoted by $\boldsymbol{e}_{\mathbf{11'}}$, or $\boldsymbol{n}$ in Newman-Penrose notation. The vector $\boldsymbol{e}_{\mathbf{11'}}$ is fixed by the four conditions

$$\boldsymbol{g}(\boldsymbol{e}_{\mathbf{11'}}, \boldsymbol{e}_{\boldsymbol{BB'}}) = \epsilon_{\mathbf{1}\boldsymbol{B}} \epsilon_{\mathbf{1'}\boldsymbol{B'}}.$$

Now, for fixed $v_\bullet$, there exists (at least locally) a unique null hypersurface $\mathscr{N}_{v_\bullet}$ in $\mathcal{M}$ satisfying $\mathscr{N}_{v_\bullet} \cap \mathscr{I}^- = \mathscr{Z}_{v_\bullet}$ such that at $\mathscr{Z}_{v_\bullet}$ the vector $\boldsymbol{e}_{\mathbf{11'}}$ is tangent to $\mathscr{N}_{v_\bullet}$ – this involves solving the eikonal equation $\boldsymbol{g}(\mathbf{d}\Phi, \mathbf{d}\Phi) = 0$ for some scalar $\Phi \in \mathcal{X}(\mathcal{W})$ near $\mathscr{I}^-$ with the appropriate initial conditions; for further details see, for example, Stewart (1991), section 4.3. By varying $v_\bullet$ one thus obtains (at least locally) a foliation of null hypersurfaces intersecting $\mathscr{I}^-$. Thence, the affine parameter v along the null generators of $\mathscr{I}^-$ can be used as a coordinate on $\mathcal{W}$. Accordingly, one sets $x^0 = v$, and has

$$\mathscr{N}_{v_\bullet} \equiv \{p \in \mathcal{W} \mid v(p) = v_\bullet\},$$

so that the normal to $\mathscr{N}_{v_\bullet}$ is given by $\mathbf{d}v$. The vector $\boldsymbol{e}_{\mathbf{11'}}$ can now be extended into $\mathcal{W}$ by requiring it to be tangent to the generators of these hypersurfaces; that is, one has

$$\boldsymbol{e}_{\mathbf{11'}} = \boldsymbol{g}^\sharp(\mathbf{d}x^0, \cdot). \tag{18.3}$$

Let r denote an affine parameter of the integral curves of $\boldsymbol{e}_{\mathbf{11'}}$ so that one can write $\boldsymbol{e}_{\mathbf{11'}} = \boldsymbol{\partial}_r$. Without loss of generality one can choose $r \simeq 0$. The coordinate system (x^μ) on $\mathcal{W}$ is then completed by setting $x^1 = r$ and by extending the coordinates $(x^{\mathcal{A}})$ on $\mathscr{Z}_v$ so that they are constant along the integral curves of $\boldsymbol{e}_{\mathbf{00'}}$ and $\boldsymbol{e}_{\mathbf{11'}}$. As a consequence of this construction one has

$$\mathscr{N} \equiv \{p \in \mathcal{W} \mid x^0(p) = 0\}, \qquad \mathscr{I}^- \equiv \{p \in \mathcal{W} \mid x^1(p) = 0\}.$$

The vectors $\boldsymbol{e}_{\mathbf{00'}}$, $\boldsymbol{e}_{\mathbf{01'}}$ and $\boldsymbol{e}_{\mathbf{10'}}$ can be extended off $\mathscr{I}^-$ by parallel propagation along the direction of $\boldsymbol{e}_{\mathbf{11'}}$. Accordingly, one has

$$\nabla_{\mathbf{11'}}\boldsymbol{e}_{\mathbf{11'}} = 0, \qquad \nabla_{\mathbf{11'}}\boldsymbol{e}_{\mathbf{01'}} = 0, \qquad \nabla_{\mathbf{11'}}\boldsymbol{e}_{\mathbf{10'}} = 0, \qquad \text{on } \mathcal{W}. \tag{18.4}$$

To obtain an explicit expression for the frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ in the coordinates $(x^\mu) = (v, r, x^{\mathcal{A}})$, it is observed that from Equation (18.3) – rewritten in the form $\boldsymbol{g}(\boldsymbol{\partial}_r, \cdot) = \langle \mathbf{d}v, \cdot\rangle$ – one obtains the pairings

$$\boldsymbol{g}(\boldsymbol{\partial}_r, \boldsymbol{\partial}_v) = 1, \qquad \boldsymbol{g}(\boldsymbol{\partial}_r, \boldsymbol{\partial}_r) = 0, \qquad \boldsymbol{g}(\boldsymbol{\partial}_r, \boldsymbol{\partial}_{\mathcal{A}}) = 0. \tag{18.5}$$

Taking into account the above, the most general form for the frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ consistent with Equations (18.1) and (18.3) is given by

$$\begin{aligned}
\boldsymbol{e}_{\mathbf{00'}} &= \boldsymbol{\partial}_v + U\boldsymbol{\partial}_r + X^{\mathcal{A}}\boldsymbol{\partial}_{\mathcal{A}}, \\
\boldsymbol{e}_{\mathbf{11'}} &= \boldsymbol{\partial}_r, \\
\boldsymbol{e}_{\mathbf{01'}} &= \omega\boldsymbol{\partial}_r + \xi^{\mathcal{A}}\boldsymbol{\partial}_{\mathcal{A}}, \\
\boldsymbol{e}_{\mathbf{10'}} &= \bar{\omega}\boldsymbol{\partial}_r + \bar{\xi}^{\mathcal{A}}\boldsymbol{\partial}_{\mathcal{A}},
\end{aligned}$$

where U and $X^{\mathcal{A}}$ are real functions and ω and $\xi^{\mathcal{A}}$ are complex functions. Observe, in particular, that because of the conditions in (18.5), $\boldsymbol{e}_{\mathbf{01'}}$ and $\boldsymbol{e}_{\mathbf{10'}}$ cannot have a v-component. Using, again, relation (18.1) one finds that the components $g^{\mu\nu} = \boldsymbol{g}^\sharp(\mathbf{d}x^\mu, \mathbf{d}x^\nu)$ are of the form

$$(g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & g^{11} & g^{1\mathcal{A}} \\ 0 & g^{\mathcal{A}1} & g^{\mathcal{A}\mathcal{B}} \end{pmatrix},$$

where

$$g^{11} = 2(U - \omega\bar{\omega}), \qquad g^{1\mathcal{A}} = X^{\mathcal{A}} - (\xi^{\mathcal{A}}\bar{\omega} + \bar{\xi}^{\mathcal{A}}\omega), \qquad g^{\mathcal{A}\mathcal{B}} = -(\xi^{\mathcal{A}}\bar{\xi}^{\mathcal{B}} + \bar{\xi}^{\mathcal{A}}\xi^{\mathcal{B}}).$$

In particular, one has that $U \simeq 0$, $X^{\mathcal{A}} \simeq 0$, $\omega \simeq 0$, consistent with the fact that $\boldsymbol{e}_{\mathbf{11'}}$ is tangent to the generators of null infinity and that v is an affine parameter; hence, $\boldsymbol{e}_{\mathbf{11'}} \simeq \boldsymbol{\partial}_v$. Observe also that $\boldsymbol{e}_{\mathbf{01'}} \simeq \xi^{\mathcal{A}}\boldsymbol{\partial}_{\mathcal{A}}$. Thus, the pull-back to $\mathscr{Z}_v$ of

$$g^{\mathcal{AB}}\boldsymbol{\partial}_{\mathcal{A}} \otimes \boldsymbol{\partial}_{\mathcal{B}} = -(\xi^{\mathcal{A}}\bar{\xi}^{\mathcal{B}} + \bar{\xi}^{\mathcal{A}}\xi^{\mathcal{B}})\boldsymbol{\partial}_{\mathcal{A}} \otimes \boldsymbol{\partial}_{\mathcal{B}},$$

to be denoted by $\boldsymbol{\varsigma}^{\sharp}$, corresponds to the two-dimensional (contravariant) metric of the sections of null infinity. Now, by assumption $\mathscr{Z}_v \approx \mathbb{S}^2$ so that $\boldsymbol{\varsigma}$ is conformal to the standard metric of $\mathbb{S}^2$.

Finally, combining the propagation conditions (18.2) and (18.4) with the definition of the spin connection coefficients – see Equations (3.31) and (3.33) – in the form

$$\Gamma_{\boldsymbol{AA'BC}} = \frac{1}{2}\epsilon_{\boldsymbol{BP}}\langle \boldsymbol{\omega}^{\boldsymbol{PQ'}}, \nabla_{\boldsymbol{AA'}}\boldsymbol{e}_{\boldsymbol{CQ'}}\rangle,$$

one finds

$$\Gamma_{\mathbf{00'10}} \simeq 0, \qquad \Gamma_{\mathbf{00'00}} \simeq 0,$$

and

$$\Gamma_{\mathbf{00'11}} = \bar{\Gamma}_{\mathbf{10'1'0'}} + \Gamma_{\mathbf{10'10}}, \quad \Gamma_{\mathbf{01'11}} = \bar{\Gamma}_{\mathbf{10'1'1'}}, \quad \Gamma_{\mathbf{11'}\boldsymbol{AB}} = 0, \quad \text{on } \mathcal{W} \subset \mathcal{M}.$$

The discussion of this section is summarised in the following

Lemma 18.1 (***frame gauge conditions for the standard characteristic problem***) *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote an asymptotically simple spacetime satisfying $\boldsymbol{Ric}[\tilde{\boldsymbol{g}}] = 0$ and let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ with $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$ be a conformal extension thereof for which the condition $\Xi = 0$ describes past null infinity $\mathscr{I}^-$. The frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ can be chosen so that, given a null hypersurface $\mathscr{N}$ intersecting $\mathscr{I}^-$ on $\mathscr{Z} \approx \mathbb{S}^2$, one has*

$$\Gamma_{\mathbf{00'11}} = \bar{\Gamma}_{\mathbf{10'1'0'}} + \Gamma_{\mathbf{10'10}},$$
$$\Gamma_{\mathbf{01'11}} = \bar{\Gamma}_{\mathbf{10'1'1'}}, \quad \Gamma_{\mathbf{11'}\boldsymbol{AB}} = 0, \quad \textit{on } \mathcal{W} \subset \mathcal{M}.$$

In addition, one has that

$$\Gamma_{\mathbf{00'01}} = \Gamma_{\mathbf{00'10}} = \Gamma_{\mathbf{00'00}} = U = X^{\mathcal{A}} = \omega = 0, \qquad \textit{on } \mathscr{I}^-.$$

Remark. The conventions used here for the vectors $\boldsymbol{e}_{\mathbf{00'}}$ and $\boldsymbol{e}_{\mathbf{11'}}$ are the opposite of those used in Kánnár (1996b). They have been chosen to agree with the standard conventions in the treatment of asymptotics as given in Penrose and Rindler (1986) and Stewart (1991) and to ease the comparison with the characteristic problem on a cone.

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18.1.2 The choice of conformal gauge

The geometric setting discussed in the previous section has an inherent conformal gauge freedom which can be exploited to simplify the analysis.

As discussed in Section 8.2.5, the Ricci scalar $R[\boldsymbol{g}]$ plays the role of a ***conformal gauge source function*** for the conformal field equations. A possible choice in the present setting is to fix the conformal factor Ξ linking the metrics $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ in such a manner that $R[\boldsymbol{g}] = 0$. To see that this can always be done, consider first a situation involving a generic conformal factor Ξ for which $R[\boldsymbol{g}] \neq 0$, and let

$$\boldsymbol{g}' \equiv \vartheta^2 \boldsymbol{g}, \tag{18.6}$$

with ϑ a positive function on $\mathcal{W}$. Defining $\Xi' \equiv \vartheta\Xi$ one finds that $\boldsymbol{g}' = \Xi'^2\tilde{\boldsymbol{g}}$. Consistent with the above conformal rescaling one considers the following transformation behaviour for the $\boldsymbol{g}$-orthonormal frame $\{\boldsymbol{e}_{\boldsymbol{AA}'}\}$:

$$\boldsymbol{e}'_{\boldsymbol{00}'} = \boldsymbol{e}_{\boldsymbol{00}'}, \qquad \boldsymbol{e}'_{\boldsymbol{11}'} = \vartheta^{-2}\boldsymbol{e}_{\boldsymbol{11}'}, \qquad \boldsymbol{e}'_{\boldsymbol{01}'} = \vartheta^{-1}\boldsymbol{e}_{\boldsymbol{01}'}, \qquad \boldsymbol{e}'_{\boldsymbol{10}'} = \vartheta^{-1}\boldsymbol{e}_{\boldsymbol{10}'}.$$

Using the transformation law under conformal rescalings for the Ricci scalar, Equation (5.6c), one finds that the requirement $R[\boldsymbol{g}'] = 0$ is equivalent to the wave equation

$$\nabla^a\nabla_a\vartheta = \frac{1}{6}R[\boldsymbol{g}]; \tag{18.7}$$

see also Equation (8.30). The general theory of the characteristic problem for wave equations ensures the existence of a unique solution to this equation in a neighbourhood $\mathcal{U}$ of $\mathscr{Z}$ in $J^+(\mathscr{Z})$ if some suitable data are prescribed on $\mathscr{N} \cup \mathscr{I}^-$; see, for example, Rendall (1990). A natural requirement on the initial data for Equation (18.7) is to have $\boldsymbol{\omega}'^{\boldsymbol{11}'} = \mathbf{d}\Xi'$ on $\mathscr{Z}$ where $\{\boldsymbol{\omega}'^{\boldsymbol{AA}'}\}$ denotes the coframe dual to $\{\boldsymbol{e}'_{\boldsymbol{AA}'}\}$. This is equivalent to setting

$$\boldsymbol{\omega}'^{\boldsymbol{11}'} = \vartheta\mathbf{d}\Xi \qquad \text{on } \mathscr{Z}.$$

By choosing $\vartheta^{-1}|_{\mathscr{Z}} = \boldsymbol{e}_{\boldsymbol{11}'}(\Xi)|_{\mathscr{Z}} = \langle\mathbf{d}\Xi, \boldsymbol{e}_{\boldsymbol{11}'}\rangle|_{\mathscr{Z}}$ one can, in fact, ensure that

$$\boldsymbol{e}'_{\boldsymbol{11}'}(\Xi') = \langle\mathbf{d}\Xi', \boldsymbol{e}'_{\boldsymbol{11}'}\rangle = 1 \qquad \text{on } \mathscr{Z}.$$

The principal part of the wave Equation (18.7), expressed in terms of frame derivatives, is given by

$$\boldsymbol{e}_{\boldsymbol{00}'}\big(\boldsymbol{e}_{\boldsymbol{11}'}(\vartheta)\big) + \boldsymbol{e}_{\boldsymbol{11}'}\big(\boldsymbol{e}_{\boldsymbol{00}'}(\vartheta)\big) - \boldsymbol{e}_{\boldsymbol{01}'}\big(\boldsymbol{e}_{\boldsymbol{10}'}(\vartheta)\big) - \boldsymbol{e}_{\boldsymbol{10}'}\big(\boldsymbol{e}_{\boldsymbol{01}'}(\vartheta)\big).$$

Thus, Equation (18.7) implies an *intrinsic propagation equation on* $\mathscr{N}$ for $\boldsymbol{e}_{\boldsymbol{00}'}(\vartheta)$ if $\boldsymbol{e}_{\boldsymbol{11}'}(\vartheta)$ is known on $\mathscr{N}$. Analogously, one has an *intrinsic propagation equation on* $\mathscr{I}^-$ for $\boldsymbol{e}_{\boldsymbol{11}'}(\vartheta)$ if $\boldsymbol{e}_{\boldsymbol{00}'}(\vartheta)$ is known on $\mathscr{I}^-$. The freedom in the specification

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of characteristic data can be exploited by observing that under the conformal rescaling (18.6) one obtains the transformation rules

$$\Gamma'_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}} = \Gamma_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}} - \vartheta^{-1}\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta), \qquad \Gamma'_{\mathbf{1}\mathbf{0}'\mathbf{0}\mathbf{0}} = \vartheta^{-2}\Gamma_{\mathbf{1}\mathbf{0}'\mathbf{0}\mathbf{0}} - \vartheta^{-3}\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\vartheta).$$

Accordingly, by setting

$$\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta) = \vartheta\Gamma_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}}, \qquad \boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\vartheta) = \vartheta\Gamma_{\mathbf{1}\mathbf{0}'\mathbf{0}\mathbf{0}} \qquad \text{on } \mathscr{Z},$$

one obtains

$$\Gamma'_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}} = 0, \qquad \Gamma'_{\mathbf{1}\mathbf{0}'\mathbf{0}\mathbf{0}} = 0, \qquad \text{on } \mathscr{Z}.$$

To propagate the freely specifiable components of $\nabla_{\boldsymbol{A}\boldsymbol{A}'}\vartheta$ along $\mathscr{N}$ and $\mathscr{I}^-$ it is convenient to consider the transformation law under conformal rescalings of the trace-free part of the Ricci tensor

$$\Phi'_{ab} - \Phi_{ab} = -2\vartheta^{-1}\bigg(\nabla_a\nabla_b\vartheta - 2\vartheta^{-1}\nabla_a\vartheta\nabla_b\vartheta - \frac{1}{4}g_{ab}(\nabla^c\nabla_c\vartheta - 2\vartheta^{-1}\nabla_c\vartheta\nabla^c\vartheta)\bigg). \tag{18.8}$$

Now, recalling that

$$\Phi_{\boldsymbol{A}\boldsymbol{A}'\boldsymbol{B}\boldsymbol{B}'} = e_{\boldsymbol{A}\boldsymbol{A}'}{}^a e_{\boldsymbol{B}\boldsymbol{B}'}{}^b \Phi_{ab}, \qquad \Phi'_{\boldsymbol{A}\boldsymbol{A}'\boldsymbol{B}\boldsymbol{B}'} = e'_{\boldsymbol{A}\boldsymbol{A}'}{}^a e'_{\boldsymbol{B}\boldsymbol{B}'}{}^b \Phi'_{ab},$$

one can consider the propagation equations

$$\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}\big(\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta)\big) - 2\vartheta^{-1}\big(\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta)\big)^2 = \vartheta\Phi_{22} \qquad \text{on } \mathscr{N}, \tag{18.9a}$$

$$\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}\big(\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\vartheta)\big) - 2\vartheta^{-1}\big(\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\vartheta)\big)^2 = \vartheta\Phi_{00} \qquad \text{on } \mathscr{I}^-. \tag{18.9b}$$

These two equations can be read as ordinary differential equations along the generators of $\mathscr{N}$ and $\mathscr{I}^-$ for $\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta)$ and $\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\vartheta)$, respectively. Accordingly, a solution exists in a neighbourhood of $\mathscr{Z}$ on $\mathscr{N}$ and, respectively, on $\mathscr{I}^-$. Comparing with Equation (18.8), one sees that these solutions ensure

$$\Phi'_{22} = 0 \qquad \text{on } \mathscr{N}, \tag{18.10a}$$

$$\Phi'_{00} = 0 \qquad \text{on } \mathscr{I}^-. \tag{18.10b}$$

Once the solutions $\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta)$ and $\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\vartheta)$ to the propagation conditions (18.9a) and (18.9b) have been obtained, one can use the intrinsic equations implied by (18.7) on $\mathscr{N} \cup \mathscr{I}^-$ to obtain $\boldsymbol{e}_{\mathbf{0}\mathbf{0}'}(\vartheta)$ on $\mathscr{N}$ and $\boldsymbol{e}_{\mathbf{1}\mathbf{1}'}(\vartheta)$ on $\mathscr{I}^-$.

The analysis of this section can be summarised in the following:

Lemma 18.2 (***conformal gauge conditions for the standard characteristic problem***) *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote an asymptotically simple spacetime satisfying $\boldsymbol{Ric}[\tilde{\boldsymbol{g}}] = 0$ and let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ with $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$ be a conformal extension thereof for which the condition $\Xi = 0$ describes past null infinity $\mathscr{I}^-$. Given the frame*

$\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ of Lemma 18.1, the conformal factor Ξ can be chosen so that given a null hypersurface $\mathscr{N}$ intersecting $\mathscr{I}^-$ on $\mathscr{Z} \approx \mathbb{S}^2$ one has

$$R[\boldsymbol{g}] = 0, \qquad \textit{in a neighbourhood } \mathcal{W} \textit{ of } \mathscr{Z} \textit{ on } J^+(\mathscr{Z}).$$

Moreover, one has the additional gauge conditions

$$\begin{aligned}
&\boldsymbol{e}_{\boldsymbol{11'}}(\Xi) = 1, \qquad \Gamma_{\boldsymbol{01'11}} = \Gamma_{\boldsymbol{10'00}} = 0, \qquad \textit{on } \mathscr{Z},\\
&\Phi_{22} = 0 \qquad \textit{on } \mathscr{N},\\
&\Sigma_{\boldsymbol{AA'}} = \boldsymbol{e}_{\boldsymbol{11'}}(\Xi)\delta_{\boldsymbol{A}}{}^{\boldsymbol{1}}\delta_{\boldsymbol{A'}}{}^{\boldsymbol{1'}}, \qquad \Phi_{00} = 0, \qquad \textit{on } \mathscr{I}^-.
\end{aligned}$$

Remark. In the gauge given by Lemma 18.2 one has that $L_{\boldsymbol{AA'BB'}} = \Phi_{\boldsymbol{AA'BB'}}$. This fact will be used repeatedly in the following without any further mention.

18.2 The conformal evolution equations in the standard characteristic initial value problem

This section analyses general aspects of the standard characteristic initial value problem for the conformal Einstein field equations with data prescribed on the null hypersurfaces $\mathscr{N}$ and $\mathscr{I}^-$. The spinorial conformal field equations, as discussed in Section 8.3.2, will be used to formulate this problem. Accordingly, on $\mathcal{W}$ it will be required that

$$\Sigma_{\boldsymbol{AA'BB'}} = 0, \qquad \Xi^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} = 0, \tag{18.11a}$$

$$\Xi_{\boldsymbol{AA'}} = 0, \qquad Z_{\boldsymbol{AA'BB'}} = 0, \qquad Z_{\boldsymbol{AA'}} = 0, \qquad Z = 0, \tag{18.11b}$$

$$\Delta_{\boldsymbol{CDBB'}} = 0, \qquad \Lambda_{\boldsymbol{BB'CD}} = 0, \tag{18.11c}$$

where, for convenience, one defines

$$\Xi_{\boldsymbol{AA'}} \equiv \Sigma_{\boldsymbol{AA'}} - \nabla_{\boldsymbol{AA'}}\Xi.$$

Following the conventions of Chapter 13 let $\mathbf{u}$ denote the collection of independent components of the unknowns appearing in the conformal field Equations (18.11a)–(18.11c) and let $\mathbf{u}_\star$ be its value on $\mathscr{N} \cup \mathscr{I}^+$.

Strictly speaking, as no hyperbolic reduction procedure has yet been applied to equations (18.11a)–(18.11c) – that is, the equations do not constitute a symmetric hyperbolic system – one does not directly obtain a characteristic problem in the sense described in Section 12.1.2. Nevertheless, the structure of the conformal evolution equations can be used to obtain a symmetric hyperbolic system for which the theory of Section 12.5 can be applied. Thus, it is necessary to analyse the properties of the conformal field equations on the hypersurfaces $\mathscr{N}$ and $\mathscr{I}^+$. When evaluated on $\mathscr{N} \cup \mathscr{I}^+$ the system (18.11a)–(18.11c) splits into a set of ***interior*** and a set of ***transverse*** equations. As the name suggests, interior equations contain only derivatives which are intrinsic to the null hypersurfaces. The interior equations divide, in turn, into ***transport equations*** containing the directional derivative along the generators of the hypersurface and ***constraint***

equations which do not contain this derivative. In the transverse equations one deals with the directional derivative transverse to the surface.

To see how this split comes about, it is convenient to recall some aspects of the hyperbolic reduction procedure for the Equations (18.11a)–(18.11c). Given a timelike vector τ^μ and a suitable set of gauge source functions $F^{\boldsymbol{a}}(x)$ and $F_{\boldsymbol{AB}}(x)$ on $\mathcal{W}$, one obtains a symmetric hyperbolic system for the independent components of the various conformal fields. As discussed in Proposition 13.1, the characteristic polynomial of this system contains factors of the form $g^{\mu\nu}\xi_\mu\xi_\nu$. Accordingly, the combined null hypersurface $\mathscr{N} \cap \mathscr{I}^+$ is a null hypersurface of the reduced evolution system. Following the discussion of Section 12.1.2, it follows that the reduced system contains equations which are intrinsic to $\mathscr{N} \cap \mathscr{I}^+$ and equations which are transverse to the initial hypersurface. In the following, it is shown how this observation can be extended to the full conformal field equations.

The interior equations on $\mathscr{N}$

The interior equations on the null hypersurface $\mathscr{N}$ should contain only the directional derivatives along the directions given by $\boldsymbol{e_{11'}}$, $\boldsymbol{e_{01'}}$ and $\boldsymbol{e_{10'}}$. Inspection shows that the subset of (18.11a)–(18.11c) with this property is given by the equations

$$\Xi_{\boldsymbol{11'}} = 0, \quad Z_{\boldsymbol{11'AA'}} = 0, \quad Z_{\boldsymbol{11'}} = 0, \tag{18.12a}$$

$$\Sigma_{\boldsymbol{11'BB'}} = 0, \quad \Xi^{\boldsymbol{C}}{}_{\boldsymbol{D11'BB'}} = 0, \tag{18.12b}$$

$$\Delta_{\boldsymbol{1DBB'}} = 0, \quad \Lambda_{\boldsymbol{B1'CD}} = 0. \tag{18.12c}$$

More explicitly, taking into account the gauge conditions given by Lemmas 18.1 and 18.2 one has the equations

$$\boldsymbol{e_{11'}}(\Xi) = \Sigma_{\boldsymbol{11'}}, \tag{18.13a}$$

$$\boldsymbol{e_{11'}}(\Sigma_{\boldsymbol{00'}}) = -\Xi\Phi_{11} - s, \quad \boldsymbol{e_{11'}}(\Sigma_{\boldsymbol{01'}}) = -\Xi\Phi_{12}, \quad \boldsymbol{e_{11'}}(\Sigma_{\boldsymbol{11'}}) = 0, \tag{18.13b}$$

$$\boldsymbol{e_{11'}}(s) = -\Phi_{11}\Sigma_{\boldsymbol{11'}} + 2\Phi_{12}\Sigma_{\boldsymbol{01'}}, \tag{18.13c}$$

$$\boldsymbol{e_{11'}}(e_{\boldsymbol{BB'}}{}^\mu) = -\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{1}}e_{\boldsymbol{C1'}}{}^\mu - \bar{\Gamma}_{\boldsymbol{B'B}}{}^{\boldsymbol{C'}}{}_{\boldsymbol{1'}}e_{\boldsymbol{1C'}}{}^\mu, \tag{18.13d}$$

$$\begin{aligned}\boldsymbol{e_{11'}}(\Gamma_{\boldsymbol{BB'CD}}) = {}& -\Gamma_{\boldsymbol{F1'CD}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{1}} - \Gamma_{\boldsymbol{1F'CD}}\bar{\Gamma}_{\boldsymbol{B'B}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{1'}} \\ & - \Xi\phi_{\boldsymbol{BCD1}}\epsilon_{\boldsymbol{1'B'}} - \Phi_{\boldsymbol{C1'DB'}}\epsilon_{\boldsymbol{1B}},\end{aligned} \tag{18.13e}$$

$$\begin{aligned}\boldsymbol{e_{11'}}(\Phi_{\boldsymbol{D0'BB'}}) = {}& \nabla_{\boldsymbol{10'}}\Phi_{\boldsymbol{D1'BB'}} - \nabla_{\boldsymbol{D1'}}\Phi_{\boldsymbol{10'BB'}} + \nabla_{\boldsymbol{D0'}}\Phi_{\boldsymbol{11'BB'}} \\ & - 2\Sigma_{\boldsymbol{1B'}}\phi_{\boldsymbol{BD01}} + 2\Sigma_{\boldsymbol{0B'}}\phi_{\boldsymbol{BD11}},\end{aligned} \tag{18.13f}$$

$$\boldsymbol{e_{11'}}(\phi_{\boldsymbol{ABC0}}) = \nabla_{\boldsymbol{01'}}\phi_{\boldsymbol{ABC1}}, \tag{18.13g}$$

with the understanding that equations for quantities already determined by gauge conditions are dropped from the list. Despite their apparent complexity, the above equations possess a delicate hierarchical structure which allows one to solve them sequentially from some basic data on $\mathscr{Z}$ and $\mathscr{N}$. This structure is briefly described in the following paragraphs.

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One starts by combining Equation (18.13a) with the third equation in (18.13b) and then using that $\boldsymbol{e}_{\mathbf{11}'} = \boldsymbol{\partial}_r$ to find that $\partial_r^2 \Xi = 0$. Hence, taking into account Lemma 18.2 one concludes that $\Xi = r$ along $\mathscr{N}$. Next, one can consider Equation (18.13e) for $\Gamma_{\mathbf{01}'\mathbf{11}}$ and $\Gamma_{\mathbf{10}'\mathbf{11}}$ (in standard Newman-Penrose (NP) notation γ and λ) which, in view of the gauge conditions, gives the subsystem

$$\partial_r \Gamma_{\mathbf{01}'\mathbf{11}} = -(\Gamma_{\mathbf{01}'\mathbf{11}})^2 - \Gamma_{\mathbf{10}'\mathbf{11}} \bar{\Gamma}_{\mathbf{1}'\mathbf{01}'\mathbf{1}'},$$
$$\partial_r \Gamma_{\mathbf{10}'\mathbf{11}} = -2\Gamma_{\mathbf{01}'\mathbf{11}} \Gamma_{\mathbf{10}'\mathbf{11}} + \Xi \phi_4.$$

The above ***Riccati system*** can be solved if ϕ_4 is known along $\mathscr{N}$. With $\Gamma_{\mathbf{01}'\mathbf{11}}$ and $\Gamma_{\mathbf{10}'\mathbf{11}}$ known, one can then make use of Equation (18.13d) for $e_{\mathbf{01}'}{}^{\mathcal{A}} = \xi^{\mathcal{A}}$ which takes the form

$$\partial_r \xi^{\mathcal{A}} = -\Gamma_{\mathbf{01}'\mathbf{11}} \xi^{\mathcal{A}} - \bar{\Gamma}_{\mathbf{1}'\mathbf{01}'\mathbf{1}'} \bar{\xi}^{\mathcal{A}}.$$

This equation together with its complex conjugate constitute a system of ordinary differential equations for $\xi^{\mathcal{A}}$ and $\bar{\xi}^{\mathcal{A}}$ which can be solved with the information already available. To determine the frame coefficient ω one considers Equation (18.13d) for $e_{\mathbf{01}'}{}^{1} = \omega$ so that

$$\partial_r \omega = -\Gamma_{\mathbf{01}'\mathbf{11}} \omega - \bar{\Gamma}_{\mathbf{1}'\mathbf{01}'\mathbf{1}'} \bar{\omega} + \Gamma_{\mathbf{01}'\mathbf{01}} + \bar{\Gamma}_{\mathbf{1}'\mathbf{00}'\mathbf{1}'}.$$

Accordingly, one also needs to consider the equations for $\Gamma_{\mathbf{01}'\mathbf{01}}$ and $\Gamma_{\mathbf{10}'\mathbf{01}}$ (β and α in NP notation), namely,

$$\partial_r \Gamma_{\mathbf{01}'\mathbf{01}} = -\Gamma_{\mathbf{01}'\mathbf{01}} \Gamma_{\mathbf{10}'\mathbf{11}} - \Gamma_{\mathbf{10}'\mathbf{01}} \bar{\Gamma}_{\mathbf{1}'\mathbf{01}'\mathbf{1}'} + \Phi_{12},$$
$$\partial_r \Gamma_{\mathbf{10}'\mathbf{01}} = -\Gamma_{\mathbf{01}'\mathbf{01}} \Gamma_{\mathbf{10}'\mathbf{11}} - \Gamma_{\mathbf{10}'\mathbf{01}} \bar{\Gamma}_{\mathbf{0}'\mathbf{11}'\mathbf{1}'} + \Xi \phi_3,$$

so that, in addition, one requires equations for ϕ_3 and Φ_{12}. These can be found to be given by

$$\partial_r \phi_3 = \omega \partial_r \phi_4 + \xi^{\mathcal{A}} \partial_{\mathcal{A}} \phi_4 - 4\Gamma_{\mathbf{01}'\mathbf{11}} \phi_3 + 4\Gamma_{\mathbf{01}'\mathbf{01}} \phi_4,$$
$$\partial_r \Phi_{12} = \Sigma_{\mathbf{01}'} \phi_4 - \Sigma_{\mathbf{11}'} \phi_3.$$

Thus, to close the system one considers the third equation in (18.13b). The key observation is that for a given choice of ϕ_4 on $\mathscr{N}$ and with the knowledge of $\Gamma_{\mathbf{01}'\mathbf{11}}$ and $\Gamma_{\mathbf{10}'\mathbf{11}}$ from a previous integration one obtains a system of ordinary differential equations along the generators of $\mathscr{N}$ for the unknowns ω, $\xi^{\mathcal{A}}$, $\Gamma_{\mathbf{01}'\mathbf{01}}$, $\Gamma_{\mathbf{10}'\mathbf{01}}$, ϕ_3, Φ_{12} and $\Sigma_{\mathbf{01}'}$.

At this point, one considers Equation (18.13d) for $e_{\mathbf{11}'}{}^{\mathcal{A}}$. One has

$$\partial_r X^{\mathcal{A}} = -\Gamma_{\mathbf{00}'\mathbf{11}} \xi^{\mathcal{A}} - \bar{\Gamma}_{\mathbf{0}'\mathbf{01}'\mathbf{1}'} \bar{\xi}^{\mathcal{A}}.$$

Recalling the gauge condition $\Gamma_{\mathbf{00}'\mathbf{11}} = \bar{\Gamma}_{\mathbf{10}'\mathbf{1}'\mathbf{0}'} + \Gamma_{\mathbf{10}'\mathbf{10}}$ one has enough information to integrate along the generators of $\mathscr{N}$. Next, one considers the equations for $\Gamma_{\mathbf{01}'\mathbf{00}}$ and $\Gamma_{\mathbf{10}'\mathbf{00}}$ (σ and ρ in NP notation):

$$\partial_r \Gamma_{\mathbf{01}'\mathbf{00}} = -\Gamma_{\mathbf{01}'\mathbf{00}} \Gamma_{\mathbf{01}'\mathbf{11}} - \Gamma_{\mathbf{10}'\mathbf{00}} \bar{\Gamma}_{\mathbf{01}'\mathbf{1}'\mathbf{1}'} + \Phi_{02},$$
$$\partial_r \Gamma_{\mathbf{10}'\mathbf{00}} = -\Gamma_{\mathbf{01}'\mathbf{00}} \Gamma_{\mathbf{10}'\mathbf{11}} - \Gamma_{\mathbf{10}'\mathbf{00}} \bar{\Gamma}_{\mathbf{10}'\mathbf{1}'\mathbf{1}'} + \Xi \phi_2.$$

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Hence, one has to couple the above to the equations for ϕ_2 and Φ_{02}:

$$\begin{aligned}
\partial_r\phi_2 &= \omega\partial_r\phi_3 + \xi^{\mathcal{A}}\partial_{\mathcal{A}}\phi_3 + \Gamma_{\mathbf{0}\mathbf{1}'\mathbf{0}\mathbf{0}}\phi_4 + 2\Gamma_{\mathbf{0}\mathbf{1}'\mathbf{0}\mathbf{1}}\phi_3 - 3\Gamma_{\mathbf{0}\mathbf{1}'\mathbf{1}\mathbf{1}}\phi_2, \\
\partial_r\Phi_{02} &= \nabla_{\mathbf{1}\mathbf{0}'}\Phi_{12} + \Sigma_{\mathbf{0}\mathbf{1}'}\phi_3 - \Sigma_{\mathbf{0}\mathbf{0}'}\phi_4.
\end{aligned}$$

Thus, it is then necessary to consider simultaneously the first equation in (18.13b) and Equation (18.13c) to determine $\Sigma_{\mathbf{0}\mathbf{0}'}$ and s – notice that at this stage one already knows all the frame and connection coefficients appearing in $\nabla_{\mathbf{1}\mathbf{0}'}$. In turn, this forces the coupling with the equation for Φ_{11} obtained from (18.13f):

$$\partial_r\Phi_{11} = \nabla_{\mathbf{1}\mathbf{0}'}\Phi_{12} - \Sigma_{\mathbf{1}\mathbf{1}'}\phi_2 + \Sigma_{\mathbf{0}\mathbf{1}'}\phi_3.$$

Recapitulating, one has obtained a further closed subsystem of ordinary differential equations along the generators of $\mathscr{N}$ for the fields $\Gamma_{\mathbf{0}\mathbf{1}'\mathbf{0}\mathbf{0}}$, $\Gamma_{\mathbf{1}\mathbf{0}'\mathbf{0}\mathbf{0}}$, ϕ_2, Φ_{02}, Φ_{11}, s and $\Sigma_{\mathbf{0}\mathbf{0}'}$. With the information obtained from the solution to this system, one can also solve for the frame coefficient U and the connection coefficient $\Gamma_{\mathbf{0}\mathbf{0}'\mathbf{0}\mathbf{1}}$ (ϵ in NP notation) via the equations

$$\begin{aligned}
\partial_r U &= -\Gamma_{\mathbf{0}\mathbf{0}'\mathbf{1}\mathbf{1}}\omega - \bar{\Gamma}_{\mathbf{0}'\mathbf{0}\mathbf{1}'\mathbf{1}'}\bar{\omega} + \Gamma_{\mathbf{0}\mathbf{0}'\mathbf{0}\mathbf{1}} + \bar{\Gamma}_{\mathbf{0}'\mathbf{0}\mathbf{0}'\mathbf{1}'}, \\
\partial_r\Gamma_{\mathbf{0}\mathbf{0}'\mathbf{0}\mathbf{1}} &= -\Gamma_{\mathbf{0}\mathbf{1}'\mathbf{0}\mathbf{1}}\Gamma_{\mathbf{0}\mathbf{0}'\mathbf{1}\mathbf{1}} - \Gamma_{\mathbf{1}\mathbf{0}'\mathbf{0}\mathbf{1}}\bar{\Gamma}_{\mathbf{0}'\mathbf{0}\mathbf{1}'\mathbf{1}'} + \Xi\phi_2 + \Phi_{11}.
\end{aligned}$$

The integration of the connection coefficients can now be completed with the equation for $\Gamma_{\mathbf{0}\mathbf{0}'\mathbf{0}\mathbf{0}}$ (κ in NP notation) dictated by (18.13e), that is,

$$\partial_r\Gamma_{\mathbf{0}\mathbf{0}'\mathbf{0}\mathbf{0}} = -\Gamma_{\mathbf{0}\mathbf{1}'\mathbf{0}\mathbf{0}}\Gamma_{\mathbf{0}\mathbf{0}'\mathbf{1}\mathbf{1}} - \Gamma_{\mathbf{1}\mathbf{0}'\mathbf{0}\mathbf{0}}\bar{\Gamma}_{\mathbf{0}'\mathbf{0}\mathbf{1}'\mathbf{1}'} + \Xi\phi_1 + \Phi_{01},$$

which needs to be supplemented by the equations for ϕ_1 and Φ_{01}:

$$\begin{aligned}
\partial_r\phi_1 &= \nabla_{\mathbf{0}\mathbf{1}'}\phi_2, \\
\partial_r\Phi_{01} &= \nabla_{\mathbf{1}\mathbf{0}'}\Phi_{11} - \Sigma_{\mathbf{0}\mathbf{1}'}\phi_2 + \Sigma_{\mathbf{0}\mathbf{0}'}\phi_3.
\end{aligned}$$

Again, one has a subsystem of ordinary differential equations along the generators of $\mathscr{N}$. The integration of the interior equations on $\mathscr{N}$ is completed by considering the equation for the rescaled Weyl spinor component ϕ_0

$$\partial_r\phi_0 = \nabla_{\mathbf{0}\mathbf{1}'}\phi_1,$$

which, too, is an ordinary differential equation, and by that for Φ_{00}:

$$\partial_r\Phi_{00} = \nabla_{\mathbf{1}\mathbf{0}'}\Phi_{01} - \nabla_{\mathbf{0}\mathbf{1}'}\Phi_{01} + 2\Sigma_{\mathbf{0}\mathbf{0}'}\phi_2 - 2\Sigma_{\mathbf{0}\mathbf{1}'}\phi_1 + \nabla_{\mathbf{0}\mathbf{0}'}\Phi_{11}.$$

This last equation is different from the other ones in the hierarchy as its last term in the right-hand side (i.e. $\nabla_{\mathbf{0}\mathbf{0}'}\Phi_{11}$) contains transverse derivatives. However, using the evolution equations in Section 18.2.2, this term can be formally computed on $\mathscr{N}$ from the available data.

The interior equations on $\mathscr{I}^-$

On $\mathscr{I}^-$ the intrinsic equations should contain only the derivatives along the directions given by $\boldsymbol{e}_{\mathbf{00'}}$, $\boldsymbol{e}_{\mathbf{10'}}$ and $\boldsymbol{e}_{\mathbf{01'}}$. The relevant subset of (18.11a)–(18.11c) is, in this case, given by

$$\Xi_{\boldsymbol{AA'}} \simeq 0, \quad Z_{\boldsymbol{AA'}} \simeq 0, \quad Z_{\boldsymbol{AA'BB'}} \simeq 0, \quad \text{for } \boldsymbol{AA'} \neq \mathbf{11'}, \tag{18.14a}$$

$$\Sigma_{\boldsymbol{AA'BB'}} \simeq 0, \quad \Xi^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} \simeq 0, \quad \text{for } \boldsymbol{AA'},\ \boldsymbol{BB'} \neq \mathbf{11'}, \tag{18.14b}$$

$$\Delta_{\mathbf{0}\boldsymbol{DBB'}} \simeq 0, \quad \Lambda_{\boldsymbol{B}\mathbf{0'}\boldsymbol{CD}} \simeq 0. \tag{18.14c}$$

More explicitly, taking into account the gauge conditions given by Lemmas 18.1 and 18.2 the above equations encode the following ***transport equations:***

$$\boldsymbol{e}_{\mathbf{00'}}(\Xi) \simeq 0, \tag{18.15a}$$

$$\boldsymbol{e}_{\mathbf{00'}}(\Sigma_{\mathbf{11'}}) \simeq -s, \tag{18.15b}$$

$$\boldsymbol{e}_{\mathbf{00'}}(s) \simeq -\Phi_{11}\Sigma_{\mathbf{11'}}, \tag{18.15c}$$

$$\boldsymbol{e}_{\mathbf{00'}}(e_{\mathbf{01'}}{}^{\mu}) \simeq \Gamma_{\mathbf{00'}}{}^{\boldsymbol{CC'}}{}_{\mathbf{01'}}e_{\boldsymbol{CC'}}{}^{\mu} - \Gamma_{\mathbf{01'}}{}^{\boldsymbol{CC'}}{}_{\mathbf{00'}}e_{\boldsymbol{CC'}}{}^{\mu}, \tag{18.15d}$$

$$P_{\boldsymbol{CD}\mathbf{00'}\boldsymbol{BB'}} \simeq -\Xi\phi_{\boldsymbol{BCD}\mathbf{0}}\epsilon_{\mathbf{0'}\boldsymbol{B'}} - \Phi_{\boldsymbol{C}\mathbf{0'}\boldsymbol{DB'}}\epsilon_{\mathbf{0}\boldsymbol{B}} \quad \boldsymbol{BB'} \neq \mathbf{11'}, \tag{18.15e}$$

$$\nabla_{\mathbf{00'}}\phi_{\boldsymbol{ABC}\mathbf{1}} \simeq \nabla_{\mathbf{10'}}\phi_{\boldsymbol{ABC}\mathbf{0}}, \tag{18.15f}$$

$$\begin{aligned}\nabla_{\mathbf{00'}}\Phi_{\boldsymbol{D}\mathbf{1'}\boldsymbol{BB'}} + \nabla_{\boldsymbol{D}\mathbf{0'}}\Phi_{\mathbf{01'}\boldsymbol{BB'}} - \nabla_{\mathbf{01'}}\Phi_{\boldsymbol{D}\mathbf{0'}\boldsymbol{BB'}}& \\ - \nabla_{\boldsymbol{D}\mathbf{1'}}\Phi_{\mathbf{00'}\boldsymbol{BB'}} \simeq 2\Sigma_{\mathbf{1}\boldsymbol{B'}}\phi_{\mathbf{0}\boldsymbol{DB}\mathbf{1}},&\end{aligned} \tag{18.15g}$$

where, following the notation of Chapter 8, the field $P_{\boldsymbol{CDAA'BB'}}$ denotes the geometric curvature. In addition to the above, Equations (18.14a)–(18.14c) also contain the ***constraint equations***

$$\boldsymbol{e}_{\mathbf{01'}}(\Xi) \simeq 0, \qquad \boldsymbol{e}_{\mathbf{01'}}(\Sigma_{\mathbf{11'}}) \simeq 0, \qquad \boldsymbol{e}_{\mathbf{01'}}(s) \simeq -\Phi_{01}\Sigma_{\mathbf{11'}}, \tag{18.16a}$$

$$\boldsymbol{e}_{\mathbf{01'}}(e_{\mathbf{10'}}{}^{\mu}) - \boldsymbol{e}_{\mathbf{10'}}(e_{\mathbf{01'}}{}^{\mu}) \simeq \Gamma_{\mathbf{01'}}{}^{\boldsymbol{CC'}}{}_{\mathbf{10'}}e_{\boldsymbol{CC'}}{}^{\mu} - \Gamma_{\mathbf{10'}}{}^{\boldsymbol{CC'}}{}_{\mathbf{01'}}e_{\boldsymbol{CC'}}{}^{\mu}, \tag{18.16b}$$

$$P_{\boldsymbol{CD}\mathbf{01'10'}} \simeq \Xi\phi_{\mathbf{1}\boldsymbol{CD}\mathbf{0}}\epsilon_{\mathbf{1'0'}} - \Phi_{\boldsymbol{C}\mathbf{1'}\boldsymbol{D}\mathbf{0'}}. \tag{18.16c}$$

If Equations (18.16a)–(18.16c) hold in a certain section of $\mathscr{I}^-$, then using an argument similar to that of the propagation of the constraints in the standard Cauchy problem, it can be shown that they will hold everywhere else on null infinity by virtue of the transport Equations (18.15a)–(18.15g). Thus, they need to be solved only on $\mathscr{Z}$.

In analogy to the transport equations on $\mathscr{N}$, the transport Equations (18.15a)–(18.15g) can be solved along the generators of $\mathscr{I}^-$ exploiting a hierarchical structure if some basic data are provided. Some inspection reveals that the basic data are given by either the connection coefficient $\Gamma_{\mathbf{10'11}}$ or the rescaled Weyl spinor component ϕ_0. The details of this construction will not be further elaborated.

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18.2.1 The freely specifiable data

The discussion of the hierarchical structure of the interior equations on $\mathscr{N} \cup \mathscr{I}^-$ allows the identification of the basic ***reduced initial data set*** $\mathbf{r}_\star$ from which the full initial data $\mathbf{u}_\star$ on $\mathscr{N} \cup \mathscr{I}^-$ for the conformal Einstein field equations can be computed. As already observed, the choice of reduced initial data sets is not unique. Two possible ways of specifying the reduced data are given in the following:

Lemma 18.3 (***freely specifiable data for the standard characteristic problem***) *Assume that the gauge conditions given by Lemmas 18.1 and 18.2 are satisfied in a neighbourhood $\mathscr{U}$ of $\mathscr{Z}$ on $\mathscr{N} \cup \mathscr{I}^-$. Initial data $\mathbf{u}_\star$ for the conformal Einstein field equations on $\mathscr{N} \cup \mathscr{I}^-$ can be computed from either of the two following reduced initial data sets:*

(i) $\mathbf{r}_{1\star}$ *consisting of*

$$\begin{aligned}
&\Gamma_{\mathbf{10'11}} \quad \textit{on } \mathscr{I}^-,\\
&\phi_4 \quad \textit{on } \mathscr{N},\\
&\phi_3, \quad \phi_2 + \bar{\phi}_2, \quad \xi^{\mathcal{A}}, \quad \textit{on } \mathscr{Z};
\end{aligned}$$

(ii) $\mathbf{r}_{2\star}$ *consisting of*

$$\begin{aligned}
&\phi_0 \quad \textit{on } \mathscr{I}^-,\\
&\phi_4 \quad \textit{on } \mathscr{N},\\
&\Gamma_{\mathbf{10'11}}, \quad \Phi_{20}, \quad \phi_3, \quad \phi_2 + \bar{\phi}_2, \quad \xi^{\mathcal{A}}, \quad \textit{on } \mathscr{Z}.
\end{aligned}$$

In both cases the field $\xi^{\mathcal{A}}$ is chosen so that $-(\xi^{\mathcal{A}}\bar{\xi}^{\mathcal{B}}+\bar{\xi}^{\mathcal{A}}\xi^{\mathcal{B}})\boldsymbol{\partial}_{\mathcal{A}}\otimes\boldsymbol{\partial}_{\mathcal{B}}$ is conformal to the standard (contravariant) metric on $\mathbb{S}^2$.

Remark. The reduced set $\mathbf{r}_{2\star}$ in (ii) has the advantage of being symmetric with respect to $\mathscr{N}$ and $\mathscr{I}^-$.

Proof The proof of this lemma follows from the discussion in the previous subsection. Further discussion can be found in Friedrich (1981a). □

18.2.2 The reduced conformal field equations

To apply the theory on the characteristic initial value problem discussed in Section 12.5 one has to extract a suitable symmetric hyperbolic system out of the conformal field Equations (18.11a)–(18.11c). Given the split between intrinsic and transverse equations, a hyperbolic reduction procedure such as the one discussed in Chapter 13 is not required. Instead, a suitable choice of ***reduced conformal field equations*** is given by the combinations

$$\Xi_{\mathbf{11'}} = 0, \qquad Z_{\mathbf{11'}} = 0, \qquad Z_{\mathbf{11'}\boldsymbol{AB'}} = 0, \tag{18.17a}$$

$$\Sigma_{\mathbf{11'}\boldsymbol{BB'}} = 0, \qquad \Xi^{\boldsymbol{C}}{}_{\boldsymbol{D}\mathbf{11'}\boldsymbol{BB'}} = 0, \tag{18.17b}$$

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$$-\Delta_{\boldsymbol{1BC0'}} = 0, \quad \Delta_{\boldsymbol{0BC0'}} - \Delta_{\boldsymbol{1BC1'}} = 0, \quad \Delta_{\boldsymbol{0BC'1'}} = 0, \tag{18.17c}$$

$$-\Lambda_{\boldsymbol{01'00}} = 0, \quad \Lambda_{\boldsymbol{00'BC}} - \Lambda_{\boldsymbol{11'BC}} = 0, \quad \Lambda_{\boldsymbol{10'11}} = 0. \tag{18.17d}$$

A more explicit form of the equations is discussed in Section 18.3. From these expressions, adopting the matricial notation of Chapter 12 and considering *suitable multiples* of the equations, the reduced conformal field equations can be written schematically in the form

$$\mathbf{A}^\mu(x,\mathbf{u})\partial_\mu \mathbf{u} + \mathbf{B}(x,\mathbf{u}) = 0, \tag{18.18}$$

with $\mathbf{A}^\mu$ Hermitian matrices and

$$\mathbf{A}^\mu(\omega^{\boldsymbol{00'}}{}_\mu + \omega^{\boldsymbol{11'}}{}_\mu) \qquad \text{positive definite.} \tag{18.19}$$

Thus, one obtains a symmetric hyperbolic system for the components of $\mathbf{u}$. Using the expressions for the principal part of the system (18.17a)–(18.17d), a computation shows that the characteristic polynomial of the reduced system contains factors of the form $g^{\mu\nu}\xi_\mu\xi_\nu$ so that the null hypersurfaces $\mathscr{N}$ and $\mathscr{I}^-$ are indeed characteristics of the system. It follows from (18.19) that the surfaces with normal $\boldsymbol{\omega}^{\boldsymbol{00'}} + \boldsymbol{\omega}^{\boldsymbol{11'}}$ are spacelike for the symmetric hyperbolic system. Although the coordinates $x^0 = v$ and $x^1 = r$ have been constructed so that they have non-negative values, the reduced Equations (18.17a)–(18.17d) also hold for negative values of the coordinates. It follows that the hypersurface

$$\mathcal{S}_\star \equiv \big\{p \in \mathbb{R}\times\mathbb{R}\times\mathbb{S}^2 \mid x^0(p) + x^1(p) = 0\big\} \tag{18.20}$$

is spacelike for Equation (18.18) in a neighbourhood of $\mathscr{Z}$.

18.3 A local existence result for characteristic problems

As discussed in Section 12.5, the existence and uniqueness of solutions to a characteristic initial value problem can be obtained via an *auxiliary Cauchy initial value problem* on a spacelike hypersurface – in the present case the hypersurface $\mathcal{S}_\star$ defined by (18.20). The formulation of this auxiliary Cauchy problem crucially depends on Whitney's extension theorem so that initial data on $\mathscr{N}\cup\mathscr{I}^-$ can be extended to a spacetime neighbourhood $\mathcal{U}$ of $\mathscr{Z}$. In turn, the application of Whitney's theorem depends on being able to evaluate all (interior and transverse) derivatives of the initial data on $\mathscr{N}\cup\mathscr{I}^-$.

18.3.1 Computation of the formal derivatives on $\mathscr{N}\cup\mathscr{I}^-$

To verify that one can compute all derivatives of the initial data on $\mathscr{N}\cup\mathscr{I}^-$ one needs to inspect the principal part of the reduced Equations (18.17a)–(18.17d).

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Borrowing the notation of Proposition 13.1, the reduced Equations (18.17a)–(18.17b) take the form

$$\partial_r \boldsymbol{\sigma} = \mathbf{G}(\boldsymbol{\sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}), \tag{18.21a}$$

$$\partial_r \boldsymbol{e} = \mathbf{H}(\boldsymbol{e}, \boldsymbol{\Gamma}), \tag{18.21b}$$

$$\partial_r \boldsymbol{\Gamma} = \mathbf{K}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}); \tag{18.21c}$$

that is, they are transport equations along the direction given by $\boldsymbol{e}_{\mathbf{00'}}$. For the equations in (18.17c) one has

$$\partial_r \Phi_{20} - \bar{\omega}\partial_r \Phi_{21} - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{21} = L_{20}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}), \tag{18.22a}$$

$$\partial_r \Phi_{10} - \bar{\omega}\partial_r \Phi_{11} - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{11} = L_{10}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}), \tag{18.22b}$$

$$\partial_r \Phi_{00} - \bar{\omega}\partial_r \Phi_{01} - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{01} = L_{00}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}), \tag{18.22c}$$

$$\begin{aligned}&\partial_r \Phi_{21} + \partial_v \Phi_{21} + U\partial_r \Phi_{21} + X^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{21}\\ &\qquad - \omega\partial_r \Phi_{20} - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{20} - \bar{\omega}\partial_r \Phi_{22} - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{22} = M_{21}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}),\end{aligned} \tag{18.22d}$$

$$\begin{aligned}&\partial_r \Phi_{11} + \partial_v \Phi_{11} + U\partial_r \Phi_{11} + X^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{11}\\ &\qquad - \omega\partial_r \Phi_{10} - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{10} - \bar{\omega}\partial_r \Phi_{12} - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{12} = M_{11}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}),\end{aligned} \tag{18.22e}$$

$$\begin{aligned}&\partial_r \Phi_{01} + \partial_v \Phi_{01} + U\partial_r \Phi_{01} + X^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{01}\\ &\qquad - \omega\partial_r \Phi_{00} - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{00} - \bar{\omega}\partial_r \Phi_{02} - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{02} = M_{01}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}),\end{aligned} \tag{18.22f}$$

$$\partial_v \Phi_{22} + U\partial_r \Phi_{22} + X^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{22} - \omega\partial_r \Phi_{21} - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{21} = N_{22}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}), \tag{18.22g}$$

$$\partial_v \Phi_{12} + U\partial_r \Phi_{12} + X^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{12} - \omega\partial_r \Phi_{11} - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{11} = N_{12}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}), \tag{18.22h}$$

$$\partial_v \Phi_{02} + U\partial_r \Phi_{02} + X^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{02} - \bar{\omega}\partial_r \Phi_{01} - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{01} = N_{02}(\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\phi}), \tag{18.22i}$$

where L_{20}, L_{10}, L_{00}, M_{21}, M_{11}, M_{01}, N_{22}, N_{12} and N_{02} are smooth functions of their arguments – their explicit form will not be required. Finally, for the Equations (18.17d) involving the components of the rescaled Weyl tensor one has

$$\partial_r \phi_0 - \omega\partial_r \phi_1 - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\phi_1 = W_0(\boldsymbol{\Gamma}, \boldsymbol{\phi}), \tag{18.23a}$$

$$\partial_r \phi_1 + \partial_v \phi_1 + U\partial_r \phi_1 + X^{\mathcal{A}}\partial_{\mathcal{A}}\phi_1 \tag{18.23b}$$

$$\qquad - \bar{\omega}\partial_r \phi_0 - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\phi_0 - \omega\partial_r \phi_2 - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\phi_2 = W_1(\boldsymbol{\Gamma}, \boldsymbol{\phi}), \tag{18.23c}$$

$$\partial_r \phi_2 + \partial_v \phi_2 + U\partial_r \phi_2 + X^{\mathcal{A}}\partial_{\mathcal{A}}\phi_2 \tag{18.23d}$$

$$\qquad - \bar{\omega}\partial_r \phi_1 - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\phi_1 - \omega\partial_r \phi_3 - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\phi_3 = W_2(\boldsymbol{\Gamma}, \boldsymbol{\phi}), \tag{18.23e}$$

$$\partial_r \phi_3 + \partial_v \phi_3 + U\partial_r \phi_3 + X^{\mathcal{A}}\partial_{\mathcal{A}}\phi_3 \tag{18.23f}$$

$$\qquad - \bar{\omega}\partial_r \phi_2 - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\phi_2 - \omega\partial_r \phi_4 - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\phi_4 = W_3(\boldsymbol{\Gamma}, \boldsymbol{\phi}), \tag{18.23g}$$

$$\partial_v \phi_4 + U\partial_r \phi_4 + X^{\mathcal{A}}\partial_{\mathcal{A}}\phi_4 - \bar{\omega}\partial_r \phi_3 - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\phi_3 = W_4(\boldsymbol{\Gamma}, \boldsymbol{\phi}), \tag{18.23h}$$

with W_0, W_1, W_2, W_3 and W_4 smooth functions of their arguments – again, their explicit form will not be required.

In what follows, it is shown that all *formal partial derivatives* on $\mathscr{N} \cup \mathscr{I}^-$ can indeed be computed from the above equations.

Computation of formal derivatives on $\mathscr{I}^-$

To compute the formal derivatives on $\mathscr{I}^-$ one first observes that the partial derivatives ∂_v, ∂_2, ∂_3 are interior, while ∂_r is transverse. In this case, direct inspection shows that except for

$$\partial_r\phi_4, \quad \partial_r\Phi_{22}, \quad \partial_r\Phi_{12}, \quad \partial_r\Phi_{02},$$

all ∂_r-derivatives of the unknown $\mathbf{u}$ can be computed using Equations (18.21a)–(18.21c), (18.22a)–(18.22f) and (18.23a)–(18.23g). The exceptional cases shown above arise due to the fact that $\omega = U = 0$ on $\mathscr{I}^-$ so that Equations (18.22g)–(18.22i) and (18.23h) evaluated at $\mathscr{I}^-$ do not, in fact, contain ∂_r-derivatives. To get around this problem one computes the ∂_r-derivative of (18.22g)–(18.22i) and (18.23h) and then evaluates on $\mathscr{I}^-$ to obtain the system

$$\begin{aligned}
&\partial_v\big(\partial_r\Phi_{22}\big) + \partial_r U\partial_r\Phi_{22} + \partial_r X^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{22} - \partial_r\omega\partial_r\Phi_{12}\\
&\qquad\qquad\qquad - \partial_r\xi^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{12} - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\partial_r\Phi_{12} \simeq \partial_r N_{22},\\
&\partial_v\big(\partial_r\Phi_{12}\big) + \partial_r U\partial_r\Phi_{12} + \partial_r X^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{12} - \partial_r\omega\partial_r\Phi_{11}\\
&\qquad\qquad\qquad - \partial_r\xi^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{11} - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\partial_r\Phi_{11} \simeq \partial_r N_{12},\\
&\partial_v\big(\partial_r\Phi_{02}\big) + \partial_r U\partial_r\Phi_{02} + \partial_r X^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{02} - \partial_r\omega\partial_r\Phi_{01}\\
&\qquad\qquad\qquad - \partial_r\xi^{\mathcal{A}}\partial_{\mathcal{A}}\Phi_{01} - \xi^{\mathcal{A}}\partial_{\mathcal{A}}\partial_r\Phi_{01} \simeq \partial_r N_{02},\\
&\partial_v\big(\partial_r\phi_4\big) + \partial_r U\partial_r\phi_4 + \partial_r X^{\mathcal{A}}\partial_{\mathcal{A}}\phi_4 - \partial_r\bar{\omega}\partial_r\phi_3\\
&\qquad\qquad\qquad - \partial_r\bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\phi_3 - \bar{\xi}^{\mathcal{A}}\partial_{\mathcal{A}}\partial_r\phi_3 \simeq \partial_r W_4.
\end{aligned}$$

The latter can be interpreted as a system of first-order linear ordinary differential equations for $\partial_r\phi_4$, $\partial_r\Phi_{22}$, $\partial_r\Phi_{12}$, $\partial_r\Phi_{02}$. The initial data on $\mathscr{Z}$ for these equations can be computed from the data on $\mathscr{N} \cup \mathscr{I}^-$. General results of the theory of ordinary differential equations ensures that this system of equations can be solved in a neighbourhood of $\mathscr{Z}$ on $\mathscr{I}^-$. Accordingly, all the first transverse derivatives on $\mathscr{I}^-$ can be explicitly computed. The argument described in this paragraph can be generalised, by repeatedly differentiating the reduced equations with respect to ∂_r, to iteratively compute higher order ∂_r-derivatives as the solution to a system of algebraic equations and linear PDEs.

Computation of formal derivatives on $\mathscr{N}$

The analysis of the formal derivatives on $\mathscr{N}$ is almost the mirror image of that on $\mathscr{I}^-$. In this case ∂_r, ∂_2, ∂_3 are interior derivatives, while ∂_v is transverse. After an inspection of the list of Equations (18.21a)–(18.21c), (18.22a)–(18.22i) and (18.23a)–(18.23h) one finds that *only*

$$\begin{gathered}
\partial_v\phi_4, \quad \partial_v\phi_3, \quad \partial_v\phi_2, \quad \partial_v\phi_1,\\
\partial_v\Phi_{22}, \quad \partial_v\Phi_{12}, \quad \partial_v\Phi_{11}, \quad \partial_v\Phi_{02}, \quad \partial_v\Phi_{01}
\end{gathered}$$

are algebraically determined by the initial data on $\mathscr{N}$. To obtain the remaining transverse derivatives, one computes the ∂_v-derivatives of Equations (18.21a)–(18.21c), (18.22a)–(18.22c) and (18.23a) and evaluates them on $\mathscr{N}$ to obtain a first-order system of ordinary differential equations along the generators of $\mathscr{N}$ for

$$\partial_v \boldsymbol{\sigma}, \quad \partial_v \boldsymbol{e}, \quad \partial_v \boldsymbol{\Gamma}, \quad \partial_v \Phi_{02}, \quad \partial_v \Phi_{01}, \quad \partial_v \Phi_{00}, \quad \partial_v \phi_0.$$

Supplementing this system with the information on $\mathscr{Z}$ implied by the initial data for the reduced equations, one finds that the general theory of ordinary differential equations ensures the existence of solutions in a neighbourhood of $\mathscr{Z}$ on $\mathscr{N}$. In this manner one obtains a complete set of first-order transverse derivatives on $\mathscr{N}$. Higher order transverse derivatives can be obtained iteratively by computing higher order ∂_v-derivatives of the reduced conformal field equations as required.

The analysis described in the previous paragraphs can be summarised in the following:

Lemma 18.4 (***computation of formal derivatives***) *Any arbitrary formal derivatives* $(\partial^{\boldsymbol{\alpha}}\mathbf{u})_\star$ *of the vector unknown* $\mathbf{u}$ *on* $\mathscr{N} \cup \mathscr{I}^-$ *can be computed from the prescribed initial data* $\mathbf{u}_\star$ *for the reduced conformal field equations on* $\mathscr{N} \cup \mathscr{I}^-$.

18.3.2 The subsidiary system

To show that the solutions of the reduced equations imply a solution to the full conformal field equations if initial data satisfying the constraints on $\mathscr{N}$ and $\mathscr{I}^-$ are prescribed, it is necessary to obtain a suitable subsidiary system for the zero quantities encoding the conformal field equations. The *propagation of the constraints* is ensured by the following:

Proposition 18.1 (***propagation of the constraints***) *A solution* $\mathbf{u}$ *of the reduced conformal field Equations* (18.17a)–(18.17d) *on a neighbourhood* $\mathcal{U}$ *of* $\mathscr{Z}$ *on* $J^+(\mathscr{Z})$ *that coincides with initial data on* $\mathscr{N} \cup \mathscr{I}^-$ *satisfying the conformal equations is a solution to the conformal field Equations* (18.11a)–(18.11c) *on* $\mathcal{U}$.

A subsidiary system adapted to the geometry of the characteristic problem described in the previous sections is obtained from the following derivatives of the zero quantities associated to the conformal field equations:

$$\nabla_{\mathbf{11}'}\Xi_{\boldsymbol{AA}'}, \qquad \nabla_{\mathbf{11}'}Z_{\boldsymbol{AA}'}, \qquad \nabla_{\mathbf{11}'}Z_{\boldsymbol{AA}'\boldsymbol{BB}'},$$
$$\nabla_{\mathbf{11}'}\Sigma_{\boldsymbol{AA}'\boldsymbol{BB}'}, \qquad \nabla_{\mathbf{11}'}\Xi_{\boldsymbol{CDAA}'\boldsymbol{BB}'}$$
$$(\nabla_{\mathbf{00}'}+\nabla_{\mathbf{11}'})\Delta_{\boldsymbol{CDBB}'}, \qquad (\nabla_{\mathbf{00}'}+\nabla_{\mathbf{11}'})\Lambda_{\boldsymbol{BB}'\boldsymbol{CD}}.$$

Using arguments similar to those employed in Sections 13.3 and 13.4.5 one rewrites the above derivatives as homogeneous expressions in the zero quantities. Further details of these lengthy calculations can be found in Friedrich (1981a).

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Once a subsidiary system of the required form has been obtained, the propagation of the constraints follows from the uniqueness of solutions to the characteristic problem.

In addition to Proposition 18.1 one has the following:

Corollary 18.1 (***preservation of the conformal gauge***) *Let* $\mathbf{u}$ *denote a solution to the characteristic problem for the conformal field equations on a neighbourhood* $\mathcal{U}$ *of* $\mathscr{Z}$ *on* $J^+(\mathscr{Z})$ *which satisfies the gauge conditions given in Lemmas 18.1 and 18.2. Then the metric* $\boldsymbol{g}$ *constructed from the components of the solution* $\mathbf{u}$ *satisfies the vacuum Einstein field equations* $R[\boldsymbol{g}] = 0$*.*

This result follows from an argument similar to the one used to prove the propagation of the *algebraic* conformal field equation encoding the transformation rule for the Ricci scalar in Lemma 8.1. Here one considers the derivative

$$\nabla_{\boldsymbol{11'}}\big(\Xi\nabla^{\boldsymbol{AA'}}\nabla_{\boldsymbol{AA'}}\Xi - 2\nabla_{\boldsymbol{AA'}}\Xi\nabla^{\boldsymbol{AA'}}\Xi\big)$$

and makes use of the conformal field equations to rewrite it as a homogeneous expression in zero quantities. In view of the transformation law of the Ricci scalar under conformal rescalings, the term in brackets coincides with $R[\boldsymbol{g}]$. Now, from the discussion leading to Lemma 18.2 one concludes that $R[\boldsymbol{g}] = 0$ on $\mathscr{N} \cup \mathscr{I}^-$. The corollary then follows from the uniqueness of the solutions to the characteristic problem.

18.3.3 The existence result

Combining the analysis developed in the previous subsections with the theory of characteristic initial value problems for symmetric hyperbolic systems of Section 12.5, one obtains the following existence result:

Theorem 18.1 (***existence and uniqueness to the standard asymptotic characteristic problem***) *Given a smooth reduced initial data set* $\mathbf{r}_\star$ *for the conformal Einstein field equations on* $\mathscr{N} \cup \mathscr{I}^-$*, there exists a unique smooth solution of the conformal field equations in a neighbourhood* $\mathcal{U}$ *of* $\mathscr{Z}$ *in* $J^+(\mathscr{Z})$ *which implies the prescribed initial data on* $\mathscr{N} \cup \mathscr{I}^-$*.*

Proof It follows from Lemma 18.4 that the formal derivatives of $\mathbf{u}$ can be computed to any arbitrary order from the reduced data $\mathbf{r}_\star$ on $\mathscr{N} \cup \mathscr{I}^-$. Hence, it is possible to formulate an auxiliary Cauchy problem for the reduced conformal field Equations (18.17a)–(18.17d) with data implied by the extension to a neighbourhood of $\mathscr{Z}$ given by Whitney's theorem. Thus, using Theorem 12.7 and the discussion in Section 12.5.3 there is a neighbourhood $\mathcal{W}$ of $\mathscr{Z}$ in $J^+(\mathscr{Z})$ in which there exists a unique solution $\mathbf{u}$ to the reduced conformal field equations which on $\mathscr{N} \cup \mathscr{I}^-$ coincides with the data $\mathbf{u}_\star$ implied by the prescribed reduced initial data – as $\mathscr{Z} \approx \mathbb{S}^2$, it is necessary to combine solutions in two different patches. Finally Proposition 18.1 and Corollary 18.1 imply that the solution to the reduced equations is, in fact, a solution to the full conformal field equations. □

The characteristic problem on $\mathscr{N}' \cup \mathscr{I}^+$

The analysis leading to Theorem 18.1 can be adapted to analyse the dual asymptotic characteristic problem with data on $\mathscr{N}' \cup \mathscr{I}^+$ where $\mathscr{N}'$ is a future-oriented null hypersurface. In this case one endeavours to find a solution in a neighbourhood $\mathcal{U}'$ of $\mathscr{Z}' = \mathscr{N}' \cap \mathscr{I}^+$ in $J^-(\mathscr{Z}')$. All the relevant expressions can be obtained from those for the characteristic problem on $\mathscr{N} \cup \mathscr{I}^-$ through the replacements $\mathbf{0} \mapsto \mathbf{1}$, $\mathbf{1} \mapsto \mathbf{0}$ in the spinorial frame indices so that

$$\boldsymbol{e}_{\mathbf{00'}} \mapsto \boldsymbol{e}_{\mathbf{11'}}, \quad \boldsymbol{e}_{\mathbf{11'}} \mapsto \boldsymbol{e}_{\mathbf{00'}}, \quad \boldsymbol{e}_{\mathbf{01'}} \mapsto \boldsymbol{e}_{\mathbf{10'}}, \quad \boldsymbol{e}_{\mathbf{10'}} \mapsto \boldsymbol{e}_{\mathbf{01'}}.$$

In particular, one has

$$\phi_0 \mapsto \phi_4, \quad \phi_1 \mapsto \phi_3, \quad \phi_2 \mapsto \phi_2, \quad \phi_3 \mapsto \phi_1, \quad \phi_4 \mapsto \phi_0$$

and

$$\omega \mapsto \bar{\omega}, \qquad \xi^{\mathcal{A}} \mapsto \bar{\xi}^{\mathcal{A}}.$$

Similarly, for the connection coefficients and the components of the trace-free Ricci spinor one has

$$\Gamma_{\mathbf{01'00}} \mapsto \Gamma_{\mathbf{10'11}}, \qquad \Phi_{12} \mapsto \Phi_{10} = \Phi_{01}, \qquad \text{and so on.}$$

For consistency, one should replace the coordinate v along the generators of $\mathscr{I}^-$ with a coordinate u along the generators of $\mathscr{I}^+$.

18.4 The asymptotic characteristic problem on a cone

As discussed in the introduction, an alternative characteristic problem for the conformal Einstein field equations consists of a configuration where initial data is prescribed in a neighbourhood of the vertex of a cone representing the timelike infinity of a Minkowski-like spacetime; see Figure 18.1, right. This type of geometric setup for a characteristic initial value problem was originally introduced in Friedrich (1986c) and is intended to model ***purely radiative spacetimes***, that is, a system describing gravitational radiation from past null infinity which interacts non-linearly with itself and eventually escapes to future null infinity. Intuitively, one would expect this type of solution to the Einstein field equations to have a smooth structure at null infinity. To ensure that the gravitational field consists only of gravitational radiation one requires that the generators of null infinity are complete and that past timelike infinity is represented by a point i^- which is *regular from the point of view of the conformal completion.*

To discuss the geometric setting in a more precise manner it is convenient to introduce some definitions.

Definition 18.1 (***spacetimes with a cone past boundary***) *A spacetime* $(\mathcal{M}, \boldsymbol{g})$ *is said to have a* ***cone past boundary*** *if:*

(i) There exists a causal, oriented and time-oriented spacetime $(\mathcal{M}', \boldsymbol{g}')$ *(the* ***ambient manifold****).*
(ii) There exists a point $o \in \mathcal{M}'$ *such that the set consisting of* o *and all points of* $\mathcal{M}'$ *which can be joined to* o *by a causal curve in* $\mathcal{M}'$ *– to be denoted by* $J^+(o, \mathcal{M}')$ *– is closed in* $\mathcal{M}'$*.*
(iii) Given $\mathscr{N}_o \equiv \partial J^+(o, \mathcal{M}')$*, then* $\mathscr{N}_o \setminus \{o\}$ *is a smooth null hypersurface of* $\mathcal{M}'$*.*
(iv) The set $\mathcal{M}$ *corresponds to* $J^+(o, \mathcal{M}')$ *together with the structures it inherits from* $(\mathcal{M}', \boldsymbol{g}')$ *– in particular,* $\boldsymbol{g}$ *is the pull-back of* $\boldsymbol{g}'$ *to* $\mathcal{M}$*.*

Given $p \in \mathcal{M}$*, the set* $\mathscr{N}_p \subset \mathcal{M}$ *is called the* ***future null cone of*** p*.*

In terms of the above notions one introduces the further notion:

Definition 18.2 (***spacetimes with a complete past null infinity cone***) *A vacuum spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *is said to be a* ***solution to the Einstein field equations with complete null cone at past timelike infinity*** i^- *if there exists a conformal extension* $(\mathcal{M}, \boldsymbol{g}, \Xi)$ *with cone-like past boundary* $\mathscr{N}_{i^-}$ *such that the conformal factor satisfies*

$$\Xi > 0 \qquad \text{on } \mathcal{M} \setminus \mathscr{N}_{i^-}, \tag{18.24a}$$

$$\Xi = 0 \qquad \text{on } \mathscr{N}_{i^-}, \tag{18.24b}$$

$$\mathbf{d}\Xi \neq 0 \qquad \text{on } \mathscr{N}_{i^-} \setminus \{i^-\}, \tag{18.24c}$$

$$\mathbf{d}\Xi = 0, \quad \boldsymbol{Hess}\,\Xi \text{ non-degenerate at } i^-, \tag{18.24d}$$

and there is a diffeomorphism by means of which the manifolds $\tilde{\mathcal{M}}$ *and* $\mathcal{M} \setminus \mathscr{N}_{i^-}$ *can be identified so that* $\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}$ *on* $\mathcal{M} \setminus \mathscr{N}_{i^-}$*. The set* $\mathscr{N}_{i^-} \setminus \{i^-\}$ *is swept by the future-directed null geodesics through* i^- *and represents the past null infinity* $\mathscr{I}^-$ *of the spacetime.*

Equipped with the above definitions, one can formulate a ***pure radiation problem*** in which one asks: *given data on a cone* $\mathscr{N}_o$*, is there a unique solution to the Einstein field equations with complete past null infinity implying fields on* $\mathscr{I}^-$ *which can be identified with the data prescribed on* $\mathscr{N}_o$ *and such that the point* o *can be identified with* i^-*?*

18.4.1 Gauge conditions

This section gives a brief discussion of the gauge specification process for the characteristic initial value problem on a cone. As is the case in all initial value

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problems concerning the conformal field equations, one has to consider three different types of gauges: conformal, coordinate and frame gauges. These are analysed in turn.

The conformal gauge

Given a null cone $\mathscr{N}_o$ with vertex o, let $\boldsymbol{l}$ denote the vector tangent to the null generators of $\mathscr{N}_o$. Consistent with conditions (18.24a)–(18.24d), it is assumed that one has a conformal factor Ξ such that

$$\Xi = 0, \qquad \mathbf{d}\Xi = 0, \qquad s \neq 0 \qquad \text{at } o.$$

Mimicking the discussion of Section 16.3, one can transvect the conformal field equations

$$\nabla_a \nabla_b \Xi = -\Xi L_{ab} + s g_{ab}, \qquad \nabla_a s = -\nabla^b \Xi L_{ba}, \tag{18.25}$$

with $\boldsymbol{l}$ to find that $\Xi = 0$ and $s \neq 0$ on $\mathscr{N}_o$ and, moreover, that $\mathbf{d}\Xi \neq 0$ on $\mathscr{N}_o \setminus \{o\}$. It is also observed that if $s|_o = 0$, then $\mathbf{d}\Xi = 0$ on $\mathscr{N}_o$. The behaviour of the conformal gauge at o can be refined by considering a rescaling as in Equation (18.6) with $\vartheta > 0$. Making use of the transformation formula for the Friedrich scalar s, Equation (8.29b), one finds that $s'|_o = (s\vartheta^{-1})|_o$. Let $\gamma(\varsigma)$ with $\varsigma \in \mathbb{R}$ denote a future-directed null geodesic on $\mathscr{N}_o$ with $\gamma(0) = o$ such that $\boldsymbol{l} = \dot{\gamma}$ and, consequently, $\nabla_{\boldsymbol{l}} \boldsymbol{l} = 0$. Setting $\boldsymbol{l}' \equiv \vartheta^{-1}\boldsymbol{l}$, one finds that $\boldsymbol{g}'(\boldsymbol{l}', \boldsymbol{l}') = 0$ and $\nabla'_{\boldsymbol{l}'} \boldsymbol{l}' = 0$ as well. Using the transformation formula for the trace-free Ricci tensor Φ_{ab}, Equation (18.8), one finds that along γ it holds that

$$\vartheta^3 l'^a l'^b \Phi'_{ab} = \vartheta^{-1} l^a l^b \Phi_{ab} + 2 l^b \nabla_b \big(l^a \nabla_a (\vartheta^{-1}) \big).$$

Thus, if the value of the component $l'^a l'^b \Phi'_{ab}$ is prescribed, the above equation can be read as an ordinary differential equation for ϑ along the null geodesic γ. The initial value of ϑ can be fixed through the specification of $s|_o$. Using the first of the equations in (18.25) one finds that

$$s|_o \boldsymbol{g}'(\boldsymbol{l}', \boldsymbol{l}')|_o = \nabla_{\boldsymbol{l}'} \nabla_{\boldsymbol{l}'} \Xi'|_o.$$

In order to have a local minimum of Ξ at o, one needs that $\nabla_{\boldsymbol{l}'} \nabla_{\boldsymbol{l}'} \Xi'|_o > 0$ forcing $s|_o > 0$ – in the signature $(+---)$. Without loss of generality, one can then set

$$s = 2 \qquad \text{at } o, \tag{18.26}$$

and

$$l^a l^b \Phi_{ab} = 0 \qquad \text{on } \mathscr{N}_o \text{ near } o. \tag{18.27}$$

In this construction there is still the freedom of specifying the value of $\mathbf{d}\vartheta$ at o. Adapting the arguments of Section 18.1.2 one can set a characteristic initial value problem on $\mathscr{N}_o$ for the wave Equation (18.7) in such a way that

$$R[\boldsymbol{g}] = 0 \qquad \text{on } J^+(\mathscr{N}_o) \text{ near } o. \tag{18.28}$$

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The coordinates and the frame near o

A convenient four-dimensional description of the null cone $\mathscr{N}_o$ is obtained using $\boldsymbol{g}$-normal coordinates $y = (y^\mu)$ centred at o; see Sections 2.4.5 and 11.6.2. Accordingly, one has that $y^\mu(o) = 0$, $g_{\mu\nu}(o) = \eta_{\mu\nu}$, $\partial_\lambda g_{\mu\nu}(o) = 0$ and $\Gamma_\mu{}^\nu{}_\lambda(o) = 0$. These properties can be more concisely summarised in the expression

$$y^\mu g_{\mu\nu} = y^\mu \eta_{\mu\nu} \qquad \text{in a neighbourhood of } o. \tag{18.29}$$

In these coordinates, for fixed $(y^\mu) \neq 0$ one has that the curve $\gamma : \varsigma \to \varsigma y^\mu$ is a geodesic through o and that

$$\mathscr{N}_o = \{y^\mu \in \mathbb{R}^4 \,|\, \eta_{\mu\nu} y^\mu y^\nu = 0,\ y^0 \geq 0\}.$$

Thus, in *these coordinates the null cone $\mathscr{N}_o$ can be thought of as being the null cone through the origin in Minkowski spacetime.*

Associated to the $\boldsymbol{g}$-normal coordinates, it is natural to consider a ***normal frame centred at*** o, that is, a frame $\{\boldsymbol{e_a}\}$ which, in a neighbourhood $\mathcal{U}$ of o, satisfies $\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}) = \eta_{\boldsymbol{ab}}$ and $\nabla_{\dot\gamma}\boldsymbol{e_a} = 0$ for any geodesic passing through o. Without loss of generality, one can assume that the frame coefficients in $\boldsymbol{e_a} = e_{\boldsymbol{a}}{}^\mu \boldsymbol{\partial}_\mu$ satisfy $e_{\boldsymbol{a}}{}^\mu(o) = \delta_{\boldsymbol{a}}{}^\mu$. Using the properties of the exponential function, it can be shown that the frame coefficients $e_{\boldsymbol{a}}{}^\mu$ depend smoothly on the coordinates (y^μ). It can then be verified that $\boldsymbol{g}(\dot\gamma, \boldsymbol{e_a})$ is constant along γ. Moreover, using that $g_{\mu\nu} = \eta_{\boldsymbol{ab}}\omega^{\boldsymbol{a}}{}_\mu \omega^{\boldsymbol{b}}{}_\nu$, it can be shown that

$$y^\mu \delta_\mu{}^{\boldsymbol{a}} e_{\boldsymbol{a}}{}^\nu(y) = y^\nu, \qquad y^\mu \eta_{\mu\nu} e_{\boldsymbol{a}}{}^\nu(y) = y^\mu \eta_{\mu\nu}\delta_{\boldsymbol{a}}{}^\nu. \tag{18.30}$$

The above conditions can be regarded as an alternative definition of normal coordinates. More precisely, if a set of coordinates $y = (y^\mu)$ and frame coefficients $\{e_{\boldsymbol{a}}{}^\mu\}$ satisfy the conditions in (18.30) the metric components $g_{\mu\nu}$ will satisfy condition (18.29).

To complete the discussion, it is convenient to introduce the vector field $\boldsymbol{y}(y) = y^\mu \boldsymbol{\partial}_\mu$ tangent to the geodesics through o. One then has

$$\boldsymbol{y}(o) = 0, \qquad (\nabla_\mu y^\nu)|_o = \delta_\mu{}^\nu, \qquad \nabla_{\boldsymbol{y}}\boldsymbol{y} = \boldsymbol{y}.$$

Writing $\boldsymbol{y}$ in terms of a $\boldsymbol{g}$-normal frame one has that $\boldsymbol{y} = y^{\boldsymbol{a}}\boldsymbol{e_a}$ where $y^{\boldsymbol{a}}(y) = \delta_\nu{}^{\boldsymbol{a}} y^\nu$. Furthermore, using $\nabla_{\boldsymbol{y}}\boldsymbol{e_a} = 0$ one concludes that

$$y^{\boldsymbol{a}}(y)\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}(y) = \delta_\nu{}^{\boldsymbol{a}} y^\nu \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}(y) = 0, \qquad \text{close to } o.$$

The coordinates $y = (y^\mu)$ and the frame $\{\boldsymbol{e_a}\}$ satisfying the conditions discussed in the previous paragraphs will be collectively known as a ***normal gauge***. This gauge system is supplemented by a normalised spin frame $\{\epsilon_{\boldsymbol{A}}{}^A\}$ satisfying $y^{AA'}\nabla_{AA'}\epsilon_{\boldsymbol{A}}{}^B = 0$ such that $\{\boldsymbol{e}_{\boldsymbol{AA'}}\} = \{\boldsymbol{\epsilon_A}\bar{\boldsymbol{\epsilon}}_{\boldsymbol{A'}}\}$ with $\boldsymbol{e}_{\boldsymbol{AA'}} = \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\boldsymbol{e_a}$ – here $y^{AA'}$ is the spinorial counterpart of the vector $\boldsymbol{y}$. In what follows, *all spinors will be expressed in components with respect to this type of frame.*

Adapted coordinates on $\mathscr{N}_o$

The coordinates $y = (y^\mu)$ introduced in the previous subsections provide a convenient spacetime description of $\mathscr{N}_o$. However, to analyse the intrinsic geometry of the cone, one needs *adapted coordinates*. The construction of these coordinates is similar to that of the coordinates $(v, r, x^{\mathcal{A}})$ used in the analysis of the characteristic problem on $\mathscr{N} \cup \mathscr{I}^-$ in Section 18.1.1. The fundamental difference is that, in the case of a cone, these adapted coordinates degenerate at the vertex o. More precisely, one can consider adapted coordinates $x = (x^\mu)$ such that $\mathscr{N}_o$ is given as a level surface by the condition $r \equiv x^1 = 0$ and $v \equiv x^0$ is a parameter along the generators with tangent $\boldsymbol{l}$ – thus, $\boldsymbol{l} = \boldsymbol{\partial}_v$. The two-dimensional spacelike surfaces $\mathscr{Z}_{v_\bullet} \equiv \{p \in \mathscr{N} \mid v(p) = v_\bullet\}$ satisfy $\mathscr{Z}_{v_\bullet} \approx \mathbb{S}^2$, except for the limit case $\mathscr{Z}_0 = \{o\}$ which is a point. As in Section 18.1.1, $(x^{\mathcal{A}})$ denote local coordinates on $\mathscr{Z}_{v_\bullet}$. On $\mathscr{N}_o$ the covector $\boldsymbol{n}^\flat \equiv \mathbf{d}r$ is null and normal to $\mathscr{N}_o$. The coordinate r can be chosen so that one has the usual normalisation $\boldsymbol{g}(\boldsymbol{l}, \boldsymbol{n}) = 1$. Finally, the vectors $\boldsymbol{l}$ and $\boldsymbol{n}$ can be completed to a frame by choosing a pair of complex conjugate vectors $\boldsymbol{m}$, $\bar{\boldsymbol{m}} \in T(\mathscr{Z}_{v_\bullet})$, for $v_\bullet \neq 0$, such that $\boldsymbol{g}(\boldsymbol{m}, \bar{\boldsymbol{m}}) = -1$. As in Section 18.1.1 the vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ can be parallelly propagated along the generators of $\mathscr{N}_o$ off some fiduciary section $\mathscr{Z}_{v_\bullet}$.

18.4.2 Null data on the cone

As in the case of the characteristic problem on $\mathscr{N} \cup \mathscr{I}^-$, there are several ways of prescribing the free data. The most physically meaningful specification consists of the so-called ***radiation field*** encoding information on the two components of the Weyl tensor with the slowest fall-off at null infinity, and can be thought of as describing *the two polarisation states of incoming radiation.*

To describe the null data, let, as in previous sections, $\boldsymbol{l}$ denote the vector tangent to the generators of the null cone $\mathscr{N}_o$. As $\boldsymbol{l}$ is a null vector, there exists a spinor κ^A such that $l^{AA'} = \kappa^A \bar{\kappa}^{A'}$ with $l^{AA'}$ the spinorial counterpart of $\boldsymbol{l}$. The spinor κ^A is defined up to a phase $\kappa^A \mapsto e^{\mathrm{i}\vartheta}\kappa^A$ with $\vartheta \in \mathbb{R}$ constant along the null generators. The ***radiation field*** is then defined as the component

$$\phi_0 \equiv \kappa^A \kappa^B \kappa^C \kappa^D \phi_{ABCD}$$

of the rescaled Weyl spinor. Due to the phase ambiguity in κ^A, the radiation field is a spin-weighted quantity. The information encoded in the radiation field is equivalent to information on the pull-back of $d_{abcd}l^a l^c$ to $\mathscr{N}_o$. More precisely, if $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ are complex vectors tangent to the sections of $\mathscr{N}_o$ such that $\boldsymbol{g}(\boldsymbol{l}, \boldsymbol{m}) = 0$, then it follows from the symmetries of the Weyl tensor that $\phi_0 = d_{abcd}l^a m^b l^c m^d$.

Solving the constraints on $\mathscr{N}_o$

In analogy to the characteristic problem on $\mathscr{N} \cup \mathscr{I}^-$, and making use of the adapted coordinates $x = (v, r, x^{\mathcal{A}})$ and of the frame $\{\boldsymbol{l},\ \boldsymbol{n},\ \boldsymbol{m},\ \bar{\boldsymbol{m}}\}$, the conformal

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Einstein field equations split into equations transverse and intrinsic to $\mathscr{N}_o$. The intrinsic equations divide, in turn, into propagation equations (i.e. ordinary differential equations) along the generators of the cone and constraints which need to be solved only at a particular cut. Assuming the conformal gauge discussed in Section 18.4.1, the knowledge of the radiation field ϕ_0 on $\mathscr{N}_o$ allows one to compute the value of the remaining conformal fields in a neighbourhood of o on $\mathscr{N}_o$. More precisely, one has the following:

Proposition 18.2 (***reduced initial data for the asymptotic characteristic problem on a cone***) *In the conformal gauge given by conditions* (18.26), (18.27) *and* (18.28)*, the transport equations induced by the conformal Einstein field equations and the structure equations on $\mathscr{N}_o$ uniquely determine the fields Ξ, s, $\Phi_{\boldsymbol{AA'BB'}}$ and $\phi_{\boldsymbol{ABCD}}$ on $\mathscr{N}_o$ once the radiation field ϕ_0 has been prescribed. The resulting fields satisfy the constraint equations on $\mathscr{N}_o$.*

Details on this result can be found in Friedrich (2014b).

Evaluating formal derivatives on $\mathscr{N}_o$

In addition to solving the constraint equations on $\mathscr{N}_o$, and in order to apply the theory of characteristic problems on a cone, given a choice of radiation field, it is necessary to show that the (formal) derivatives of any order of the conformal fields can be determined on the null cone along the generators of $\mathscr{N}_o$. This analysis is analogous to the one discussed in Section 18.3.1 for the characteristic problem on $\mathscr{N} \cup \mathscr{I}^-$. In the present case, however, the analysis is more delicate as the set $\mathscr{Z} = \mathscr{N} \cap \mathscr{I}^-$ shrinks to a point, so that the information for the integration along the generators has to be extracted solely from the null data. The key result is the following (see Friedrich (2014b)):

Proposition 18.3 (***computation of formal derivatives at the vertex***) *In a neighbourhood of the point o let the fields Ξ, s, $\Phi_{\boldsymbol{AA'BB'}}$ $\phi_{\boldsymbol{ABCD}}$, $e_{\boldsymbol{AA'}}{}^{\mu}$, $\Gamma_{\boldsymbol{AA'BC}}$ be smooth and be expressed in an o-centred normal gauge and a conformal gauge satisfying Equations* (18.26), (18.27) *and* (18.28)*. If the above fields satisfy the conformal field equations, then the Taylor expansions of the fields Ξ, s, $\Phi_{\boldsymbol{AA'BB'}}$ and $\phi_{\boldsymbol{ABCD}}$ in a suitable neighbourhood of o are determined by the null datum ϕ_0.*

Remark. In the above proposition, the neighbourhoods of o are spacetime neighbourhoods in the ambient manifold $\mathcal{M}'$ containing the cone $\mathscr{N}_o$.

18.4.3 The existence result

The setting described in the previous paragraphs leads to the following existence result, adapted from Chruściel and Paetz (2013):

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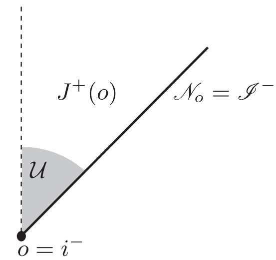

Figure 18.4 Schematic representation of the set up for the asymptotic characteristic problem on a cone. The existence results are restricted to a neighbourhood $\mathcal{U}$ of o in $J^+(o)$.

Theorem 18.2 (***local existence for the asymptotic characteristic problem on a cone***) *For any smooth prescription of the radiation field ϕ_0 on the null cone at the origin of the Minkowski spacetime, $\mathscr{N}_o$, there exists a neighbourhood $\mathcal{U} \subset J^+(o)$ of o, a smooth metric $\boldsymbol{g}$ and a smooth function Ξ such that:*

(i) $\mathscr{N}_o$ *is the light cone of* o *for* $\boldsymbol{g}$.
(ii) $\Xi = 0$ *on* $\mathscr{N}_o$.
(iii) $\mathbf{d}\Xi = 0$, $\boldsymbol{Hess}\,\Xi \neq 0$ *on* o.
(iv) $\mathbf{d}\Xi \neq 0$ *on* $\partial J^+(o) \cap \mathcal{U} \setminus \{o\}$.
(v) *The function* Ξ *has no zeros on* $\mathcal{U} \cap I^+(o)$ *and the metric* $\tilde{\boldsymbol{g}} = \Xi^{-2}\boldsymbol{g}$ *satisfies the vacuum Einstein field equations on* $\mathcal{U} \cap I^+(o)$.

Moreover, the rescaled Weyl spinor ϕ_{ABCD} of the pair $(\boldsymbol{g}, \Xi)$ extends smoothly across $\mathscr{N}_o$ and the restriction of $\phi_{ABCD}\epsilon_{\mathbf{0}}{}^A\epsilon_{\mathbf{0}}{}^B\epsilon_{\mathbf{0}}{}^C\epsilon_{\mathbf{0}}{}^D$ to $\mathscr{N}_o \setminus \{o\}$ coincides with the prescribed radiation field ϕ_0. The solution is unique up to isometries.

Remark. It follows from points (ii), (iii) and (iv) that the set $\mathscr{N}_o \setminus \{o\}$ corresponds to the past null infinity $\mathscr{I}^-$ of the resulting spacetime, while the vertex o is its past timelike infinity i^-. A schematic representation of the set up of the above theorem is given in Figure 18.4.

The proof of the above theorem, as given in Chruściel and Paetz (2013), makes use of the metric version of the conformal field equations and the associated wave equations discussed in Paetz (2015); see also Section 13.5.2. The reason behind the use of a hyperbolic reduction based on wave equations – as opposed, say, to the first-order symmetric hyperbolic systems used throughout this book – lies in the fact that the available theory of characteristic problems on a cone is well understood for this type of equations; see Dossa (1986, 2002).

18.5 Further reading

Characteristic problems in general relativity have a long history. The first systematic discussion has been given in Sachs (1962c). Further classical discussions can be found in Penrose (1965, 1980) and Müller zu Hagen and Seifert (1977).

A review on the various approaches to the problem, including an analysis of the possible choices of free data, can be found in Chruściel and Paetz (2012); this reference provides a convenient point of entry to the literature on the subject.

The basic theory of asymptotic characteristic initial value problems for the conformal field equations has been developed in the articles by Friedrich (1981a,b). A version of Theorem 18.1 in the analytic setting was given in Friedrich (1982). This result has been extended to the smooth setting in Kánnár (1996b) using the reduction to an auxiliary Cauchy problem given in Rendall (1990). The geometric set up for the asymptotic characteristic problem on a cone has first been given in Friedrich (1986c). The relation between Taylor expansions at the vertex of the null cone and the interior equations implied by the conformal Einstein field equations has been examined in Friedrich (2014b). The existence result for the characteristic problem in the cone has been given in Chruściel and Paetz (2013). Characteristic problems on a cone are less studied than those on intersecting null hypersurfaces. A good point of entry to the literature is Choquet-Bruhat et al. (2011).

Characteristic problems provide a natural approach to the construction of solutions to the Einstein equations by means of numerical methods. An advantage of this formulation is its clear-cut connection with the notion of gravitational radiation; see, for example, Damour and Schmidt (1990). A review on the subject can be found in Winicour (2012).

The characteristic initial value problem has been used in the seminal work by Christodoulou on the collapse of a spherically symmetric self-gravitating scalar field; see Christodoulou (1986).