Third Meeting, January 8th, 1886.

Dr Ferguson, F.R.S.E., President, in the Chair.

Two Mechanical Integrators or Planimeters.

By A. Y. Fraser, M.A., F.R.S.E.

The Measurement of Areas.

For the measurement of a plane area, bounded by an irregular curve, various methods are adopted. Besides the well-known methods of approximation in use among land measurers, the following may be mentioned—

The area is divided by parallel ordinates. These are measured, and the results are then treated in several ways to produce more or less accurate results. See Williamson's *Integral Calculus*, third edition, p. 211.

The area may be drawn on paper ruled in small squares ("plotting paper"), or a sheet of glass ruled in squares is placed over the area. The complete squares included are counted, and the incomplete squares estimated.

The area is drawn on cardboard or sheet metal, cut out, and then weighed. The weight of a known area is then ascertained, and the required result got. This method gives very good results. A variation of the same method is to cover the area with small shot, which is weighed or counted.

A planimeter or mechanical integrator may be used. Here a pointer has merely to be taken round the boundary, and the instrument records the area traced out. Many planimeters have been invented; but for all practical purposes, the beautiful little instrument invented by Prof. Amsler thirty years ago, holds its own against all comers.

Planimeters.

The problem, in general, attempted by planimeters, will be understood by a reference to figure 24. While the tracing point of the

https://doi.org/10.1017/S0013091500029849 Published online by Cambridge University Press

instrument moves over any arc aa', an index records the area $aa'\nu'\nu$. Suppose, now, the pointer starts at A, the extreme right of the figure, and passes round ABC to C; the area recorded will be ABCEF. Suppose, again, that in bringing the pointer round CDA it is arranged that the reading given by the index is subtracted from, and not added to, the former reading (and the instrument can easily be made to effect this), then the area recorded, when the pointer has returned to A, will be ABCEF − CDAFE, that is the area ABCD.

This in fact is the mechanical integration of $\int y dx$ between the limits $x = OF$ and $x = OE$.

It is a matter of indifference where the commencement with the tracing point is made, so long as the point is stopped at the same place.

For a full discussion of the problem to be solved by the inventor of a planimeter, the reader may be referred to the papers of John Sang* and Clerk Maxwell,* and, especially to the full account of the whole matter by Professor Shaw.† Clerk Maxwell points out the division of planimeters into two kinds—(1) Those that involve slipping in some of their parts; (2) Those that involve rolling only. He enforces the objection to the former class (errors on account of friction), and gives a design for an instrument of the second kind—a design which, so far as I know, never came to anything practically.

Of the two instruments to be described here, the first may be taken to belong to the first class, that is, it involves rolling and slipping in its movements, while the second belongs to the second or purely rolling class. While it is not for a moment supposed that these can compete as practical instruments with Amsler's, they are brought forward because they are of an entirely new design, and both are of considerable mathematical interest.

The Single Roller Planimeter.

The essential portion of this instrument is a tapering roller or spindle (fig. 25) mounted to turn on an axis, and made to turn by placing the paper, on which the area is traced, on the surface of the roller, pressing the paper against the surface with a pencil or other

* Transactions of the Royal Scottish Society of Arts, Vol. IV., p. 119; p. 420.

† Journal of the Institute of Civil Engineers, Vol. lxxxii., 1884-85, Part IV., pp. 75-164; on "Mechanical Integrators."

pointed object, and pulling or pushing the paper in a direction at right angles to the axis of the roller, and tangentially to the surface.

The more easily to understand this motion, the reader may place on a table in front of him a tumbler on its side with its bottom towards him, lay a sheet of paper over the tumbler and press it gently down on the tumbler with, say, a finger nail of the right hand. Then on pulling the paper away to the left with the left hand, the tumbler will roll along the table. If the tumbler could be mounted to rotate in the same way, about an axis passing longitudinally through it, the resemblance would be complete.

The dimensions of the spindle, of which a section is shown in fig. 25, as actually constructed are: Length GH, measured along the surface, 6 inches; circumference at H, 10 inches; at G, $2\frac{1}{2}$ inches. The circumference at any point K is got thus: Let O be a point 2 inches from H; measure the distance along the surface from K to O, n inches say, that is, $n-2$ inches from K to H, then the circumference at K is $20/n$ inches.

The method of mounting the roller will be seen from figures 26, 27. LT is a box of $\frac{5}{8}$-in. mahogany. Fig. 26 shows the box as looked at when about to be used; fig. 27, the back of the box through which the wide end of the roller, which is graduated, is seen. The top of the box is covered with two slips of tin, R, S, which slide in from each end, and do not meet in the middle by about half an inch. The tins are bent to the shape of the roller, and through the middle opening of half an inch, the roller projects so as to show a strip very slightly above the level of the lids. A guiding strip of wood is fastened from V to W, so that a pencil pressed vertically down on the roller and against this strip of wood, will always be vertically above the axis of the roller. The dimensions of the box are LM 16 ins., LN 5 ins., QP $3\frac{1}{2}$ ins., NP 7 ins.

To find the area of any surface, we proceed as follows:—The roller is first turned round so that the zero point of the scale (fig. 27) may be under the pointer. A small mirror laid over against the box may be used to keep the scale always in view. Suppose the area in fig. 24 is to be integrated. The slip of paper on which the area is traced is passed along the top RS of the box, under the guide VW. The paper is placed so that A can be brought under the point of a pencil resting against the guide VW. The pencil is then put down so as to press the paper gently against the surface of the roller. The paper is pushed away to the right, the projecting ledge QY of the box, fig. 26, guiding the paper, the pencil all the time being slid up and down against the guide VW so as to follow the curve ABC. When the pencil has reached C, the area ABCEF has been recorded on the scale. The paper is now pulled to the left, and the pencil is made to follow CDA, the roller meanwhile rotating backwards, and the indicator recording the area CDAFE in the negative sense. The indicator has now recorded the area ABCFE – CDAFE, that is the area ABCD; the result being given in square inches, and whatever fraction of an inch the scale may be made to show. We shall now prove that the result is as stated.

If, while the paper lies on the top of the box, a line (OF, fig. 24) be drawn on the paper parallel to $\lambda\mu$ (fig. 26), and two inches beyond it, that is, through the point O of figure 25, we can prove that the instrument, in passing over the arc aa', records the area $aa'\nu'\nu$. Suppose the length of the arc aa' is s, and that it is n inches from OF; the area of $aa'\nu'\nu$, if s be taken small, will be sn inches. To prove that the reading on the scale is sn.

The roller immediately below aa' (at K, say, fig. 25) is $20/n$ inches in circumference, and s inches of this circumference passes round in tracing the arc aa'. Now s inches is $s/(20/n)$ of the whole circumference at that place, that is $sn/20$ths, and in making this movement there will be $sn/20$ths of a complete revolution of the roller. But in a whole revolution the index records 20 inches (fig. 27), therefore in making $sn/20$ths of a revolution the record will be $(sn/20) \times 20$ inches, that is sn inches. And aa' is any portion of the boundary, so that the proof is complete.*

* Strictly speaking, perhaps, the arc aa' should be considered as the hypotenuse of a right-angled triangle, having one side s parallel, the other t perpendicular to OF. The pointer would then be supposed to be taken over s during which the roller would rotate (with the result as above), then over t during which there would be no rotation. Considered in this way the total area accounted for is less than the area ABCD by the sum of the elementary triangles round the curve (if we suppose the triangles always made to the inside of the figure); and the length of the path traversed by the pointer is greater than the curve ABCD by a corresponding sum of small lengths. Finally, if the triangles be taken small, the limit to the areas thus considered is the area ABCD, and at the same time the limit to the curve as thus traced is the curve ABCD; therefore in passing round ABCD the area integrated is the area ABCD.

Of course instead of 20 as the fundamental constant for the roller, any other quantity might have been taken. The part of the curve also which is taken for the spindle may be varied, only at the low end it must be steep enough to show a change in the readings, and at the other end it must not be inconveniently steep.

The general equation in rectangular co-ordinates to the curve GKH (fig. 25), can be determined from the defining property, which may be put as follows:—Let the arc OK $= s$ then the circumference at K is $20/s$, say c/s. If y be the ordinate at K we have $y = c/2\pi s$, from which the equation to GKH can be found in terms of x and y.

The general result, which I get in what for the part of the curve to be used is a convergent series of powers of y, is not of any interest. The particular equation, terms beyond a certain distance being omitted, from which the actual curve under discussion was traced is

$$x = \frac{c}{y} + \frac{1}{6}\frac{y^3}{c} + \frac{1}{56}\frac{y^7}{c^3} - 2{\cdot}2258$$

where $c = 10/\pi = 3{\cdot}183$.

The curve reduces to an equilateral hyperbola whenever the terms after the first can be neglected.

Though there is slipping in this instrument in the working parts, that is, between the pencil and the paper, there is none where it would be of any consequence, namely, between the paper and the roller.

The difficulty with this planimeter, beyond the difficulty of constructing the roller, which can be got over well enough, is in co-ordinating the movements of the right and left hands so that the curve is accurately followed. This difficulty of avoiding small departures from the curve, led to the attempt to devise a modification of the instrument, where the paper should rest on a table and a pointer could be guided over the curve with a single motion. This attempt resulted in the construction of a much more interesting planimeter, of which a brief account will now be given.

The Double Roller Planimeter.

The arrangement of this instrument is shown in figure 28, which is an outline sketch of the principal parts. The motion is imparted to the lower spindle by the upper spiral roller, which is pivoted about H and G. The axis HG is kept always in the same horizontal plane by means of the two supports GC, HC′, running on castors, and the pointer P, which is rigidly attached to the frame FFF, carrying H, G; and HG is kept always parallel to itself and at right angles to the axis of the lower spindle by means of a linkwork contrivance (a pair of parallelograms) not shown in the figure. While the pointer P traces out the area, the spiral roller moves about, subject to the restrictions just mentioned, and acts as a driver of the spindle; the connection between the two being always one of *pure rolling contact.*

Inasmuch as the area to be integrated has not now to be bent along the surface of the roller, but lies in a horizontal plane, the different circumferences of the under roller are determined not now by distances measured along its surface, but by distances measured along its axis. This change being made, the numbers already given for the last spindle will apply here. The most interesting change is in the equation to the curve. For instead of $y = c/2\pi s$ we have $y = c/2\pi x$ that is

$$xy = \text{ a constant,}$$

and therefore the curve is an equilateral hyperbola.

The proof that this instrument does integrate the area is so very little different from that already given that it may be dispensed with here.

A little consideration will show that it is theoretically indifferent whether the spiral roller move with the pointer, the other roller being fixed, or the under roller move with the pointer while the spiral is fixed.

The spiral roller has one peculiarity not yet touched on. In order to secure that when the pointer is moved back an inch say, the point of contact of the two rollers should also move back the same horizontal distance, it is arranged that the point of contact is in all positions of the rollers vertically below the axis HG. This property makes the finding of the equation to the spiral a very interesting problem. We may conclude with the discussion of this problem.

To find the Equation to the Spiral.

As S (fig. 29) moves along the line $y = b$, the spiral, whose centre is S, rolls along the hyperbola $xy = c^2$ so as to have the point of contact Q always vertically below S.

Let x, y be the rectangular co-ordinates of Q considered as a point on the hyperbola, and r, θ the polar co-ordinates of the same point considered as belonging to the spiral.

Then $QN = rd\theta = dx$;

PN $= dr = -dy$ (got either from the figure directly or from $y = b - r$),

therefore $$\frac{rd\theta}{dr} = -\frac{dx}{dy} \qquad (1).$$

But $$xy = c^2$$

therefore $$\frac{dx}{dy} = -\frac{c^2}{y^2} = -\frac{c^2}{(b-r)^2}$$

whence substituting in (1) we get

$$\frac{rd\theta}{dr} = \frac{c^2}{(b-r)^2} \qquad (2);$$

therefore $$\int d\theta = c^2 \int \frac{dr}{r(b-r)^2}$$

$$= c^2 \int \left\{ \frac{1}{b^2 r} + \frac{1}{b^2(b-r)} + \frac{1}{b(b-r)^2} \right\} dr;$$

therefore $$\theta = c^2 \left\{ \frac{1}{b^2}\log r - \frac{1}{b^2}\log(b-r) + \frac{1}{b(b-r)} \right\} + k$$

$$= \frac{c^2}{b^2}\log\frac{r}{b-r} + \frac{c^2}{b(b-r)} + k \qquad (3)$$

$$= \frac{c^2}{b^2}\log_e 10 \log_{10}\frac{r}{b-r} + \frac{c^2}{b(b-r)} + k$$

$$= \frac{c^2}{b^2}\left\{ 2 \cdot 3049 \log_{10}\frac{a}{1-a} + \frac{1}{1-a} + \lambda \right\} \qquad (4);$$

where $2 \cdot 3049 = \log_e 10$

$$a = \frac{r}{b}$$

$$\lambda = k \div \frac{c^2}{b^2}.$$

Equation (3) is the general equation to the spiral; (4) is the more convenient form from which to trace the curve.

The spiral is seen to have an infinite number of windings about the point S, and to be asymptotic to a circle of radius b.

In the spiral as actually constructed the constants had to be chosen so as to be practically convenient, and after a number of trials, TA (fig. 29), that is the shortest radius of the part of the spiral to be used, was taken equal to $\frac{1}{5}b$, and b is therefore $\frac{5}{4}$ of AB, which again is the radius of a circle 10 inches in circumference. The equation actually used for tracing the spiral is

$$\theta = 1 \cdot 8533 \log\frac{a}{1-a} + \cdot 8041 \frac{1}{1-a} + \cdot 1107 \qquad (5)$$

Going from $r=8b/40$ by fortieths of b up to $r=32b/40$, using equation (5), and changing from circular measure to degrees, we get the following values:—

r (inches)	θ (degrees)	$\Delta\theta$	r (inches)	θ (degrees)	$\Delta\theta$
·398	0·00				
·448	8·74	8·74	1·044	107·95	9·46
·497	17·11	8·37	1·094	117·98	10·03
·547	25·19	8·08	1·144	128·74	10·76
·597	33·09	7·90	1·194	140·21	11·47
·647	40·89	7·80	1·243	152·76	12·55
·696	48·68	7·79	1·293	166·52	13·76
·746	56·50	7·82	1·343	181·81	15·29
·796	64·43	7·93	1·393	198·99	17·18
·846	72·52	8·09	1·442	218·58	19·59
·895	80·86	8·34	1·492	241·28	22·70
·945	89·48	8·62	1·542	268·15	26·87
·995	98·49	9·01	1·592	300·64	32·49

Only one remark will be made on the above table. As r increases uniformly the column for $\Delta\theta$ shows that θ is not increasing uniformly, and that in fact there is a minimum value for $d\theta/dr$ between $r=13b/40$ and $r=15b/40$. It may be interesting to show that this can be deduced from one of our equations.

From (2) we have

$$\frac{d\theta}{dr}=\frac{c^2}{r(b-r)^2}=\frac{c^2}{b^2r-2br^2+r^3};$$

therefore

$$\frac{d^2\theta}{dr^2}=-c^2\frac{b^2-4br+3r^2}{r^2(b-r)^4}=-c^2\frac{(b-3r)(b-r)}{r(b-r)^4}$$

$$=-c^2\frac{b-3r}{r^2(b-r)^3} \qquad (6)$$

Equating (6) to zero to find maximum or minimum values for $d\theta/dr$ we have $b-3r=0$, that is $r=\frac{1}{3}b=13\frac{1}{3}b/40$ which accords with the result in the table.