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SURVIVAL PROBABILITIES BASED ON PARETO CLAIM DISTRIBUTIONS

COMMENT

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1. INTRODUCTION

In a recent paper SEAL (1980) calculated numerically survival probabilities based on Pareto claim distributions.

The Pareto density may be written as

$$(1) \qquad p(x) = \frac{q}{b}\left(1+\frac{x}{b}\right)^{-q-1}, \qquad \begin{array}{l} 0 < x < \infty \\ b, q > 0 \end{array}.$$

Generalizing, the Pareto distribution may be regarded as a special case of the so-called beta-prime distribution (KEEPING, 1962, p. 83) with density function

$$(2) \qquad f(x) = \frac{1}{B(p, q)}\, x^{p-1}\,(1+x)^{-p-q}, \qquad \begin{array}{l} 0 < x < \infty \\ p, q > 0 \end{array},$$

where $B(p, q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$ is the beta function.

In his paper SEAL (1980, Appendix 1) arrived at a contradiction concerning this beta-prime distribution. He found on one side that all derivatives of the characteristic function exist at the origin and on the other side that only the moments of order $n < q$ exist. In this note we will show that this contradiction is due to the use of an incorrect expression for the characteristic function of the beta-prime distribution, which was taken over from JOHNSON and KOTZ (1970, Ch. 26) and OBERHETTINGER (1973, Table A).

2. THE CONFLUENT HYPERGEOMETRIC FUNCTIONS

For easy reference we list some basic properties of confluent hypergeometric functions (see e.g. SLATER, 1960).

There are two types of confluent hypergeometric functions, namely [1]

[1] Another notation for the series (3) is ${}_1F_1\,(a, b, z)$.

(3) $$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!},$$

where $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)} = a(a+1) \ldots (a+n-1)$, $(a)_0 = 1$,

and

(4) $$U(a, b, z) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(a, b, z)}{\Gamma(1+a-b)\,\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\,\Gamma(2-b)} \right\}.$$

The series (3) is absolutely convergent for all values of a, b and z, real or complex, excluding $b = 0, -1, -2, \ldots$. The function $U(a, b, z)$ is a many-valued function with principal branch given by $-\pi < \arg z \leqslant \pi$. This function is analytic for all values of a, b and z, even when b is zero or a negative integer. It can be represented as

(5) $$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt}\, t^{a-1} (1+t)^{b-a-1}\, dt,$$

for those values of a, b and z for which the integral exists.

If we differentiate $U(a, b, z)$ we get for the n^{th} derivative

(6) $$\frac{d^n}{dz^n} U(a, b, z) = (-1)^n (a)_n\, U(a+n, b+n, z).$$

Further we have for real b (which will be our case), the following behavior of $U(a, b, z)$ as $z \to 0$

(7.a.) $$U(a, b, z) \sim \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \qquad \text{if } b < 1,$$

(7.b.) $$\sim -\frac{1}{\Gamma(a)} [\ln z + \psi(a) - 2\gamma] \qquad \text{if } b = 1,$$

(7.c.) $$\sim \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} \qquad \text{if } b > 1,$$

where $\psi(a) = \dfrac{\Gamma'(a)}{\Gamma(a)}$ is the psi-function and $\gamma =$ Euler's constant.

3. THE CHARACTERISTIC FUNCTION OF THE BETA-PRIME DISTRIBUTION

The characteristic function of the beta-prime distribution is given by

$$\phi(t) = \frac{1}{B(p, q)} \int_0^{\infty} e^{itx}\, x^{p-1} (1+x)^{-p-q}\, dx,$$

and has, according to (5), the following representation in terms of confluent hypergeometric functions.

$$\phi(t) = \frac{\Gamma(p+q)}{\Gamma(q)} U(p, 1-q, -it). \tag{8}$$

Now we have from (6)

$$\frac{1}{i^n} \frac{d^n \phi(t)}{dt^n} = \frac{\Gamma(p+q)}{\Gamma(q)} (p)_n U(p+n, 1-q+n, -it).$$

Using (7), this gives as $t \to 0$ for the n^{th} moment about zero

$$\mu'_n = \begin{cases} \dfrac{(p)_n}{(q-1)^{(n)}} & \text{if } n < q, \\ \infty & \text{if } n \geqslant q, \end{cases} \tag{9}$$

where $(q-1)^{(n)} = \dfrac{\Gamma(q)}{\Gamma(q-n)} = (q-1)(q-2)\dots(q-n)$.

As a special case the characteristic function and the moments of the Pareto distribution can be obtained by putting $p=1$ and introducing the scale factor b.

Using the relation

$$U(1, 2-\nu, z) = e^z E_\nu(z),$$

(see e.g. MAGNUS et al., 1966, p. 338), where $E_\nu(z)$ is the generalized exponential integral, it is easely seen that in this case our formula (8) specializes to the expression (3) of SEAL (1980).

Finally, let us remark that for q not an integer, formula (8) can be rewritten, by means of (4), in the following form

$$\phi(t) = M(p, 1-q, -it) + |t|^q e^{-i\frac{\pi}{2}q} \frac{\Gamma(-q)}{B(p, q)} M(p+q, 1+q, -it). \tag{10}$$

Comparing with Seal's Appendix 1, we see that there only the first term of the characteristic function, namely $M(p, 1-q, -it)$, was considered, which clears up the contradiction.

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