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The classification of triangulated subcategories

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1. Introduction

The first main result of this paper is a bijective correspondence between the strictly full triangulated subcategories dense in a given triangulated category and the subgroups of its Grothendieck group (Thm. 2.1). Since every strictly full triangulated subcategory is dense in a uniquely determined thick triangulated subcategory, this result refines any known classification of thick subcategories to a classification of all strictly full triangulated ones. For example, one can thus refine the famous classification of the thick subcategories of the finite stable homotopy category given by the work of Devinatz–Hopkins–Smith ([Ho], [DHS], [HS] Thm. 7, [Ra] 3.4.3), which is responsible for most of the recent advances in stable homotopy theory. One can likewise refine the analogous classification given by Hopkins and Neeman ([Ho] Sect. 4, [Ne] 1.5) of the thick subcategories of $D(R)_{\text{parf}}$, the chain homotopy category of bounded complexes of finitely generated projective R-modules, where R is a commutative noetherian ring.

The second main result is a generalization of this classification result of Hopkins and Neeman to schemes, and in particular to non-noetherian rings. Let X be a quasi-compact and quasi-separated scheme, e.g. any commutative ring or algebraic variety. Denote by $D(X)_{\text{parf}}$ the derived category of perfect complexes, the homotopy category of those complexes of sheaves of $\mathcal{O}_X$-modules which are locally quasi-isomorphic to a bounded complex of free $\mathcal{O}_X$-modules of finite type. I say a thick triangulated subcategory $\mathcal{A} \subseteq D(X)_{\text{parf}}$ is a $\otimes$-subcategory if for each object E in $D(X)_{\text{parf}}$ and each A in $\mathcal{A}$, the derived tensor product $E \otimes A$ is also in $\mathcal{A}$. This is a mild condition on $\mathcal{A}$. If X has an ample line bundle $\mathcal{L}$ (e.g. for X a classical, hence quasi-projective variety), it suffices by 3.11.1 below to check that $\mathcal{L}^{-1} \otimes \mathcal{A} \subseteq \mathcal{A}$. If $X = \text{Spec}(R)$ for R a commutative ring, all thick subcategories $\mathcal{A} \subseteq D(R)_{\text{parf}}$ are $\otimes$-subcategories. The second main result

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(Thm. 3.15) gives for X a quasi-compact and quasi-separated scheme a bijective correspondence between the thick triangulated $\otimes$-subcategories of $D(X)_{\text{parf}}$ and the subspaces $Y \subseteq X$ which are unions of closed subspaces $Y_\alpha \subseteq X$ with $X - Y_\alpha$ quasi-compact. To such a Y corresponds the thick $\otimes$-subcategory of those perfect complexes acyclic at each point of $X - Y$. For X noetherian, all subspaces are quasi-compact, and the remaining condition that Y be a union of subspaces closed in X is usually expressed by saying that 'Y is closed under specialization'. For a detailed comparison with the previous work of Hopkins and Neeman see 3.17 below.

Refining the second main result by the first, I obtain (Thm. 4.1) a classification of all strictly full triangulated $\otimes$-subcategories $\mathcal{A}$ of $D(X)_{\text{parf}}$: they correspond bijectively to data (Y, H) where $Y \subseteq X$ is a subspace as above and $H \subseteq K_0(X \text{ on } Y)$ is a $K_0(X)$-submodule of the Grothendieck group of perfect complexes acyclic off Y. Thus $\mathcal{A}$ is determined by a condition Y on the supports and a condition H on the multiplicities. This classification has been my personal motivation for developing the results of this paper. I seek to define a good intersection ring of 'algebraic cycles' on schemes X where the classical construction of the Chow ring fails, for example on singular algebraic varieties or on regular schemes flat and of finite type over $\mathbb{Z}$. Inspired by the superiority of Cartier divisors over Weil divisors and by recent progress in local intersection theory, I believe the good notion of 'algebraic n-cycle' is that of those perfect complexes in some triangulated subcategory $\mathcal{A}^n \subseteq D(X)_{\text{parf}}$ which remains to be defined. Technological secrets about 'moving lemmas' demand that $\mathcal{A}^n$ should be a $\otimes$-subcategory, and show it cannot be thick in general. The classification has proved to be very helpful here in clarifying the issues to be resolved.

Other results presented here worth mentioning are the Tensor Nilpotence Theorem (Thms. 3.6 and 3.8), generalizing to schemes a result of Hopkins and Neeman for noetherian rings ([Ho] Thm. 10, [Ne] 1.1) analogous to the Nilpotence Theorem of Devinatz–Hopkins–Smith ([DHS] Thm. 1) in stable homotopy, and a useful but little-known necessary and sufficient condition for the equality of two classes in the Grothendieck group of a triangulated category (Lemma 2.4), due to Landsburg inspired by Heller.

I have tried to make each theorem, proposition, and lemma resistant to misinterpretation on being 'zapped' out of its context by endlessly restating, or at least referencing, all hypotheses and definitions explicitly. However, I have followed a customary global convention that 'ring' always means 'commutative ring'.

2. 'Rappels'

For the convenience of the reader and to eliminate ambiguities, I will briefly recall some basic definitions and results of the theory of triangulated categories. Useful general references are the classic treatments [Ha] I and II, and [Ve], as well as

[SGA4] XVII Section 1.2, [SGA5] VIII, [SGA6], and the more recent [BBD] Section 1, [Ri], [Sp], and [BN].

1.1. Let $\mathcal{T}$ be a triangulated category. A *full triangulated subcategory* of $\mathcal{T}$ is a full subcategory $\mathcal{A}$ with the structure of a triangulated category such that the inclusion functor $i: \mathcal{A} \to \mathcal{T}$ is a triangulated (a.k.a. 'exact') functor, i.e. such that it preserves distinguished ('exact') triangles and commutes with the suspension ('translation') endofunctors. Since the axioms of a triangulated category imply that any morphism is an edge of an exact triangle which is unique up to isomorphism of the opposite vertex ([Ha] I Sect. 1 TR1, TR3, Prop. 1.1.c, or [Ve] I Sect. 1 no. 1), and since the functor i is fully faithful so two triangles in $\mathcal{A}$ are isomorphic in $\mathcal{T}$ if and only if they are isomorphic in $\mathcal{A}$, a triangle in $\mathcal{A}$ is exact if and only if its image is exact in $\mathcal{T}$. Thus the triangulated category structure of $\mathcal{A}$ is uniquely determined by that on $\mathcal{T}$.

Thus the definition above is equivalent to: a full triangulated subcategory of $\mathcal{T}$ is a full non-empty subcategory $\mathcal{A}$ of $\mathcal{T}$ such that for every exact triangle of $\mathcal{T}$ of which two vertices are in $\mathcal{A}$, then the third vertex is isomorphic to an object of $\mathcal{A}$ (c.f. [Ve] I Sect. 1 no. 2–3).

1.2. A *strictly full triangulated subcategory* $\mathcal{A}$ of $\mathcal{T}$ is a full triangulated subcategory such that $\mathcal{A}$ contains every object of $\mathcal{T}$ which is isomorphic to an object of $\mathcal{A}$.

1.3. A *thick* (a.k.a *épaisse*) *triangulated subcategory* $\mathcal{A}$ of $\mathcal{T}$ is a strictly full triangulated subcategory such that every direct summand in $\mathcal{T}$ of an object of $\mathcal{A}$ is itself an object of $\mathcal{A}$. Rickard ([Ri] Prop. 1.3) showed that this definition is equivalent to the somewhat more complicated classic definition given by Verdier ([Ve] I Sect. 2 1–1). Using his definition Verdier showed that a full subcategory $\mathcal{A}$ of $\mathcal{T}$ is a thick triangulated subcategory if and only if there exists a triangulated category $\mathcal{T}'$ and a triangulated functor $\mathcal{T} \to \mathcal{T}'$ such that $\mathcal{A}$ is the full subcategory of those objects whose image in $\mathcal{T}'$ is isomorphic to 0. (c.f. [Ve] I Sect. 2 nos. $1, 2, 3$).

1.4. A full triangulated subcategory $\mathcal{A}$ of a triangulated category $\mathcal{D}$ is *dense* in $\mathcal{D}$ if each object of $\mathcal{D}$ is a direct summand of an object isomorphic to an object in $\mathcal{A}$.

REMARK 1.5. If $\mathcal{A}$ is a full triangulated subcategory of a triangulated category $\mathcal{T}$, the intersection of all the thick triangulated subcategories of $\mathcal{T}$ containing $\mathcal{A}$ is a thick triangulated subcategory $\mathcal{D}$. This $\mathcal{D}$ is the smallest thick triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{A}$. On the other hand, one checks easily that the strictly full subcategory $\widetilde{\mathcal{A}}$ of all objects of $\mathcal{T}$ which are summands of objects isomorphic to objects in $\mathcal{A}$ is a thick triangulated subcategory of $\mathcal{T}$. Thus $\mathcal{D} \subseteq \widetilde{\mathcal{A}}$. But by Rickard's definition of thick one has $\widetilde{\mathcal{A}} \subseteq \mathcal{D}$ since $\mathcal{D}$ is thick and contains $\mathcal{A}$. Thus $\mathcal{D} = \widetilde{\mathcal{A}}$.

Hence $\mathcal{A}$ is dense in a thick triangulated subcategory $\mathcal{D}$ of $\mathcal{T}$ if and only if $\mathcal{D}$ is the smallest thick triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{A}$.

1.6. Let $\mathcal{T}$ be a triangulated category, which we suppose is essentially small (1.7) and so replace with an equivalent triangulated category which has a set of objects. The *Grothendieck group* $K_0(\mathcal{T})$ is the quotient group of the free abelian group on the set of isomorphism classes of objects of $\mathcal{T}$ by the *Euler relations*: $[B] = [A] + [C]$ whenever there is an exact triangle in $\mathcal{T}$

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \nwarrow \qquad \swarrow & \\ & C & \end{array} \tag{1.6.1}$$

See [SGA6] IV Section 1, [SGA5] VIII Section 2. The Grothendieck group has the universal mapping property that any function from the set of isomorphism classes of objects of $\mathcal{T}$ to an abelian group G such that the Euler relations hold in G factors through a unique homomorphism $K_0(\mathcal{T}) \to G$. $K_0(\ )$ is covariant for triangulated functors. One has $[A] + [B] = [A \oplus B]$ from the exact triangle $A \to A \oplus B \to B \rightsquigarrow \cdots$. Then $[A] + [0] = [A]$, which implies $[0] = 0$ in the Grothendieck group. Note that if ΣA is the suspension of A in $\mathcal{T}$, so there exists an exact triangle $A \to 0 \to \Sigma A \rightsquigarrow \cdots$, one has $[A] + [\Sigma A] = [0] = 0$, so $[\Sigma A] = -[A]$. From all this it follows that *every* element of $K_0(\mathcal{T})$ is of the form $[C]$ for some object C of the triangulated category $\mathcal{T}$. The analogous statement for the Grothendieck group of an abelian or exact category would not be true; the $[C]$'s only generate the group in these cases.

1.7. A category $\mathcal{T}$ is *essentially small* if it is equivalent to a small category, i.e. if there exists a set of objects of $\mathcal{T}$ (as opposed to a class of objects in the sense of Gödel–Bernays set theory) such that every object of $\mathcal{T}$ is isomorphic to an object in this set.

Note that if $\mathcal{T}$ is an essentially small triangulated category then any full subcategory, any localization, and in particular any Verdier quotient by a thick triangulated subcategory ([Ve] I Sect. 2 no. 3) is essentially small. The stable homotopy category of finite CW spectra is essentially small. For any quasi-compact and quasi-separated scheme X (e.g. for any noetherian scheme) the triangulated category of perfect complexes (3.1) on $X, D(X)_{\text{parf}}$, is essentially small ([TT] Appendix F). If $\mathcal{A}$ is an essentially small abelian category then the derived category $D(\mathcal{A})$ is also essentially small. However, neither the category of all abelian groups, nor its derived category $D(\mathbb{Z}\text{-mod})$ is essentially small: there are too many non-isomorphic abelian groups. Indeed these exist in every cardinality.

3. Classification of dense strictly full triangulated subcategories

As in 1.5, each strictly full triangulated subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is dense in a uniquely determined thick subcategory $\mathcal{D}$ of $\mathcal{T}$. Thus to classify all such $\mathcal{A}$ in $\mathcal{T}$ it suffices to classify: (1) the thick subcategories $\mathcal{D}$ in $\mathcal{T}$, and (2) the strictly full dense triangulated subcategories $\mathcal{A}$ of $\mathcal{D}$. The second part of the classification is given by:

THEOREM 2.1 *Let $\mathcal{D}$ be an essentially small* (1.7) *triangulated category. Then there is a one-to-one correspondence between the strictly full dense* (1.2, 1.4) *triangulated subcategories $\mathcal{A}$ in $\mathcal{D}$ and the subgroups H of the Grothendieck group $K_0(\mathcal{D})$.*

To $\mathcal{A}$ corresponds the subgroup which is the image of $K_0(\mathcal{A})$ in $K_0(\mathcal{D})$. To H corresponds the full subcategory $\mathcal{A}_H$ whose objects are those A in $\mathcal{D}$ such that $[A] \in H \subseteq K_0(\mathcal{D})$.

Proof. One checks that these formulas define functions between the set of strictly full dense triangulated subcategories and the set of subgroups: Trivially im $K_0(\mathcal{A})$ is a subgroup of $K_0(\mathcal{D})$. The Euler relation in $K_0(\mathcal{D})$ gives readily that $\mathcal{A}_H$ is a strictly full triangulated subcategory of $\mathcal{D}$. And $\mathcal{A}_H$ is dense in $\mathcal{D}$ since for all $D \in \mathcal{D}$ one has $D \oplus \Sigma D \in \mathcal{A}_H$ as $[D \oplus \Sigma D] = [D] + [\Sigma D] = [D] - [D] = 0 \in H$.

It remains to be checked that the two functions are inverse to each other. For H a subgroup, clearly im $K_0(\mathcal{A}_H) \subseteq H$. But as in 1.6, any element of $H \subseteq K_0(\mathcal{D})$ is of the form $[D]$ for some $D \in \mathcal{D}$, and then $D \in \mathcal{A}_H$ since $[D] \in H$. Thus H im $K_0(\mathcal{A}_H)$, and the composition of the two functions gives the identity on the set of subgroups. To see that the reverse composition gives the identity on the set of subcategories, completing the proof of the theorem, one must check that for each $D \in \mathcal{D}$ one has $D \in \mathcal{A}$ if and only if $[D] \in$ im $K_0(\mathcal{A}) \subseteq \mathcal{K}_0(\mathcal{D})$. But this is given by the following lemma.

LEMMA 2.2 *Let $\mathcal{A}$ be a strictly full dense triangulated subcategory of the essentially small triangulated category $\mathcal{D}$. Then for any object D of $\mathcal{D}$, one has that $D \in \mathcal{A}$ if and only if $[D] = 0$ in $K_0(\mathcal{D})/\text{im } K_0(\mathcal{A})$.*

Proof. Passing to an equivalent triangulated category, I may assume that $\mathcal{D}$ has a set of objects. Consider the relation $\sim$ on the set of isomorphism classes of objects of $\mathcal{D}$ defined by $D \sim D'$ iff there exist A and A' in $\mathcal{A}$ such that there is an isomorphism $D \oplus A \cong D' \oplus A'$. One checks easily that the relation $\sim$ is an equivalence relation. Denote by G the quotient by $\sim$ of the set of isomorphism classes of objects. Denote by $\langle D \rangle$ the class in G of the object D of $\mathcal{D}$.

I claim $D \in \mathcal{A}$ iff $\langle D \rangle \sim \langle 0 \rangle$ in G. For clearly $D \in \mathcal{A}$ implies $\langle D \rangle \sim \langle 0 \rangle$. Conversely if $\langle D \rangle \sim \langle 0 \rangle$, there are $A, A' \in \mathcal{A}$ and an isomorphism $D \oplus A \cong 0 \oplus A' = A'$. Then $D \oplus A \in \mathcal{A}$, and as two of the three vertices of the exact triangle $A \to D \oplus A \to D \rightsquigarrow \Sigma A$ are in the strictly full triangulated subcategory $\mathcal{A}$, at the third vertex one has also $D \in \mathcal{A}$. This proves the claim.

To complete the proof of the lemma, it remains to show that the set G is bijective with the group $K_0(\mathcal{D})/\mathrm{im}\, K_0(\mathcal{A})$ via $\langle D\rangle \leftrightarrow [D]$.

But G has a structure of an abelian monoid, with sum induced by the operation of direct sum in $\mathcal{D}$. $\langle 0\rangle$ is the zero. In fact, G has inverses and so is an abelian group. For given any element $\langle D\rangle \in G$, as $\mathcal{A}$ is dense in $\mathcal{D}$ there is a $D' \in \mathcal{D}$ such that $D \oplus D' \in \mathcal{A}$. Then $\langle D\rangle + \langle D'\rangle = \langle D \oplus D'\rangle \sim \langle 0\rangle = 0$.

Further the Euler relation holds in G, so that the surjection $\mathrm{Obj}\,\mathcal{D} \twoheadrightarrow G$ induces a surjection of abelian groups $K_0(\mathcal{D}) \twoheadrightarrow G$ sending $[D]$ to $\langle D\rangle$. For let $A \to B \to C \rightsquigarrow \Sigma A$ be an exact triangle in $\mathcal{D}$. As above, there are objects A', C' in $\mathcal{D}$ such that $A \oplus A'$, $C \oplus C'$ are in $\mathcal{A}$ so $\langle A \oplus A'\rangle = 0 = \langle C \oplus C'\rangle$. Taking the direct sum of the exact triangle $A \to B \to C \rightsquigarrow \Sigma A$ with the exact triangles $A' \to A' \to 0 \rightsquigarrow \Sigma A'$ and $0 \to C' \to C' \rightsquigarrow \Sigma 0$ gives an exact triangle $A \oplus A' \to B \oplus A' \oplus C' \to C \oplus C' \rightsquigarrow \Sigma(A \oplus A')$. The vertices $A \oplus A'$ and $C \oplus C'$ are in the strictly full triangulated subcategory $\mathcal{A}$, and hence so is the third vertex $B \oplus A' \oplus C'$. Thus in G one has $0 = \langle B \oplus A' \oplus C'\rangle = \langle B\rangle + \langle A'\rangle + \langle C'\rangle = \langle B\rangle - \langle A\rangle - \langle C\rangle$, proving the Euler relation holds there.

Thus one has a surjective homomorphism $K_0(\mathcal{D}) \twoheadrightarrow G$. Since each element of $K_0(\mathcal{D})$ is of the form $[D]$ for some $D \in \mathcal{D}$ by 1.6, the kernel of this surjection consists of all $[D]$ such that $\langle D\rangle \sim 0$, i.e. such that $D \in \mathcal{A}$. So the kernel is $\mathrm{im}\, K_0(\mathcal{A})$ and $G \cong K_0(\mathcal{D})/\mathrm{im}\, K_0(\mathcal{A})$. □

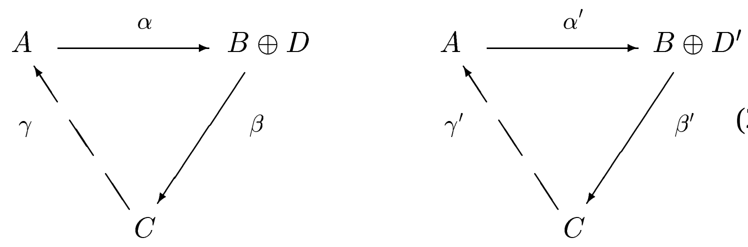
COROLLARY 2.3 *Let $\mathcal{A}$ be a full dense triangulated subcategory of the essentially small triangulated category $\mathcal{D}$. Then the homomorphism induced on $K_0(\ )$ by the inclusion of categories is a monomorphism of groups $K_0(\mathcal{A}) \rightarrowtail K_0(\mathcal{D})$.*

Proof. Let $\widetilde{\mathcal{A}}$ be the strictly full triangulated subcategory of $\mathcal{D}$ whose objects are those objects of $\mathcal{D}$ isomorphic to objects in $\mathcal{A}$. The inclusion $\mathcal{A} \subseteq \widetilde{\mathcal{A}}$ is an equivalence of triangulated categories and so induces an isomorphism $K_0(\mathcal{A}) \cong K_0(\widetilde{\mathcal{A}})$. Hence I may replace $\mathcal{A}$ by $\widetilde{\mathcal{A}}$ and so may assume $\mathcal{A}$ is a strictly full dense triangulated subcategory of $\mathcal{D}$.

Let $N = \ker(K_0(\mathcal{A}) \to K_0(\mathcal{D}))$. By Theorem 2.1 the subgroups 0 and N of $K_0(\mathcal{A})$ correspond to strictly full dense triangulated subcategories $\mathcal{Z}$ and $\mathcal{N}$ of $\mathcal{A}$. But then $\mathcal{Z}$ and $\mathcal{N}$ are also strictly full dense triangulated subcategories of $\mathcal{D}$. But $\mathrm{im}\, K_0(\mathcal{Z}) = 0 = \mathrm{im}\, N = \mathrm{im}\, K_0(\mathcal{N})$ in $K_0(\mathcal{D})$. Then by Theorem 2.1, $\mathcal{Z} = \mathcal{N} \subseteq \mathcal{D}$, whence $0 = K_0(\mathcal{Z}) = K_0 = (\mathcal{N})N$. □

The reader may find the indirectness of the proof of this useful corollary psychologically uncomfortable. If so, rather than dosing himself with a benzodiazepine, he may find relief in deducing Corollary 2.3 from the following criterion for equality of two classes in $K_0(\mathcal{D})$, whose proof is very similar to that of Lemma 2.2. This criterion is due to Landsburg ([La]), inspired by an analog for K_0 of exact categories due to Heller ([He] 2.1).

LEMMA 2.4. (Landsburg) *Let $\mathcal{D}$ be an essentially small triangulated category,* *and let D, D' be two objects of $\mathcal{D}$. Then $[D] = [D']$ in $K_0(\mathcal{D})$ if and only if there are objects $A, B, C \in \mathcal{D}$ and two exact triangles*



(2.4.1)

Proof. First I replace $\mathcal{D}$ by an equivalent triangulated category with a set of objects. Now let $\sim$ be the relation on the set of objects defined by $D \sim D'$ if there exist A, B, C and two exact triangles as in (2.4.1). The relation $\sim$ is an equivalence relation: symmetry and reflexivity are clear; and as for transitivity, note that if $D \sim D'$ because of exact triangles $A_1 \xrightarrow{\alpha_1} B_1 \oplus D \xrightarrow{\beta_1} C_1 \rightsquigarrow$ and $A_1 \xrightarrow{\alpha_1'} B_1 \oplus D' \xrightarrow{\beta_1'} C_1 \rightsquigarrow$, while $D' \sim D''$ because of the exact triangles $A_2 \xrightarrow{\alpha_2} B_2 \oplus D' \xrightarrow{\beta_2} C_2 \rightsquigarrow$ and $A_2 \xrightarrow{\alpha_2'} B_2 \oplus D'' \xrightarrow{\beta_2'} C_2 \rightsquigarrow$, then $D \sim D''$ because of the exact triangles $A_1 \oplus A_2 \xrightarrow{\alpha_1 \oplus \alpha_2} B_1 \oplus D \oplus B_2 \oplus D' \xrightarrow{\beta_1 \oplus \beta_2} C_1 \oplus C_2 \rightsquigarrow$ and $A_1 \oplus A_2 \xrightarrow{\alpha_1' \oplus \alpha_2'} B_1 \oplus D' \oplus B_2 \oplus D'' \xrightarrow{\beta_1' \oplus \beta_2'} C_1 \oplus C_2 \rightsquigarrow$, conjugated by the isomorphisms $(B_1 \oplus B_2 \oplus D') \oplus D \cong B_1 \oplus D \oplus B_2 \oplus D'$ and $(B_1 \oplus B_2 \oplus D') \oplus D'' \cong B_1 \oplus D' \oplus B_2 \oplus D''$. Let G be the quotient of the set of objects of $\mathcal{D}$ by the equivalence relation $\sim$. It remains to show that $G \cong K_0(\mathcal{D})$ via $\langle D \rangle \leftrightarrow [D]$.

G has a structure of an abelian monoid, the sum being induced by direct sum in $\mathcal{D}$. In fact, G is an abelian group. For given an element $\langle D \rangle$ of G, it has an inverse $\langle \Sigma D \rangle$ since $\langle D \oplus \Sigma D \rangle \sim \langle 0 \rangle = 0$ because of the exact triangles $D \to D \oplus \Sigma D \to \Sigma D \rightsquigarrow$ and $D \to 0 \to \Sigma D \rightsquigarrow$. The Euler relation holds in G. For if $A \to D \to C \rightsquigarrow$ is an exact triangle, considering it alongside the exact triangle $A \to A \oplus C \to C \rightsquigarrow$ gives $\langle D \rangle \sim \langle A \oplus C \rangle = \langle A \rangle + \langle C \rangle$.

Thus by the universal property of the Grothendieck group one has a homomorphism, clearly surjective, $K_0(\mathcal{D}) \twoheadrightarrow G$ sending $[D]$ to $\langle D \rangle$. This surjection is injective, and so is an isomorphism of abelian groups. For given two elements in $K_0(\mathcal{D})$, as in 1.6 they have the form $[D]$, $[D']$ for objects D, D' of $\mathcal{D}$. If they go to the same element of G, then $D \sim D'$ and there exist two exact triangles (2.4.1). Then in $K_0(\mathcal{D})$ the Euler relations give $[B \oplus D] = [A] + [C] = [B \oplus D']$, so $[B] + [D] = [B] + [D']$, and $[D] = [D']$ in this group. □

EXAMPLE 2.5 The proof of Lemma 2.2, the backbone of Theorem 2.1, was already implicit in the proof ([TT] 5.5.4) of a key special case of the K_0-criterion of Thomason–Trobaugh for the extension of perfect complexes (3.1) on an open subscheme U to perfect complexes on the ambient scheme X. As an example and since it will be needed in Section 3, I state this criterion:

EXTENSION LEMMA. *Let X be a quasi-compact and quasi-separated scheme, $Y \subseteq X$ a closed subspace such that $X - Y$ is quasi-compact, and $U \subseteq X$ a quasi-compact open subscheme. Denote by $K_0(X \,\mathrm{on}\, Y)$ the Grothendieck group of the triangulated category of perfect complexes on X acyclic off Y. Let $E^{\cdot}$ be a perfect complex on U acyclic off $U \cap Y$. Then there exists on X a perfect complex $F^{\cdot}$ acyclic off Y and such that the restriction $F^{\cdot}|U$ is quasi-isomorphic to $E^{\cdot}$, if and only if the class $[E^{\cdot}]$ in $K_0(U \,\mathrm{on}\, U \cap Y)$ is in the image of $K_0(X \,\mathrm{on}\, Y)$.*

Proof. This statement is [TT] 5.2.2, whose proof is spread out through [TT] Section 5. In essence, one reduces to the case $Y = X$ and where X has an ample family of line bundles by an inductive argument. In this case, a Koszul complex trick for extending to X morphisms defined on U between perfect complexes on X shows that the perfect complexes on U which extend are the objects of a strictly full triangulated subcategory. Trobaugh's revelation ([TT] 5.5.1) is that this subcategory is dense. Now Lemma 2.2 finishes the proof.

4. Classification of thick subcategories of perfect complexes

3.1. '*Rappels*': Let X be a quasi-compact and quasi-separated scheme. Recall that any classical algebraic variety, more generally any noetherian scheme, and the affine scheme Spec(R) for any commutative ring is quasi-compact and quasi-separated ([EGA] I).

A *strict perfect complex* on X is a bounded complex of locally free $\mathcal{O}_X$-modules of finite type ([SGA6] I 2.1, or [TT] 2.2.2). A *perfect complex* on X is a complex $E^{\cdot}$ of sheaves of $\mathcal{O}_X$-modules such that there is an open cover of X by U's such that $E^{\cdot}|U$ is quasi-isomorphic to a strict perfect complex on U ([SGA6] I 4.7, or [TT] 2.2.10).

One denotes by $D(X)$ the derived category of the abelian category of sheaves of $\mathcal{O}_X$-modules, and by $D(X)_{\mathrm{parf}}$ its strictly full subcategory whose objects are the perfect complexes on X. $D(X)_{\mathrm{parf}}$ is a thick triangulated subcategory of $D(X)$ by ([SGA6] I 4.9, 4.10, 4.17 or [TT] 2.2.13). It is contained in the thick subcategory $D^+(X)_{qc}$ of complexes cohomologically bounded below with quasi-coherent cohomology sheaves.

The objects of $D(X)_{\mathrm{parf}}$ are characterized by the following 'finite presentation' condition ([ThLG] Prop. 1.1, c.f. [TT] 2.4.1): an object $E^{\cdot}$ of $D^+(X)_{qc}$ is a perfect complex iff the functor $\mathrm{Mor}_{D(X)}(E^{\cdot})$ takes all direct sums in $D^+(X)_{qc}$, even those with infinitely many factors, to direct sums of abelian groups. Note there is a similar characterization of the homotopy finite CW spectra in the full stable homotopy category (e.g., [ThSH] 2.5, with the dual to 2.6). This fact is part of the parallel between the classification Theorem 3.15 below and the classification theorem [HS] Theorem 7 in the finite stable homotopy category.

If the scheme X has an ample family of line bundles ([SGA6] II 2.2.4 or [TT] 2.1.1); for example, if X is an affine scheme Spec(R), or is quasi-projective over a Spec(R), or is a separated regular noetherian scheme, ([TT] 2.1.2), then any perfect complex on X is globally quasi-isomorphic to a strict perfect complex ([SGA6] II 2.2.8, or [TT] 2.3.1). In particular $D(\text{Spec}(R))_{\text{parf}}$ is equivalent to the triangulated category obtained from the category of bounded chain complexes of finitely generated projective R-modules by inverting the quasi-isomorphisms. This is the category abusively denoted $D^b(R)$ in [Ne], in flagrant incompatibility with the standard use of this symbol as in [Ha], [Ve], and all the works of Grothendieck.

I refer to [Ha] II, [SGA6] I, and [Sp] for the standard notations and results on operations like the total derived tensor product $\otimes^L_{\mathcal{O}_X}$ and the total derived inverse image Lf^*. Note that generally in [Ha] and [SGA6], $\otimes^L_{\mathcal{O}_X}$ and Lf^* were defined only on $D^-(X)$, i.e. on cohomologically bounded above complexes, but they have been extended to all $D(X)$ by Spaltenstein ([Sp] 6.5, 6.7). See also [BN]. I will sometimes abbreviate $\otimes^L_{\mathcal{O}_X}$ and Lf^* simply as $\otimes$ and f^*, leaving the X understood and suppressing the 'L' when by the context these clearly refer to functors between derived categories rather than the inducing functors on categories of complexes.

Finally I note that I usually follow the customs of the tribe of algebraic geometers, for whom the differentials in a complex increase degree, ∂: $E^n \to E^{n+1}$, and who call $\ker\partial/\text{im}\,\partial$ cohomology. For topologists, differentials decrease degree and $\ker\partial/\text{im}\,\partial$ is homology. To translate between the languages of these antipodal peoples one reindexes the algebraic geometer's complex by setting $E^n = E_{-n}$, so reindexing $H^n(E^\cdot) = H_{-n}(E_\cdot)$. For example, I say a complex $E^\cdot$ is cohomologically bounded above if $H^n(E^\cdot) \cong 0$ for $n \gg 0$. Among the topologists one would say this complex $E_\cdot$ is homologically bounded below.

DEFINITION 3.2 Let X be a scheme and $E^\cdot$ a complex of sheaves of $\mathcal{O}_X$-modules. The *cohomological support* of $E^\cdot$ is the subspace $\text{Supph}(E^\cdot) \subseteq X$ of those points $x \in X$ at which the stalk complex of $\mathcal{O}_{X,x}$-modules $(E^\cdot_x)$ is not acyclic.

Thus $\text{Supph}(E^\cdot) \bigcup_{n\in\mathbb{Z}} \text{Supp}H^n(E^\cdot)$ is the union of the supports in the classic sense ([EGA] O_{I} 3.1.5) of the cohomology sheaves of $E^\cdot$.

LEMMA 3.3 *Let X be a quasi-compact and quasi-separated scheme. Let $E^\cdot$ be a perfect complex on X.*

(a) *For any $x \in X$, $E^\cdot_x$ is an acyclic complex of $\mathcal{O}_{X,x}$-modules if and only if $E^\cdot \otimes^L_{\mathcal{O}_X} k(x)$ is an acyclic complex of $k(x)$-modules.*
(b) *If Y is a quasi-compact and quasi-separated scheme and $f : Y \to X$ a morphism of schemes, then*

$$\text{Supph}(Lf^*E^\cdot) = f^{-1}(\text{Supph}(E^\cdot))$$

(c) $\text{Supph}(E^\cdot)$ *is closed in X, and $X - \text{Supph}(E^\cdot)$ is quasi-compact.*

Proof. (a) Consider the strongly converging Künneth spectral sequence (e.g. [EGA] III 6.3.2)

$$E^2_{p,q} = \text{Tor}^p_{\mathcal{O}_{X,x}}(H_q(E^\cdot_x), \quad k(x)) \Longrightarrow H_{p+q}(E^\cdot \otimes^L_{\mathcal{O}_X} k(x)).$$

This shows at once that if $E^{\cdot}_x$ is acyclic, i.e. if $H_*(E^{\cdot}_x) = 0$, then $E^{\cdot} \otimes^L_{\mathcal{O}_X} k(x)$ is acyclic. Conversely, suppose $E^{\cdot}_x$ is not acyclic. As the perfect $E^{\cdot}$ is cohomologically bounded there is a least N such that $H_N(E^{\cdot}_x) \neq 0$. Then in the spectral sequence $E^2_{p,q} = 0$ for $q < N$ as well as for $p < 0$. Then for $p+q = N$ the spectral sequence gives in the corner $p = 0$, $q = N$ an isomorphism $H_N(E^{\cdot}_x) \otimes_{\mathcal{O}_{X,x}} k(x) \cong H_N(E^{\cdot} \otimes^L_{\mathcal{O}_X} k(x))$. As $E^{\cdot}_x$ is perfect and $H_q(E^{\cdot}_x) = 0$ for $q < N$, $H_N(E^{\cdot}_x)$ is a finitely generated $\mathcal{O}_{X,x}$-module ([SGA6] I 2.10b, 5.8.1, or [TT] 2.2.3, 2.2.12). Then by Nakayama's lemma $H_N(E^{\cdot}_x) \neq 0$ implies $H_q(E^{\cdot}_x) \otimes_{\mathcal{O}_{X,x}} k(x) \neq 0$, so $E^{\cdot} \otimes^L_{\mathcal{O}_X} k(x)$ is not acyclic. This proves (a).

To prove (b), note by (a) that $y \in \mathrm{Supph}(Lf^*E^{\cdot})$ iff $(Lf^*E^{\cdot}) \otimes^L_{\mathcal{O}_Y} k(y)$ is not acyclic. But for $x = f(y) \in X$ one has $(Lf^*E^{\cdot}) \otimes^L_{\mathcal{O}_Y} k(y) \cong (E^{\cdot} \otimes^L_{\mathcal{O}_X} k(x)) \otimes^L_{k(x)} k(y)$. As $k(y)$ is an extension field of $k(x)$ it is faithfully flat over $k(x)$, and $(E^{\cdot} \otimes^L_{\mathcal{O}_X} k(x)) \otimes^L_{k(x)} k(y)$ is not acyclic iff $E^{\cdot} \otimes^L_{\mathcal{O}_X} k(x)$ is not acyclic, that is, by (a) again, iff $x \in \mathrm{Supph}(E^{\cdot})$. This proves (b).

It remains to prove (c). Suppose first that X is a noetherian scheme. The cohomology sheaves of the perfect complex $E^{\cdot}$ are coherent $\mathcal{O}_X$-modules ([SGA6] I 3.5, or [TT] 2.2.8, 2.2.12). Thus each $\mathrm{Supp}(H^n(E^{\cdot}))$ is closed in X ([EGA] I 6.8.5), and as $(E^{\cdot})$ is cohomologically bounded, for all but finitely many n one has $\mathrm{Supp}(H^n(E^{\cdot})) = \mathrm{Supp}\ 0 = \emptyset$. Thus $\mathrm{Supph}(E^{\cdot}) = \bigcup \mathrm{Supp}(H^n(E^{\cdot}))$ is closed in X. For X a noetherian scheme, any subspace, e.g. $X - \mathrm{Supph}(E^{\cdot})$ is quasi-compact ([EGA] I 2.7.1, [B-AC] II Sect. 4 no. 2 Prop. 9). This proves (c) for X noetherian.

Now suppose only that X is quasi-compact and quasi-separated. By absolute noetherian approximation ([TT] C.9, 3.20) there exist a noetherian scheme X', an affine map $g: X \to X'$, and a perfect complex $F^{\cdot}$ on X' such that $E^{\cdot} \cong Lg^*F^{\cdot}$. By (b), $\mathrm{Supph}(E^{\cdot}) = g^{-1}(\mathrm{Supph}(F^{\cdot})$, and so is closed in X as $\mathrm{Supph}(F^{\cdot})$ is closed in the noetherian X'. As the affine map g is a quasi-compact map ([EGA] I 6.1.1), $X - \mathrm{Supph}(E^{\cdot}) = g^{-1}(X' - \mathrm{Supph}(F^{\cdot}))$ is quasi-compact. This proves (c) and completes the proof of the lemma.

LEMMA 3.4 *Let X be a quasi-compact and quasi-separated scheme. Let $Y \subseteq X$ be a closed subspace such that $X - Y$ is quasi-compact. Then there exists a perfect complex $E^{\cdot}$ on X such that* $\mathrm{Supph}(E^{\cdot})Y$.

Proof. By absolute noetherian approximation ([TT] C.9, [EGA] IV 8.3.11) there exist a noetherian scheme X' of finite type over $\mathrm{Spec}(\mathbb{Z})$, a closed subspace $Y' \subseteq X'$, and an affine map $f: X \to X'$ such that $Y = f^{-1}(Y')$. If there is a perfect complex $E'^{\cdot}$ on X' such that $Y' = \mathrm{Supph}(E'^{\cdot})$, then $E^{\cdot} = Lf^*E'^{\cdot}$ will be a perfect complex on X with $Y = \mathrm{Supph}(E^{\cdot})$ by 3.3b. Thus it suffices to prove the result when X is a noetherian scheme.

For X noetherian, the closed Y has finitely many irreducible components, with generic points $y_1, \ldots, y_k$ ([B-AC] II Sect. 4 no. 2, [EGA] I 2.1.5, 2.7). If there are perfect complexes $E^{\cdot}_i$ on X such that $\mathrm{Supph}(E^{\cdot}_i) = \overline{y_i}$, then $E^{\cdot} = \oplus^k_{i=1} E^{\cdot}_i$ is perfect with $\mathrm{Supph}(E^\cdot) = \bigcup_{i=1}^{k} \overline{y_i} = Y$. Thus it suffices to prove the case where $Y = \overline{y}$ is irreducible closed with generic point y.

Then let $U = \mathrm{Spec}(A)$ be an affine open neighborhood of y in X. Let $\{f_1, \dots, f_n\}$ be a finite set of generators for the ideal in the noetherian ring A corresponding to the reduced closed subscheme $Y \cap U \subseteq \mathrm{Spec}(A)$ ([EGA] I 4.6.1, 4.2). Consider the Koszul complex $K^\cdot = K^\cdot(f_1, \dots, f_n) = \otimes_{i1}^{n}(A \xrightarrow{f_i} A)$, the tensor product of chain complexes $(A \xrightarrow{f_i} A)$ which are 0 except in degrees -1 and 0, and have A in those two degrees between which the differential is given by multiplication by f_i. $K^\cdot(f_1, \dots, f_n)$ is a strict perfect complex on $\mathrm{Spec}(A)$. On one hand $\mathrm{Supph}(K^\cdot(f_1, \dots, f_n)) \subseteq (f_1 = 0) \cap \cdots \cap (f_n = 0) = \mathrm{Supp}(A/(f_1, \dots, f_n)) = Y \cap U$. On the other hand, $\mathrm{Supph}(K^\cdot(f_1, \dots, f_n)) \supseteq \mathrm{Supp}(H^0(K(f_1, \dots, f_n))) = \mathrm{Supp}(A/(f_1, \dots, f_n))Y \cap U$. So $\mathrm{Supph}(K^\cdot) = Y \cap U$.

Let $\Sigma K^\cdot$ be the suspension of the Koszul complex $K^\cdot$, so $\Sigma K^\cdot$ is also perfect with $\mathrm{Supph}(\Sigma K^\cdot) = \mathrm{Supph}(K^\cdot) = Y \cap U$. The sum $K^\cdot \oplus \Sigma K^\cdot$ is perfect with cohomological support $Y \cap U$. In $K_0(U \text{ on } Y \cap U)$ one has $[K^\cdot \oplus \Sigma K^\cdot] = [K^\cdot] + [\Sigma K^\cdot] = [K^\cdot] - [K^\cdot] = 0$. By the K_0-extension criterion of Thomason–Trobaugh (2.5, or [TT] 5.2.2) there is a perfect complex $E^\cdot$ on X with $\mathrm{Supph}(E^\cdot) \subseteq Y$ and such that $E^\cdot|U \simeq K^\cdot \oplus \Sigma K^\cdot$. Then $y \in U \cap \mathrm{Supph}(E^\cdot)$, giving $\overline{y} = Y \subset \mathrm{Supph}(E^\cdot)$ since $\mathrm{Supph}(E^\cdot)$ is closed (3.3). Thus $Y = \mathrm{Supph}(E^\cdot)$. □

LEMMA 3.5 (Mayer-Vietoris). *Let X be a scheme, $i: U \to X$ and $j: V \to X$ two open immersions, and denote by $k: U \cap V \to X$ the open immersion of the intersection. Then for $E^\cdot, F^\cdot \in D(U \cup V)$ there is a natural long exact sequence:*

$$\begin{array}{c}
\downarrow \\
\mathrm{Mor}_{D(U\cap V)}(\Sigma k^* E^\cdot, k^* F^\cdot) \\
\downarrow \delta \\
\mathrm{Mor}_{D(U\cup V)}(E^\cdot, F^\cdot) \\
\downarrow \\
\mathrm{Mor}_{D(U)}(i^* E^\cdot, i^* F^\cdot) \oplus \mathrm{Mor}_{D(V)}(j^* E^\cdot, j^* F^\cdot) \\
\downarrow \\
\mathrm{Mor}_{D(U\cap V)}(k^* E^\cdot, k^* F^\cdot) \\
\downarrow \delta \\
\mathrm{Mor}_{D(U\cup V)}(\Sigma^{-1} E^\cdot, F^\cdot) \\
\downarrow
\end{array} \tag{3.5.1}$$

(Note by a standard abuse one has replaced X by $U \cup V$ as the target of the immersions i, j, k.)

Proof. For $l\colon P \to X$ an open immersion let $i_!\colon \mathcal{O}_P\text{-Mod} \to \mathcal{O}_X\text{-Mod}$ be the functor 'extension by 0' which sends a sheaf of $\mathcal{O}_P$-modules $\mathcal{F}$ to the sheaf of $\mathcal{O}_X$-modules given by, for $W \subseteq X$ open

$$(l_!\mathcal{F})(W) \begin{cases} \mathcal{F}(W) & \text{if } W \subseteq P \\ 0 & \text{if } W \not\subseteq P \end{cases}.$$

Extend $l_!$ to complexes by applying $l_!$ in each degree of the complex. The adjunction between the functors $l_!$ and l^* on sheaves induces an adjunction $l_! \dashv l^*$ on complexes. As $l_!$ and l^* are exact functors on sheaves, these functors preserve quasi-isomorphisms of complexes, so on passing to the derived category one still has an adjoint pair, and in particular an adjunction isomorphism

$$\mathrm{Mor}_{D(X)}(l_!l^*E^{\cdot}, F^{\cdot}) \cong \mathrm{Mor}_{D(P)}(l^*E^{\cdot}, l^*F^{\cdot}). \tag{3.5.2}$$

Consider the sequence of complexes on $U \cup V$, where the morphisms are induced by $\pm$ the various adjunction maps $\varepsilon\colon l_!l^* \to 1$ or by the $l'_!\varepsilon l'^*$ for the immersions between $U \cap V$, U, V, and $U \cup V$

$$0 \to k_!k^*E^{\cdot} \to i_!i^*E^{\cdot} \oplus j_!j^*E^{\cdot} \to E^{\cdot} \to 0. \tag{3.5.3}$$

This sequence is exact on $U \cup V$, being locally split exact on U and on V since the relevant adjunction maps restrict to natural isomorphisms $k_!k^*E^{\cdot}|U \cong j_!j^*E^{\cdot}|U$, $i_!i^*E^{\cdot}|U \cong E^{\cdot}|U$, etc. Thus the sequence gives an exact triangle in $D(U \cup V)$. Applying to this exact triangle the contravariant functor $\mathrm{Mor}_{D(U\cup V)}(\quad, F^{\cdot})$ yields a canonical long exact 'Puppe' sequence ([Ha] I 6.1, or [Ve] I Sect. 1 1–2), which conjugated by the isomorphisms (3.5.2) for $l = i, j, k$ yields (3.5.1). □

THEOREM 3.6 (Tensor nilpotence). *Let X be a quasi-compact and quasi-separated scheme. Let $E^{\cdot}$ be a perfect complex on X, and $F^{\cdot}$ a complex of sheaves of $\mathcal{O}_X$-modules which has quasi-coherent cohomology (i.e., $F^{\cdot} \in D(X)_{qc}$). Let $f\colon E^{\cdot} \to F^{\cdot}$ be a morphism in $D(X)$.*

Suppose for all $x \in X$ that $f \otimes^L_{\mathcal{O}_X} k(x) = 0$ in $D(k(x))$. Then there exists a positive integer n such that $\otimes^n f\colon \otimes^n_{\mathcal{O}_X} E^{\cdot} \to \otimes^n_{\mathcal{O}_X} F^{\cdot}$ is 0 in $D(X)$.

Proof. (c.f. [Ho] Thm. 10, [Ne] 1.1) Recall that $\otimes^n F^{\cdot} \equiv {}^L\otimes^n_{\mathcal{O}_X} F^{\cdot}$ exists for any $F^{\cdot} \in D(X)$, not just for $F^{\cdot} \in D^-(X)$, by [Sp] 6.5, 5.9.

3.6.1. I claim the conclusion '$\exists n$ such that $\otimes^n f \simeq 0$' is a local question on X. For suppose X is covered by opens U_α such that for each of these opens $\exists n_\alpha$ with $\otimes^{n_\alpha} f|U_\alpha = 0$ in $D(U_\alpha)$. Passing to a refinement of this cover, I may assume the U_α are affine and hence quasi-compact. As X is quasi-compact, there is a finite subcover $X = \bigcup_{i=1}^k U_i$. I will show by induction on k that there is an n such that $\otimes^n f = 0$ in $D(X)$. To start the induction at $k = 1$, note then that $X = U_1$ so $\otimes^{n_1} f = 0$. To do the induction step assuming the result for $k-1$, set $V = \bigcup_{i=2}^k U_i$. By the induction hypothesis, $\exists n'$ such that $\otimes^{n'} f = 0$ in $D(V)$. Set

$m = \max\{n_1, n'\}$ so $\otimes^m f|U_1 = 0$ in $D(U_1)$ and $\otimes^m f|V = 0$ in $D(V)$, as well as $\otimes^m f|U_1 \cap V = 0|U_1 \cap V = 0$ in $D(U_1 \cap V)$. Since $X = U_1 \cup V$, the Mayer–Vietoris sequence (3.5.1) then implies that $\otimes^m f \in \mathrm{Mor}_{D(X)}(\otimes^m E^{\cdot}, \otimes^m F^{\cdot})$ is δ of some morphism $t \in \mathrm{Mor}_{D(U_1\cap V)}(\Sigma k^* \otimes^m E^{\cdot}, k^* \otimes^m F^{\cdot})$. That is to say, on unfolding the proof of 3.5.1, that $\otimes^m f \colon \otimes^m E^{\cdot} \to \otimes^m F^{\cdot}$ factors in $D(X)$ as

$$\begin{array}{ccc} \bigotimes^m E^{\cdot} & \xrightarrow{\otimes^m f} & \bigotimes^m F^{\cdot} \\ {\scriptstyle \delta(\otimes^m E^{\cdot})}\big\downarrow & & \big\uparrow{\scriptstyle \varepsilon(\otimes^m F^{\cdot})}, \\ k_!\Sigma k^* \bigotimes^m E^{\cdot} & \xrightarrow[k_!(t)]{} & k_! k^* \bigotimes^m F^{\cdot} \end{array} \tag{3.6.1.1}$$

where $\delta \colon \mathrm{Id} \to \Sigma k_! k^* \cong k_! \Sigma k^*$ is the composite of the obvious isomorphism and the natural third edge of the exact triangle induced by the exact sequence (3.5.3), and where $\epsilon \colon k_! k^* \to \mathrm{Id}$ is the adjunction map. But then for $n = 2m$, $\otimes^n f \cong (\otimes^m f) \otimes (\otimes^m f) = 0$ in $D(X)$, as it has an induced factorization through $(\otimes^m f) \otimes (k_! t)$, which identifies under natural isomorphisms to $k_!((k^* \otimes^m f) \otimes t)$ and hence to 0 since $k^* \otimes^m f = 0$ in $D(U_1 \cap V)$. This proves the induction step, and hence the claim.

3.6.2. The desired conclusion $\exists n \otimes^n f = 0$ being local, and the hypotheses over X implying the hypotheses over any quasi-compact open subscheme, to prove the theorem I may and do restrict to the affine case, $X = \mathrm{Spec}(R)$. Then I may (3.1) assume $E^{\cdot}$ is a strict perfect complex. Let $E^{\cdot\vee}$ be the dual complex $E^{\cdot\vee}\mathcal{H}om(E^{\cdot}, \mathcal{O}_X) \simeq R\mathcal{H}om(E^{\cdot}, \mathcal{O}_X)$. Then $E^{\cdot\vee} \otimes_{\mathcal{O}_X} F^{\cdot} \simeq E^{\cdot\vee} \otimes^L_{\mathcal{O}_X} F^{\cdot}$ is a complex with quasi-coherent cohomology. There is a natural isomorphism

$$\begin{aligned} \mathrm{Mor}_{D(X)}(E^{\cdot}, F^{\cdot}) &\cong \mathrm{Mor}_{D(X)}(\mathcal{O}_X, R\mathcal{H}om(E^{\cdot}, F^{\cdot})) \\ &\cong \mathrm{Mor}_{\mathcal{D}(X)}(\mathcal{O}_X, E^{\cdot\vee} \otimes F^{\cdot}), \end{aligned} \tag{3.6.2.1}$$

induced by the adjunction $E^{\cdot} \otimes^L (\quad) \dashv R\mathcal{H}om(E^{\cdot}, \quad)$ and the natural isomorphism $R\mathcal{H}om(E^{\cdot}, F^{\cdot}) \simeq R\mathcal{H}om(E^{\cdot}, \mathcal{O}_X) \otimes^L F^{\cdot} \simeq E^{\cdot\vee} \otimes F^{\cdot}$ for $E^{\cdot}$ strict perfect ([SGA6] I 7.4, 7.7, boosted by [TT] 2.4.1.a,b and [Ha] I Sect. 7). Under the isomorphism (3.6.2.1) $f \colon E^{\cdot} \to F^{\cdot}$ corresponds to a $f^{\flat} \colon \mathcal{O}_X \to E^{\cdot\vee} \otimes F^{\cdot}$ in $D(X)_{qc}$. Under other instances of this natural isomorphism $\otimes^n f$ corresponds to $\otimes^n f^{\flat} \colon \mathcal{O}_X \cong \otimes^n \mathcal{O}_X \to \otimes^n (E^{\cdot\vee} \otimes F^{\cdot})$ and for all $x \in X$, $f \otimes k(x)$ corresponds to $f^{\flat} \otimes k(x)$. Thus replacing $F^{\cdot}$ by $E^{\cdot\vee} \otimes F^{\cdot}$ and $E^{\cdot}$ by $\mathcal{O}_X$, I reduce to the case $E^{\cdot}\mathcal{O}_X$.

By [BN] 5.1 (c.f. [SGA6] II Sect. 3 or [TT] B.16, B.17 for $F^{\cdot} \in D^+(X)_{qc}$) $F^{\cdot}$ is quasi-isomorphic to a complex of quasi-coherent $\mathcal{O}_X$-modules, so I may assume $F^{\cdot}$ is such a complex. Under the equivalence of the category of quasi-coherent $\mathcal{O}_X$-modules on $X = \mathrm{Spec}(R)$ and the category of R-modules, $F^{\cdot}$ is identified to a complex of R-modules, and $f: \mathcal{O}_X \to F^\cdot$ is identified to a map $f: R \to F^\cdot$ in $D(R\text{-Mod})$, i.e. to a class $f \in H^0(F^\cdot)$.

Thus it remains to show that:

3.6.2.2. If $F^\cdot$ is a complex of R-modules, and $f \in H^0(F^\cdot)$ is such that for all $x \in \mathrm{Spec}(R)\, f \otimes k(x) = 0$ in $H^0(F^\cdot \otimes k(x))$, then there is a positive integer n such that $\otimes^n f = 0$ in $H^0(\otimes^n F^\cdot)$.

3.6.3. I next reduce to the case where $F^\cdot$ is strict perfect. For $F^\cdot \in D(R\text{-Mod})$ is quasi-isomorphic to a complex of free R-modules, which in turn is the direct colimit of its strict perfect subcomplexes ([TT] 2.3.2). So I may assume that $F^\cdot \varinjlim F^\cdot_\alpha$ for $\{F^\cdot_\alpha\}$ a directed system of strict perfect complexes. For such a filtering system $H^0(F^\cdot) \cong \varinjlim H^0(F^\cdot_\alpha)$, so $f \in H^0(F^\cdot)$ is the image of a $f_\beta \in H^0(F^\cdot_\beta)$ for some β. Passing to the cofinal directed subsystem of $\alpha \succ \beta$ and setting f_α equal to the image of f_β, I may assume that there is a family of $f_\alpha \in H^0(F^\cdot_\alpha)$ compatible under the structure maps of the system and inducing f in the colimit $H^0(F^\cdot)$. Note that now all $F^\cdot_\alpha$ and so also $F^\cdot$ are complexes of flat R-modules, whence the derived tensor product over R, $F^\cdot \otimes^L k(x)$, is represented by the tensor product of chain complexes $F^\cdot \otimes k(x)$.

Set $T_\alpha = \{x \in \mathrm{Spec}(R) \mid f_\alpha \otimes k(x) = 0 \text{ in } H^0(F^\cdot_\alpha \otimes k(x))\}$ Note $T_\alpha \subseteq T_{\alpha'}$ if $\alpha \prec \alpha'$. Also $\bigcup T_\alpha = \{x \in \mathrm{Spec}(R) \mid f \otimes k(x) = 0 \text{ in } H^0(F^\cdot \otimes k(x))\} = \mathrm{Spec}(R)$. I claim that each T_α is a constructible set in $\mathrm{Spec}(R)$. For, representing the homology class $f_\alpha \in H^0(F^\cdot_\alpha)$ by an element $\tilde{f}_\alpha \in F^0_\alpha$, one has $f_\alpha \otimes k(x) = 0$ in $H^0(F^\cdot_\alpha \otimes k(x))$ iff $\tilde{f}_\alpha \otimes k(x)$ is a boundary in $F^\cdot_\alpha \otimes k(x)$, i.e. iff $0 = \tilde{f}_\alpha \otimes k(x): R \to (F^0_\alpha / \partial F^{-1}_\alpha) \otimes k(x)$. As $F^0_\alpha / \partial F^{-1}_\alpha$ is a finitely presented R-module, by [EGA] IV 9.4.5 on setting $X = S = \mathrm{Spec}(R)$ one gets that T_α is a locally constructible subset of $\mathrm{Spec}(R)$. As $\mathrm{Spec}(R)$ is quasi-compact and separated, T_α is then constructible ([EGA] 0_{III} 9.1.12), proving the claim. But then the two remarks preceding the claim imply that there exists an α such that $T_\alpha = \mathrm{Spec}(R)$ by [EGA] IV 1.9.9, 1.9.4. For this α, $f_\alpha \in H^0(F^\cdot_\alpha)$ satisfies $f_\alpha \otimes k(x) = 0$ for all $x \in \mathrm{Spec}(R)$. If for the strict perfect $F^\cdot_\alpha$ this implies that for some n $\otimes^n f_\alpha = 0$ in $H^0(\otimes^n F^\cdot_\alpha)$, one gets $\otimes^n f = 0$ in $H^0(\otimes^n F^\cdot)$ by taking the image under $F^\cdot_\alpha \to F^\cdot$. This completes the reduction to proving (3.6.2.2) for $F^\cdot$ strict perfect over the ring R.

3.6.4. Next I will reduce to the case where R is a noetherian ring of finite Krull dimension. Any commutative ring R is the direct colimit $\varinjlim R_\alpha$ of its subrings of finite type over $\mathbb{Z}$. These R_α are noetherian rings of finite dimension ([EGA] IV 5.5.4, 0_{IV} 14.1.4). As the strict perfect complex $F^\cdot$ consists of a finite number of nonzero direct summands of finitely generated free modules and finitely many maps ∂ between them, the projections on the summands and the maps between them being represented by finitely many matrices with values in R, there exists a β and a strict perfect complex $F^\cdot_\beta$ over R_β such that $F^\cdot \cong R \otimes_{R_\beta} F^\cdot_\beta$. Passing to the cofinal system of $\alpha \succ \beta$, I get a family $F^{\cdot}_{\alpha} = R_{\alpha} \otimes_{R_{\beta}} F^{\cdot}_{\beta}$ of strict perfect complexes over the R_{α}, compatible under the structure maps of the system and such that $F^{\cdot} \cong \varinjlim F^{\cdot}_{\alpha}$. Then as $H^0(F^{\cdot}) \cong \varinjlim H^0(F^{\cdot}_{\alpha})$, there is a β' such that $f \in H^0(F^{\cdot})$ is the image of a $f_{\beta'} \in H^0(F^{\cdot}_{\beta'})$. Restricting to the cofinal system of $\alpha \succ \beta'$ and setting f_{α} to be the image of $f_{\beta'}$, I may assume I have a compatible family of $f_{\alpha} \in H^0(F^{\cdot}_{\alpha})$. Note as $F^{\cdot}_{\alpha}$ is strict perfect, the tensor product of complexes $F^{\cdot}_{\alpha} \otimes (\quad)$ represents the total derived functor $F^{\cdot}_{\alpha} \otimes^{L}_{R_{\alpha}} (\quad)$.

Let $T_{\alpha} \subseteq \operatorname{Spec}(R_{\alpha})$ be the subset of those $x \in \operatorname{Spec}(R_{\alpha})$ such that $f_{\alpha} \otimes k(x) = 0$ in $H^0(F^{\cdot}_{\alpha} \otimes k(x))$. As in 3.6.3 above, T_{α} is constructible. Denoting by $\pi_{\gamma\alpha}\colon \operatorname{Spec}(R_{\gamma}) \to \operatorname{Spec}(R_{\alpha})$ the structure maps of the system and by $\pi_{\alpha}\colon \operatorname{Spec}(R) = \varprojlim \operatorname{Spec}(R_{\gamma}) \to \operatorname{Spec}(R_{\alpha})$ the projections from the inverse limit of schemes, one clearly has $\pi^{-1}_{\gamma\alpha}(T_{\alpha}) \subseteq T_{\gamma}$ and $\bigcup \pi^{-1}_{\alpha}(T_{\alpha}) = \{x \mid f \otimes k(x) = 0 \text{ in } H^0(F^{\cdot} \otimes k(x))\} = \operatorname{Spec}(R)$. Then by [EGA] IV 8.3.4 there is an α such that $T_{\alpha} = \operatorname{Spec}(R_{\alpha})$; i.e., such that for all $x \in \operatorname{Spec}(R_{\alpha})$, $f_{\alpha} \otimes k(x) = 0$. If $\otimes^n f_{\alpha}$ in $H^0(\otimes^n F^{\cdot}_{\alpha})$ for the noetherian R_{α}, then $\otimes^n f = \pi^*_{\alpha}(\otimes^n f_{\alpha}) = 0$ in $H^0(\otimes^n F^{\cdot})$. This completes the reduction to the case where R is noetherian of finite dimension.

3.6.5. Now it remains to show: *if R is a noetherian ring of finite Krull dimension, $F^{\cdot}$ a bounded complex of finitely generated projective R-modules, and $f \in H^0(F^{\cdot})$ is such that $\forall x \in \operatorname{Spec}(R)\, f \otimes k(x) = 0$ in $H^0(F^{\cdot} \otimes k(x))$, then $\exists n > 0$ such that $\otimes^n f = 0$ in $H^0(\otimes^n F^{\cdot})$.* To prove this, following the strategy of [Ne], I will induct on the dimension of R.

3.6.6. To prepare the induction, I claim that if $N \subseteq R$ is the ideal of all nilpotent elements of R, and if the statement 3.6.5 holds for R/N, then it holds for R. For then $\exists n_1 > 0$ such that $\otimes^{n_1} f = 0$ in $H^0((\otimes^{n_1} F^{\cdot}) \otimes_R R/N)$. That is, on choosing a representative of $\otimes^{n_1} f$ in $(\otimes^{n_1} F^{\cdot})^0$, $\exists x \in (\otimes^{n_1} F^{\cdot})^{-1}$ and $\exists y = \sum_1^k r_i f_i \in N(\otimes^{n_1} F^{\cdot})^0$ with $r_i \in N \subseteq R$, $f_i \in (\otimes^{n_1} F^{\cdot})^0$ such that $\otimes^{n_1} f = \partial x + y$. As each $r_i \in N$ is nilpotent, $\exists n_2 > 0$ such that for the finitely generated ideal $(r_1, \ldots, r_k) \subseteq N$, $(r_1, \ldots, r_k)^{n_2} = 0$. Then $\otimes^{n_2} y \in (r_1, \ldots r_k)^{n_2} (\otimes^{n_2} \otimes^{n_1} F^{\cdot})^0 = 0$. The element $\otimes^{n_2}(\otimes^{n_1} f) = \otimes^{n_2}(\partial x + y)$ is a sum of terms $\otimes^{n_2} y = 0$ and of terms of the form $a \otimes \partial x$, $b \otimes \partial x$ and $a \otimes \partial x \otimes b$ where a and b are products of ∂x and y. As $\partial y = \partial(\otimes^{n_1} f - \partial x) = \partial \otimes^{n_1} f - \partial\partial x = 0$, in any case $\partial a = \partial b = 0$ and $a \otimes \partial x \otimes b = \partial(a \otimes x \otimes b)$. Thus $\otimes^{n_1+n_2} f \in \partial(\otimes^{n_1+n_2} F)^{-1}$, so $\otimes^{n_1+n_2} f = 0$ in $H^0(\otimes^{n_1+n_2} F^{\cdot})$. This proves the claim.

Note $\dim R = \dim R/N$ as $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R/N) = \operatorname{Spec}(R)_{\mathrm{red}}$ have homeomorphic underlying spaces ([EGA] I 4.5, IV 5.1).

3.6.7. Now I finish the proof of 3.6.5 with the inductive argument. To start the induction with the case $\dim R = 0$, I may by the claim 3.6.6 replace R by the reduced noetherian ring of dimension 0 which is R/N. But thereafter R is the product of its finitely many residue fields $k(x)$ (e.g., [B-AC] VII Sect. 1 no. 3

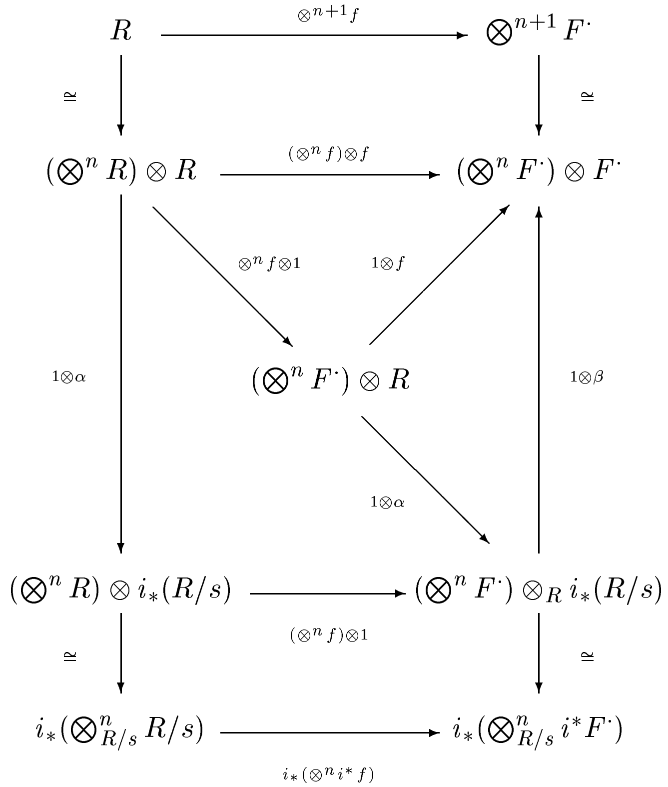
Exemple 1, IV Sect. 2 no. 5 Prop. 9, II Sect. 3 no. 5 Prop. 16). Thus $H^0(F^\cdot) \cong \bigoplus_x H^0(F^\cdot \otimes k(x))$ and as $f \otimes k(x) = 0$ for all x, one has $f = 0$.

To do the induction step, suppose $\dim R = d > 0$ and that 3.6.5 is already demonstrated for noetherian rings of dimension $\leq d-1$. Again replacing R by its quotient R/N by the nilradical, I may suppose R is reduced. Let $R \to \prod_{i=1}^m k(\eta_i)$ be the map of R to the product of the residue fields at the finitely many minimal primes of R, $\eta_i \in \mathrm{Spec}(R)$. As R is reduced, each such residue field $k(\eta_i)$ is in fact the local ring R_{η_i}. Indeed $\prod_{i=1}^m k(\eta_i)$ is isomorphic to the localization $S^{-1}R$ for S the set of non zero-divisors of R, those elements contained in no minimal prime ([B-AC] II Sect. 4 no. 3 Cor. 3 à Prop. 14, IV Sect. 2 no. 5 Prop. 10). By hypothesis, for each $\eta_i f \otimes k(\eta_i) = 0$, so $f \otimes S^{-1}R = 0$ in $H^0(F^\cdot \otimes_R S^{-1}R = S^{-1}F^\cdot)$. If $f \in F^0$ is a representative of the class $f \in H^0(F^\cdot)$, this means there exists a $y' \in S^{-1}F^{-1}$ such that $\partial y' = f$ in $S^{-1}F^0$. Then for some $s \in S$ there exists $y \in F^{-1}$ such that $sf = \partial y$ in F^0. The elements $y\colon R \to F^{-1}$ and $f\colon R \to F^0$ determine a map of complexes $\beta\colon (R \xrightarrow{s} R) \to F^\cdot$, where $(R \xrightarrow{s} R)$ is the complex with R in degrees 0 and -1, 0 in the other degrees, and where the non-trivial differential is given by s. As $s \in S$ is a non zero-divisor, this complex of R-modules is quasi-isomorphic to the complex which is R/sR concentrated in degree 0. Again as s is a non zero-divisor the closed immersion $i\colon \mathrm{Spec}(R/s) \to \mathrm{Spec}(R)$ is a regular closed immersion and the direct image functor $i_*\colon D(R/s)_{qc} \to D(R)_{qc}$ preserves perfect complexes and so restricts to a triangulated functor $i_*\colon D(R/s)_{\mathrm{parf}} \to D(R)_{\mathrm{parf}}$ ([SGA6] VII 1.4, 1.9, III 4.8.1, 2.5). Let $\alpha\colon R \to i_* R/s = R/s$ be the canonical quotient map. Replacing $\beta\colon (R \xrightarrow{s} R) \to F^\cdot$ by its composite with the isomorphism in $D(R)_{\mathrm{parf}}$ $i_*(R/s) \simeq (R \xrightarrow{s} R)$ whose composite with α is the inclusion of R into degree 0 of $(R \xrightarrow{s} R)$, I get a factorization in $D(R)_{\mathrm{parf}}$ of $f = R \xrightarrow{\alpha} i_*(R/s) \xrightarrow{\beta} F^\cdot$. This factorization and the naturality of various standard quasi-isomorphisms gives the following commutative diagram in $D(R)_{\mathrm{parf}}$ for all $n \geqslant 0$

This diagram gives a factorization of $\otimes_R^{n+1} f$ through $i_*(\otimes_{R/s}^n i^* f)$. But R/s is a noetherian ring, and as s is a non zero-divisor one has $\dim(R/s) \leqslant \dim(R) - 1 \leqslant d - 1$ ([EGA] 0_{IV} 16.1.2.2). Then by the induction hypothesis that 3.6.5 is known for dimensions $\leqslant d-1$ and since $i^* f \otimes_{R/s} k(x) = f \otimes_R k(x) = 0$ in $D(k(x))$ for all $x \in \mathrm{Spec}(R/s) \subseteq \mathrm{Spec}(R)$, one gets that there exists an $n > 0$ such that $\otimes_{R/s}^n i^* f = 0$ in $D(R/s)_{\mathrm{parf}}$. Then the factorization gives $\otimes_R^{n+1} f = 0$ in $D(R)_{\mathrm{parf}}$. This completes the proof of the induction step for 3.6.5, and hence the proof of the theorem.

REMARK 3.7. For k a field the derived category $D(k)$ is equivalent (by the functor sending a complex to its cohomology groups) to the category of $\mathbb{Z}$-graded k-vector spaces. Thus the Theorem 3.6 says that given a map in $D(X)_{qc}$, $f\colon E^\cdot \to F^\cdot$, with $E^\cdot$ perfect and such that for all $x \in X H^*(f \otimes^L k(x))$ is the zero map, then there exists an integer $n > 0$ such that $\otimes^n f = 0$ in $D(X)$. To the more sophisticated this is a surprisingly strong conclusion, and in any case the naive attempts to strengthen it fail.

(3.6.7.1)



For example, it is not sufficient to suppose that $H^*(f) = 0$ rather than $H^*(f \otimes k(x)) = 0$ for all x. For let $X = \operatorname{Spec}(\mathbb{Z}_{(p)})$, and let the complexes $E^{\cdot}$, $F^{\cdot}$ be given by the rows in the diagram below, with f given by the columns and where the center column is in degree 0

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_{(p)} & \xrightarrow{p} & \mathbb{Z}_{(p)} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_{(p)} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array} \tag{3.7.1}$$

Then $H^*(f) = 0$, but for all $n > 0$ $H^0((\otimes^n f) \otimes \mathbb{Z}/p)$ is an isomorphism $\mathbb{Z}/p \cong \mathbb{Z}/p$.

Nor can one strengthen the conclusion to $f = 0$ in $D(X)$ rather than $\exists n \otimes^n f = 0$, even if one supposes $H^*(f) = 0$ in addition to $f \otimes k(x) = 0$ for all x. For again with $X = \operatorname{Spec}(\mathbb{Z}_{(p)})$ consider the map of complexes give by

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_{(p)} & \xrightarrow{p^2} & \mathbb{Z}_{(p)} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & {\scriptstyle p}\downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_{(p)} & \xrightarrow{p^2} & \mathbb{Z}_{(p)} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array} \tag{3.7.2}$$

Here $f \neq 0$ in $D(\mathbb{Z}_{(p)})_{\text{parf}}$ as $H^0(f \otimes \mathbb{Z}/p^2) = p\colon \mathbb{Z}/p^2 \to \mathbb{Z}/p^2$, although $f \otimes \mathbb{Z}/p$ and $f \otimes \mathbb{Q}$ are 0.

In both counterexamples $X = \operatorname{Spec}(\mathbb{Z}_{(p)})$ is about as nice a scheme as one could imagine except for fields and $\emptyset$, and the counterexamples generalize to $\operatorname{Spec}(\mathbb{Z})$ and to $\operatorname{Spec}(k[T])$. Having entered this far into the question, one should abandon all hope.

THEOREM 3.8 (Tensor nilpotence with parameters). *Let X be a quasi-compact and quasi-separated scheme, $E^{\cdot}$ and $G^{\cdot}$ perfect complexes on X, and $F^{\cdot}$ a complex of sheaves of $\mathcal{O}_X$-modules with quasi-coherent cohomology. Let $f\colon E^{\cdot} \to F^{\cdot}$ be a morphism in $D(X)_{qc}$. Suppose for all $x \in$ Supph$(G^{\cdot})$ that $f \otimes k(x) = 0$ in $D(k(x))$. Then there is an integer $n > 0$ such that $G^{\cdot} \otimes (\otimes^n f) = 0$ as a morphism $G^{\cdot} \otimes (\otimes^n E^{\cdot}) \to G^{\cdot} \otimes (\otimes^n F^{\cdot})$ in $D(X)$.*

Proof. (c.f. [Ho] Thm. 10ii) For each $x \in X (G^{\cdot} \otimes f) \otimes k(x) = 0$ in $D(k(x))$ as either $G^{\cdot} \otimes k(x) \simeq 0$ or else $x \in \operatorname{Supph}(G^{\cdot})$ and $f \otimes k(x) = 0$. So by Tensor Nilpotence 3.6 there is an $n > 0$ such that $\otimes^n(G^{\cdot} \otimes f) = 0$ in $D(X)$. A fortiori $(\otimes^{n-1} R\mathcal{H}om(G^{\cdot}, \mathcal{O}_X)) \otimes (\otimes^n G^{\cdot}) \otimes (\otimes^n f) = 0$. I will show that $G^{\cdot} \otimes (\otimes^n f)$ is a retract of this morphism and so is also 0.

For this, it suffices to show by induction on n that $G^{\cdot}$ is a direct summand in $D(X)_{\text{parf}}$ of $(\otimes^{n-1} R\mathcal{H}om(G^{\cdot}, \mathcal{O}_X)) \otimes (\otimes^n G^{\cdot})$. For $n = 1$ this is trivial. To do the induction step to prove it for $n \geqslant 2$ it suffices to prove that $(\otimes^{n-2} R\mathcal{H}om(G^{\cdot}, \mathcal{O}_X)) \otimes (\otimes^{n-1} G^{\cdot})$ is a direct summand of $(\otimes^{n-1} R\mathcal{H}om(G^{\cdot}, \mathcal{O}_X)) \otimes (\otimes^n G^{\cdot})$. For $n \geq 3$ this follows upon tensoring with $(\otimes^{n-3} R\mathcal{H}om(G^{\cdot}, \mathcal{O}_X)) \otimes (\otimes^{n-2} G^{\cdot})$ from the case $n = 2$, namely that $G^{\cdot}$ is a direct summand of $R\mathcal{H}om(G^{\cdot}, \mathcal{O}_X) \otimes (\otimes^2 G^{\cdot})$. Hence it suffices to prove the latter.

As $G^{\cdot}$ is perfect, there is in $D(X)$ a natural isomorphism (easy, or [SGA6] I 7.7)

$$\begin{aligned} \theta\colon R\mathcal{H}om(G^{\cdot}, \mathcal{O}_X) \otimes^L G^{\cdot} &\xrightarrow{\sim} R\mathcal{H}om(G^{\cdot}, \mathcal{O}_X) \otimes^L \\ R\mathcal{H}om(\mathcal{O}_X, G^{\cdot}) &\xrightarrow{\sim} R\mathcal{H}om(G^{\cdot}, G^{\cdot}) \end{aligned} \tag{3.8.1}$$

Thus the problem is to show $G^{\cdot}$ is a direct summand of $R\mathcal{H}om(G^{\cdot}, G^{\cdot}) \otimes G^{\cdot}$. But if $\eta\colon G^{\cdot} \to R\mathcal{H}om(G^{\cdot}, G^{\cdot}) \otimes G^{\cdot}$ is $(\quad) \otimes G^{\cdot}$ of the morphism $\mathcal{O}_X \to R\mathcal{H}om(G^{\cdot}, G^{\cdot})$ corresponding to $1_{G^{\cdot}}\colon G^{\cdot} \to G^{\cdot}$, and if eval: $R\mathcal{H}om(G^{\cdot}, G^{\cdot}) \otimes$

$G^{\cdot} \to G^{\cdot}$ is the evaluation map, then $\mathrm{eval} \circ \eta = 1_{G^{\cdot}}$. So $G^{\cdot}$ splits off of $R\mathcal{H}om(G^{\cdot}, G^{\cdot}) \otimes G^{\cdot}$. □

DEFINITION 3.9 Let $\mathcal{T}$ be a triangulated category with a fixed functor $\otimes\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ which is a covariant triangulated functor in each variable.

A *full triangulated left-⊗-subcategory* $\mathcal{A}$ of $\mathcal{T}$ is a full triangulated subcategory of $\mathcal{T}$ such that for all objects $T \in \mathcal{T}$ and $A \in \mathcal{A}$, $T \otimes A$ is also an object in the subcategory $\mathcal{A}$. Similarly a *full triangulated right-⊗-subcategory* has $A \otimes T \in \mathcal{A}$ whenever $A \in \mathcal{A}$ and $T \in \mathcal{T}$.

A full triangulated left-⊗-subcategory is said to be respectively *strictly full*, *thick*, or *dense* if it is such as a triangulated subcategory (1.2–1.4).

The condition of being a *strictly* full left-⊗-subcategory is invariant under replacing $\otimes$ by any naturally isomorphic functor $\otimes'$.

When $\otimes$ is commutative up to isomorphism, the concepts of strictly full triangulated left-⊗-subcategory and of strictly full triangulated right-⊗-subcategory are equivalent, and one says simply strictly full triangulated ⊗-subcategory. I emphasize that despite what this terminology suggests, the condition of $\mathcal{A}$ being a ⊗-subcategory is stronger than merely requiring that $\otimes$ on $\mathcal{T}$ restricts to a functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$; it says $\mathcal{A}$ is a sort of 'ideal' in the 'ring' $\mathcal{T}$ with $\otimes$ as 'multiplication'. (However, use of the term 'ideal' for such an $\mathcal{A}$ as in 3.9.1 with Y a closed subscheme would lead immediately to a bad terminological singularity at 'the ideal associated to Y'.)

For X a quasi-compact and quasi-separated scheme and $\mathcal{T} = D(X)_{\mathrm{parf}}$, henceforth I consider strictly full triangulated ⊗-subcategories where by default $\otimes = \otimes^{L}_{\mathcal{O}_X}$ is the usual derived tensor product.

EXAMPLE 3.9.1 If X is a quasi-compact and quasi-separated scheme and $Y \subseteq X$ is any subspace, the full subcategory of $D(X)_{\mathrm{parf}}$ consisting of those perfect complexes $E^{\cdot}$ such that $\mathrm{Supph}(E^{\cdot}) \subseteq Y$ is a thick triangulated ⊗-subcategory of $D(X)_{\mathrm{parf}}$.

REMARKS 3.10 (a) The intersection of any set of full (resp. strictly full, resp. thick) triangulated left-⊗-subcategories of $\mathcal{T}$ is again a full (resp. strictly full, resp. thick) triangulated left-⊗-subcategory of $\mathcal{T}$. Thus each subcategory of $\mathcal{T}$ has a smallest full (resp. strictly full, resp. thick) left-⊗-subcategory containing it.

(b) Let $\mathcal{A}$ be a full triangulated left-⊗-subcategory of $\mathcal{T}$. By 1.5, the smallest thick triangulated subcategory $\widetilde{\mathcal{A}}$ of $\mathcal{T}$ containing $\mathcal{A}$ has as objects all direct summands in $\mathcal{T}$ of objects isomorphic to objects in $\mathcal{A}$. So an easy check shows $\widetilde{\mathcal{A}}$ is in fact a thick triangulated left-⊗-subcategory of $\mathcal{T}$, and is also the smallest such containing $\mathcal{A}$.

PROPOSITION 3.11 (⊗-subcategory test). *Let X be a quasi-compact and quasi-*

separated scheme with an ample family of line bundles $\{\mathcal{L}_\alpha\}$ ([SGA6] II 2.2.4). *Suppose* $\mathcal{A} \subseteq \mathcal{D}(X)_{\text{parf}}$ *is a thick triangulated subcategory, and that for each* $\mathcal{L}_\alpha$*, a line bundle in the given ample family, and for each* $A^\cdot$ *in* $\mathcal{A}$ *one has that* $\mathcal{L}_\alpha^{-1} \otimes A^\cdot$ *is in* $\mathcal{A}$*. Then* $\mathcal{A}$ *is a thick triangulated* $\otimes$*-subcategory of* $D(X)_{\text{parf}}$.

COROLLARY 3.11.1 (a) *If* $X = \text{Spec}(R)$ *is an affine scheme, all thick triangulated subcategories of* $D(X)_{\text{parf}}$ *are thick triangulated* $\otimes$*-subcategories.*

(b) *If* X *is a quasi-compact and quasi-separated scheme with an ample line bundle* $\mathcal{O}_X(1)$ ([EGA] II 4.5), *e.g. any scheme quasi-projective over some ring, and if* $\mathcal{A} \subseteq D(X)_{\text{parf}}$ *is a thick triangulated subcategory such that* $\mathcal{O}_X(-1) \otimes \mathcal{A} \subseteq \mathcal{A}$*, then* $\mathcal{A}$ *is a* $\otimes$*-subcategory.*

(c) *Let* X *be as in* 3.11 *with an ample family of line bundles, and let* $\mathcal{A} \subseteq D(X)_{\text{parf}}$ *be a thick triangulated subcategory which is locally determined. (That is, one supposes that if* $E^\cdot \in D(X)_{\text{parf}}$ *and if there exists an open cover* $\{U_\beta\}$ *of* X *such that for each* U_β *there is an* $F^\cdot_\beta \in \mathcal{A}$ *and an isomorphism* $E^\cdot|U_\beta \cong F^\cdot_\beta|U_\beta$ *in* $D(U_\beta)_{\text{parf}}$*, then* $E^\cdot \in \mathcal{A}$*.) Then* $\mathcal{A}$ *is a* $\otimes$*-subcategory.*

Proof of Cor. (c) Suppose $A^\cdot \in \mathcal{A}$. As any line bundle $\mathcal{L}^{-1}$ is locally isomorphic to $\mathcal{O}_X$, $\mathcal{L}^{-1} \otimes A^\cdot$ is locally isomorphic to $A^\cdot$. As $\mathcal{A}$ is locally determined, this gives $\mathcal{L}^{-1} \otimes A^\cdot \in \mathcal{A}$ for any line bundle $\mathcal{L}$ and then Proposition 3.11 yields the result.

(b) If $\mathcal{O}_X(1)$ is an ample line bundle the set $\{\mathcal{O}_X(1)\}$ consisting of $\mathcal{O}_X(1)$ alone is an ample family, and (b) gives the result.

(a) If X is affine, $\mathcal{O}_X$ is an ample line bundle. As $\mathcal{O}_X \otimes A^\cdot \cong A^\cdot$ 3.11 applies to give the result.

Proof of Proposition. Let $\mathcal{B} \subseteq D(X)_{\text{parf}}$ be the full subcategory of those $B^\cdot$ such that for each $A^\cdot \in \mathcal{A}$ one has $B^\cdot \otimes A^\cdot \in \mathcal{A}$. As $\mathcal{A}$ is a strictly full triangulated subcategory of $D(X)_{\text{parf}}$, $\mathcal{B}$ is also a strictly full triangulated subcategory. As $\mathcal{A}$ is thick, $\mathcal{B}$ is also closed under the taking of direct summands, i.e. a thick subcategory of $D(X)_{\text{parf}}$. The proposition asserts that $\mathcal{B}D(X)_{\text{parf}}$. By hypothesis, for all $\mathcal{L}_\alpha$ in the ample family one has $\mathcal{L}_\alpha^{-1} \otimes \mathcal{A} \subseteq \mathcal{A}$. Then by induction on $k \geqslant 1$ one gets $\mathcal{L}_\alpha^{-k} \otimes \mathcal{A} \subseteq \mathcal{A}$, i.e. $\mathcal{L}_\alpha^{-k} \in \mathcal{B}$. The proof of the proposition is thus completed by the following Lemma 3.12.

LEMMA 3.12 *Let* X *be a quasi-compact and quasi-separated scheme with an ample family of line bundles* $\{\mathcal{L}_\alpha\}$*. Let* $\mathcal{B} \subseteq D(X)_{\text{parf}}$ *be a thick triangulated subcategory such that for all integers* $k > 0$ *and all* α, $\mathcal{L}_\alpha^{-k} \in \mathcal{B}$*. Then* $\mathcal{B}D(X)_{\text{parf}}$.

Proof. For any complex of sheaves of $\mathcal{O}_X$-modules $B^\cdot$ quasi-isomorphic to a bounded complex which in each degree is a finite direct sum of $\mathcal{L}_\alpha^{-k}$ for various α and $k > 0$, one has $B^\cdot \in \mathcal{B}$. One sees this by an easy induction on the total number of factors $\mathcal{L}_\alpha^{-k}$ in the direct sums.

By the global resolution of perfect complexes on schemes with an ample family of line bundles ([SGA6] II 2.2.8, or as a porism to [TT] 2.3.1), each perfect complex $E^\cdot$ on X is quasi-isomorphic to a bounded above complex $B^\cdot$ which in each degree is a finite direct sum of various $\mathcal{L}_\alpha^{-k}$. For each integer n denote by $\sigma^{\geqslant n} B^{\cdot}$ the brutal truncation of $B^{\cdot}$, the subcomplex given by

$$(\sigma^{\geqslant n} B^{\cdot})^i \begin{cases} B^i & \text{if } i \geqslant n \\ 0 & \text{if } i < n \end{cases} . \tag{3.12.1}$$

Then $\sigma^{\geqslant n} B^{\cdot}$ is bounded and by the preceding paragraph $\sigma^{\geqslant n} B^{\cdot} \in \mathcal{B}$. I claim that for all n sufficiently less than 0 that $E^{\cdot} \simeq B^{\cdot}$ is a direct summand of $\sigma^{\geqslant n} B^{\cdot}$ in $D(X)_{\text{parf}}$. As $\mathcal{B}$ is thick this will imply $E^{\cdot} \in \mathcal{B}$. Thus $\mathcal{B}D(X)_{\text{parf}}$, which will prove the lemma. Thus it suffices to prove the claim.

For this, I will show for $n \ll 0$ that the inclusion of complexes $\sigma^{\geqslant n} B^{\cdot} \rightarrowtail B^{\cdot}$ is a split *epimorphism* in $D(X)_{\text{parf}}$, that the identity morphism $1 \colon B^{\cdot} \to B^{\cdot}$ factors through it. Denote by $\sigma^{\leqslant n-1} B^{\cdot}$ the opposite brutal truncation, $B^{\cdot}/\sigma^{\geqslant n} B^{\cdot}$. There is an exact triangle in $D(X)$parf: $\sigma^{\geqslant n} B^{\cdot} \to B^{\cdot} \to \sigma^{\leqslant n-1} B^{\cdot} \rightsquigarrow$. By the long exact Puppe sequence ([Ve] I Sect. 1 1–2, [Ha] I 1.1)

$$\begin{aligned} \cdots \to\ & \mathrm{Mor}_{D(X)}(B^{\cdot}, \sigma^{\geqslant n} B^{\cdot}) \to \mathrm{Mor}_{D(X)}(B^{\cdot}, B^{\cdot}) \\ \to\ & \mathrm{Mor}_{D(X)}(B^{\cdot}, \sigma^{\leqslant n-1} B^{\cdot}) \to \cdots \end{aligned} \tag{3.12.2}$$

it suffices to show for $n \ll 0$ that $B^{\cdot} \to \sigma^{\leqslant n-1} B^{\cdot}$ is 0 in $D(X)_{\text{parf}}$.

For any $F^{\cdot} \in D^+(X)_{qc}$ there is a natural strongly converging spectral sequence (e.g. [TT] 2.4.1.5–6)

$$\begin{aligned} E_2^{p,q} &= H^p(X\,;\, cH^q(R\mathcal{H}om(B^{\cdot}, F^{\cdot}))) \Longrightarrow \mathrm{Ext}^{p+q}_{D(X)}(B^{\cdot}\,,\, F^{\cdot}) \\ &= \mathrm{Mor}_{D(X)}(\Sigma^{-p-q} B^{\cdot}, F^{\cdot}) \end{aligned} \tag{3.12.3}$$

As $B^{\cdot}$ is perfect the functor $R\mathcal{H}om(B^{\cdot},\quad)$ has finite cohomological dimension (e.g. [TT] 2.4.1.b), and in particular there exists an integer a such that for all $F^{\cdot} \in D^+(X)_{qc}$ with m an integer such that $\mathcal{H}^q(F^{\cdot}) = 0$ if $q \geqslant m$, then $\mathcal{H}^q(R\mathcal{H}om(B^{\cdot}, F^{\cdot})) = 0$ for $q \geqslant m - a$ Moreover, for all $q\mathcal{H}^q(R\mathcal{H}om(B^{\cdot}, F^{\cdot}))$ is a quasi-coherent sheaf on X (e.g. [TT] 2.4.1.c). As X is quasi-compact and quasi-separated there exists an integer b such that for all quasi-coherent sheaves $\mathcal{F}$ on X and all $p \geqslant b$ one has $H^p(X\,;\,\mathcal{F}) = 0$ ([EGA] III 1.4.12 applied to $X \to \mathrm{Spec}(\mathbb{Z})$ and boosted by IV 1.7.21 or [TT] B.11). Then if $F^{\cdot} \in D^+(X)_{qc}$ with $\mathcal{H}^q(F^{\cdot}) = 0$ for $q \geqslant a - b$, then $\mathcal{H}^q(R\mathcal{H}om(B^{\cdot}, F^{\cdot})) = 0$ for $q \geqslant -b$, and $H^p(X\,;\,\mathcal{H}^q(R\mathcal{H}om(B^{\cdot}, F^{\cdot}))) = 0$ for $p + q \geqslant 0$, indeed unless $p \leqslant b$ and $q < -b$. Thus the spectral sequence (3.12.3) gives $\mathrm{Mor}_{D(X)}(B^{\cdot}, F^{\cdot}) = 0$ for such $F^{\cdot}$. In particular $\mathrm{Mor}_{D(X)}(B^{\cdot}, \sigma^{\leqslant n-1} B^{\cdot}) = 0$ if $n \leqslant a - b$. □

EXAMPLE 3.13 Proposition 3.11 and its corollary show that the 'usual kinds' of thick subcategories of $D(X)_{\text{parf}}$ are triangulated $\otimes$-subcategories. Here I give an example of the existence of an 'exotic' thick subcategory which is not a $\otimes$-subcategory.

Let $X = \mathbb{P}^1_k$ be the projective line over a field k, $\pi\colon \mathbb{P}^1_k \to \mathrm{Spec}(k)$ its structure map, and $\mathcal{O}(1)$ its fundamental line bundle. As π is smooth and projective the derived direct image functor $R\pi_*$ preserves perfect complexes and induces a triangulated functor $D(\mathbb{P}^1_k)_{\text{parf}} \to D(k)_{\text{parf}}$ ([SGA6] III 4.8.1, 2.5 or [TT] 2.5.4).

Let $\mathcal{A} \subseteq D(\mathbb{P}^1_k)_{\text{parf}}$ be the thick triangulated subcategory whose objects are those $E^{\cdot}$ for which the natural adjunction morphism $\varepsilon\colon L\pi^* R\pi_* E^{\cdot} \to E^{\cdot}$ is an isomorphism in $D(\mathbb{P}^1_k)_{\text{parf}}$. As the other adjunction morphism $\eta\colon \mathrm{Id} \to R\pi_* L\pi^*$ is always a natural isomorphism in $D(k)_{\text{parf}}$ (e.g. [ThFP] Lemme 3), using the adjunction identities $R\pi_*\varepsilon \circ \eta R\pi_* = 1$ and $\varepsilon L\pi^* \circ L\pi^*\eta = 1$ one checks easily that $L\pi^* D(k)_{\text{parf}} \subseteq \mathcal{A}$, and that in fact $\mathcal{A}$ is equivalent to $D(k)_{\text{parf}}$ via the functors $R\pi_*$ and $L\pi^*$.

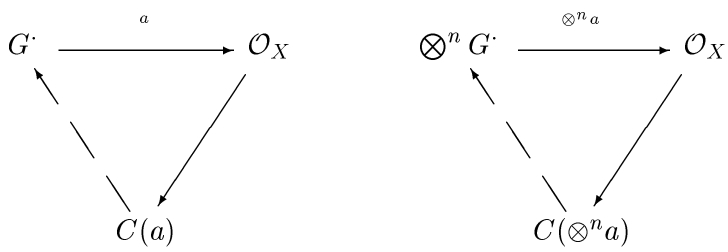
But $\mathcal{A}$ is not a $\otimes$-subcategory of $D(\mathbb{P}^1_k)_{\text{parf}}$. For if $F^{\cdot} \not\simeq 0$ is any nonzero object of $D(k)_{\text{parf}}$, $L\pi^* F^{\cdot} \in \mathcal{A}$ while $\mathcal{O}(-1) \otimes L\pi^* F^{\cdot} \notin \mathcal{A}$. Indeed $L\pi^* R\pi_*(\mathcal{O}(-1) \otimes L\pi^* F^{\cdot}) \simeq 0$ since by the projection formula ([SGA6] III 3.7, or [Ha] II 5.6) $R\pi_*(\mathcal{O}(-1) \otimes L\pi^* F^{\cdot}) \simeq (R\pi_*\mathcal{O}(-1)) \otimes F^{\cdot}$ and $R\pi_*\mathcal{O}(-1) \simeq 0$ by Serre ([EGA] III 2.12–16).

The mechanism of this example might suggest that the classification of thick $\otimes$-subcategories of $D(X)_{\text{parf}}$ given by 3.15 below might be extended to classify all thick subcategories on an X with an ample family of line bundles by incorporating some additional structure related to $\mathrm{Pic}(X)$. However, it seems difficult to find a candidate structure which reduces to the usual classification when X is affine and so all thick subcategories are $\otimes$-subcategories.

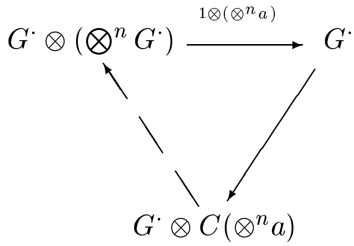
LEMMA 3.14 *Let X be a quasi-compact and quasi-separated scheme, and $E^{\cdot}, F^{\cdot} \in D(X)_{\text{parf}}$ two perfect complexes on X. Suppose that* $\mathrm{Supph}(E^{\cdot}) \subseteq \mathrm{Supph}(F^{\cdot})$. *Then $E^{\cdot}$ is in the smallest thick triangulated $\otimes$-subcategory of $D(X)_{\text{parf}}$ containing $F^{\cdot}$.*

Proof. (c.f. [Ho] proof of Thm. 7, [Ne] 1.2) Denote by $\mathcal{A}$ this smallest thick triangulated $\otimes$-subcategory containing $F^{\cdot}$.

For a morphism $a\colon G^{\cdot} \to \mathcal{O}_X$ in $D(X)_{\text{parf}}$, denote by $C(a)$ the cone of a. Then by definition and the minor abuse of considering $\mathcal{O}_X \otimes (\quad)(\quad)$, one has exact triangles for $n \geqslant 1$

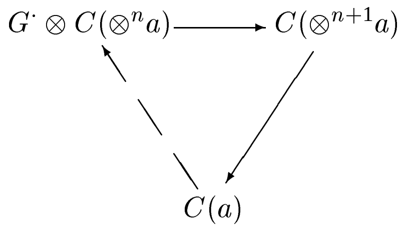


(3.14.1)



As $\otimes^{n+1} a$ is identified to the composition $a \circ (G^{\cdot} \otimes (\otimes^n a))$, the octahedral axiom ([Ve] I Sect. no.1–1 TR4, or [Ha] I Sect. TR4) gives from these a further exact triangle for all $n \geqslant 1$

(3.14.2)



Using the exact triangle resulting from tensoring this one with $E^{\cdot}$, by induction on n one sees that if $E^{\cdot} \otimes C(a)$ is in the thick triangulated $\otimes$-subcategory $\mathcal{A}$, then so is $E^{\cdot} \otimes C(\otimes^n a)$ for all $n \geqslant 1$.

As $F^{\cdot}$ is perfect, there is an isomorphism (3.8.1) $R\mathcal{H}om(F^{\cdot}, \mathcal{O}_X) \otimes F^{\cdot} \cong R\mathcal{H}om(F^{\cdot}, F^{\cdot})$, so $R\mathcal{H}om(F^{\cdot}, F^{\cdot})$ is an object of the $\otimes$-subcategory $\mathcal{A}$. Let $f\colon \mathcal{O}_X \to R\mathcal{H}om(F^{\cdot}, F^{\cdot})$ be the morphism corresponding to $1\colon F^{\cdot} \to F^{\cdot}$, and let $a\colon G^{\cdot} \to \mathcal{O}_X$ be the 'homotopy fibre' of f, the edge opposite the vertex $R\mathcal{H}om(F^{\cdot}, F^{\cdot})$ in the exact triangle having f as an edge. Then $C(a) \simeq R\mathcal{H}om(F^{\cdot}, F^{\cdot})$ is in the $\otimes$-subcategory $\mathcal{A}$, so $E^{\cdot} \otimes C(a) \in \mathcal{A}$ and as above, for all $n \geqslant 1 E^{\cdot} \otimes C(\otimes^n a) \in \mathcal{A}$.

I claim there exists an $n \geqslant 1$ such that $E^{\cdot} \otimes (\otimes^n a) = 0$ in $D(X)_{\text{parf}}$. Then considering the long exact Puppe sequence which results from applying $\text{Mor}_{D(X)}(\quad, E^{\cdot})$ to the exact triangle which is the tensor with $E^{\cdot}$ of the second exact triangle of (3.14.1), one will obtain that $E^{\cdot} \cong E^{\cdot} \otimes \mathcal{O}_X \to E^{\cdot} \otimes C(\otimes^n a)$ is a split monomorphism, i.e. that $E^{\cdot}$ is a direct summand of $E^{\cdot} \otimes C(\otimes^n a)$. As $\mathcal{A}$ is thick, this will imply that $E^{\cdot}$ is in $\mathcal{A}$, proving the lemma.

It remains to prove the claim. By tensor nilpotence with parameters (Thm. 3.8), it suffices to show for all $x \in \text{Supph}(E^{\cdot})$ that $a \otimes k(x) = 0$ in $D(k(x))$. But if $x \in \text{Supph}(E^{\cdot}) \subseteq \text{Supph}(F^{\cdot})$, then $F^{\cdot} \otimes k(x) \not\simeq 0$. Thus the map $k(x) \to R\mathcal{H}om_{k(x)}(F^{\cdot} \otimes k(x), F^{\cdot} \otimes k(x))$ corresponding to $1\colon F^{\cdot} \otimes k(x) \to F^{\cdot} \otimes k(x)$ is not zero, and so is a split monomorphism in $D(k(x))_{\text{parf}}$ as any non null-homotopic map from the field $k(x)$ to a chain complex of $k(x)$-vector spaces splits. But for $F^{\cdot}$ perfect one has an easy isomorphism (e.g. [SGA6] I 7.1.2) $R\mathcal{H}om_{\mathcal{O}_X}(F^{\cdot}, F^{\cdot}) \otimes k(x) \cong R\mathcal{H}om_{k(x)}(F^{\cdot} \otimes k(x), F^{\cdot} \otimes k(x))$, under which this split monomorphism is identified to $f \otimes k(x)$. As $fa = 0$ since a is the homotopy fibre of f, and as $f \otimes k(x)$ is a split monomorphism, one gets $a \otimes k(x) = 0$. $\square$

THEOREM 3.15 (Classification of thick $\otimes$-subcategories). *Let X be a quasi-compact and quasi-separated scheme. Denote by $\mathfrak{C}$ the set of thick triangulated $\otimes$-subcategories* (3.9) *of the derived category $D(X)_{\text{parf}}$ of perfect complexes* (3.1) *on X. Denote by $\mathfrak{S}$ the set of those subspaces $Y \subseteq X$ such that $Y = \bigcup Y_\alpha$ is a union of closed subspaces Y_α of X such that $X - Y_\alpha$ is quasi-compact.*

Then there is a bijective correspondence between $\mathfrak{C}$ and $\mathfrak{S}$.

The bijection $\psi: \mathfrak{S} \to \mathfrak{C}$ sends a subspace $Y \subseteq X$ to the thick subcategory whose objects are those perfect $E^\cdot$ such that $\text{Supph}(E^\cdot) \subseteq Y$, *i.e. which are acyclic off Y. The inverse bijection $\varphi: \mathfrak{C} \to \mathfrak{S}$ sends a triangulated subcategory $\mathcal{A} \subseteq D(X)_{\text{parf}}$ to the subspace $Y = \bigcup_{E^\cdot \in \mathcal{A}}$* $\text{Supph}(E^\cdot)$.

Proof. (c.f. [Ho] Thm. 11, [Ne] 1.5). Lemma 3.3 gives that the subspace $\varphi(\mathcal{A})$ is in $\mathfrak{S}$ for any subcategory $\mathcal{A}$ of $D(X)_{\text{parf}}$, and clearly for any subspace $Y \subseteq X$ $\psi(Y)$ is a thick triangulated $\otimes$-subcategory, and so is in $\mathfrak{C}$. So the functions φ and ψ are defined. It is easy to see that both φ and ψ preserve the partial orders on $\mathfrak{C}$ and on $\mathfrak{S}$ given by inclusions of subcategories and of subspaces, and that $\varphi\psi(Y) \subseteq Y$ and $\psi\varphi(\mathcal{A}) \supseteq \mathcal{A}$. It remains to show φ and ψ are mutually inverse, for which it suffices to prove the reverse inclusions $\varphi\psi(Y) \supseteq Y$ for $Y \in \mathfrak{S}$ and $\psi\varphi(\mathcal{A}) \subseteq \mathcal{A}$ for $\mathcal{A} \in \mathfrak{C}$.

To prove $\varphi\psi(Y) \supseteq Y$, since $Y \in \mathfrak{S}$ is a union of closed $Y_\alpha \subseteq X$ with $X - Y_\alpha$ quasi-compact, it suffices to show for each such Y_α there exists a perfect complex $E^\cdot$ such that $\text{Supph}(E^\cdot) \subseteq Y_\alpha$, such an $E^\cdot$ necessarily being in $\psi(Y)$ since $Y_\alpha \subseteq Y$. But the existence of such an $E^\cdot$ is given by Lemma 3.4.

To show $\psi\varphi(\mathcal{A}) \subseteq \mathcal{A}$ for $\mathcal{A}$ a thick triangulated $\otimes$-subcategory, let an arbitrary $E^\cdot \in \psi\varphi(\mathcal{A})$ be given. I will show $E^\cdot \in \mathcal{A}$. By the definition of ψ,φ, there are $F^\cdot_\alpha$ in $\mathcal{A}$ such that $\text{Supph}(E^\cdot) \subseteq \bigcup \text{Supph}(F^\cdot_\alpha)$. The closed subspaces $\text{Supph}(E^\cdot)$, $\text{Supph}(F^\cdot_\alpha)$ being the complements of quasi-compact opens (Lemma 3.3), they are constructible subsets of X by definition ([EGA] 0_{III} 9.1.2, IV 1.8.1). Then by [EGA] IV 1.9.9 there is a finite subset $\{\text{Supph}(F^\cdot_i)\}$ of $\{\text{Supph}(F^\cdot_\alpha)\}$ such that $\text{Supph}E^\cdot \subseteq \bigcup_{i1}^n \text{Supph}(F^\cdot_i)$. So $\text{Supph}(E^\cdot) \subseteq \text{Supph}(\oplus_{i1}^n F^\cdot_i)$, and as $\oplus_{i=1}^n F^\cdot_i \in \mathcal{A}$, Lemma 3.14 gives $E^\cdot \in \mathcal{A}$. $\square$

REMARKS 3.16.1 Recall from Corollary 3.11.1 that if X has an ample line bundle $\mathcal{O}_X(1)$, then a thick triangulated subcategory $\mathcal{A}$ of $D(X)_{\text{parf}}$ is a $\otimes$-subcategory if $\mathcal{O}_X(-1) \otimes \mathcal{A} \subseteq \mathcal{A}$. This condition is satisfied for the most 'natural' $\mathcal{A}$, e.g. if membership in $\mathcal{A}$ is locally determined (3.11.1.c), and is satisfied for all thick $\mathcal{A}$ when X is $\text{Spec}(R)$ for a ring R (3.11.1.a).

3.16.2. If Y is an arbitrary subspace of a quasi-compact and quasi-separated X, one does have a thick triangulated $\otimes$-subcategory $\psi(Y)$ of $D(X)_{\text{parf}}$, whose objects are the perfect complexes acyclic off Y. But the subspace corresponding under

3.15 to this thick $\otimes$-subcategory is not in general Y, but rather $Y' \subseteq Y$ where Y' is the union of all $Y_\alpha \subseteq Y$ such that Y_α is closed in X with quasi-compact complement. For example, consider the polynomial ring in infinitely many variables $\mathbb{C}[T_1, T_2, \ldots]$, set $X = \mathrm{Spec}(\mathbb{C}[T_1, T_2, \ldots])$ and let Y be the closed point corresponding to the maximal ideal $(T_1, T_2, \ldots)$. Then $X - Y$ is not quasi-compact, and indeed $Y' = \emptyset$, i.e. any perfect complex acyclic off Y is acyclic everywhere. Thus this closed subspace Y is not φ of a thick triangulated $\otimes$-subcategory of $D(X)_{\mathrm{parf}}$. This example is due to Neeman ([Ne] 4.1).

But when X is noetherian, all subspaces are quasi-compact ([B-AC] II Sect. 4 no. 2 Prop. 9, [EGA] I 2.7.1), and the subspaces $Y \subseteq X$ in $\mathfrak{S}$ are just those Y which are unions of closed subspaces of X, the Y 'closed under specialization'.

HISTORY 3.17 Hopkins told me that he proved versions of the nilpotence theorems 3.6 and 3.8, and of the classification Theorem 3.15 in the case $X = \mathrm{Spec}(R)$ for R a noetherian ring, but that while writing them up in his expository article on similar results in stable homotopy, he was inspired with a 'more obvious proof' for 3.6, which yielded statements for R any commutative ring ([Ho] Thm. 10, Thm. 11). As usual, 'obvious' means wrong. Neeman pointed out the error ([Ne], after 1.4), and gave a proof of 3.6 for $X = \mathrm{Spec}(R)$ of a noetherian ring R ([Ne] 1.1). In fact 3.6.6–3.6.7 is essentially Neeman's proof, which Hopkins says is essentially the same as his first unpublished proof. Neeman asks ([Ne] 1.6) if tensor nilpotence holds on $\mathrm{Spec}(R)$ when R is not noetherian; Theorem 3.6 shows that it does (c.f. 3.6.2.2 vs. [Ne] 1.1).

Hopkins used his version of tensor nilpotence to deduce a classification of the thick triangulated subcategories of $D(R)_{\mathrm{parf}}$; Theorem 11 of [Ho] states that each thick triangulated subcategory of $D(R)_{\mathrm{parf}}$ is the thick subcategory of perfect complexes acyclic off some subspace $Y \subseteq \mathrm{Spec}(R)$ closed under specialization. Hopkins says he had a valid proof of this for R noetherian, and in fact the literal statement of ([Ho] Thm. 11) is true for all R by Theorem 3.15 and 3.16.2 above. But the remarks succeeding to Hopkins' statement suggest he believed he had established a bijective correspondence between the thick subcategories and the subspaces Y closed under specialization. This is Neeman's interpretation of the statement as given in ([Ne] Sect. 0), and proved by Neeman for R noetherian ([Ne] 1.5). Neeman also pointed out that this bijective correspondence fails for certain non-noetherian R ([Ne] 4.1). My Theorem 3.15, boosted by 3.11.1.a, gives the correct bijective correspondence for arbitrary commutative rings R.

5. Classification of all strictly full $\otimes$-subcategories of $D(X)_{\mathrm{parf}}$

THEOREM 4.1 *Let X be a quasi-compact and quasi-separated scheme. Recall the definitions of* 1.2, 3.1, *and* 3.9. *Then:*

There is a bijective correspondence between the set of strictly full triangulated $\otimes$-subcategories $\mathcal{A}$ of $D(X)_{\mathrm{parf}}$, and the set of data (Y, H), where Y is a subspace *of X which is a union of closed subspaces Y_α of X with $X - Y_\alpha$ quasi-compact, and $H \subseteq K_0(X \text{ on } Y)$ is a $K_0(X)$-submodule of the Grothendieck group of the triangulated category of perfect complexes on X acyclic off Y.*

To (Y, H) corresponds the triangulated subcategory $\mathcal{A} \subseteq D(X)_{\text{parf}}$ whose objects are those perfect complexes $E^{\cdot}$ acyclic off Y and such that the class $[E^{\cdot}]$ in $K_0(X \text{ on } Y)$ lies in H.

Proof. The theorem results easily on combining 2.1 and 3.15. For the strictly full triangulated subcategory $\mathcal{A}$ is dense in the smallest thick triangulated subcategory $\widetilde{\mathcal{A}}$ containing it by 1.5. As $\mathcal{A}$ is a $\otimes$-subcategory of $D(X)_{\text{parf}}$, $\widetilde{\mathcal{A}}$ is also a $\otimes$-subcategory (3.10.b). Then by 3.15, $\widetilde{\mathcal{A}}$ corresponds to some subspace Y of the type stated. By 2.1, the strictly full dense triangulated subcategories of $\widetilde{\mathcal{A}}$ correspond bijectively to subgroups of $K_0(\widetilde{\mathcal{A}})\ K_0(X \text{ on } Y)$. As the ring structure on $K_0(X) = K_0(D(X)_{\text{parf}})$ and its action on $K_0(X \text{ on } Y)$ are induced by the derived tensor product $\otimes$, one checks easily that the strictly full dense triangulated $\otimes$-subcategories $\mathcal{A}$ of $\widetilde{\mathcal{A}}$ correspond to those subgroups $H \subseteq K_0(X \text{ on } Y)$ which are $K_0(X)$-submodules. □

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