CONGRUENCES ON SIMPLE REGULAR ω-SEMIGROUPS

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The set E of idempotents of a semigroup S can be partially ordered by defining $e \leqq f$ if and only if

$$ef = fe = e \ (e, f \in E).$$

If $E = \{e_i : i = 0, 1, \cdots\}$ and under this ordering

$$e_0 > e_1 > e_2 \cdots,$$

then we call S an ω-semigroup. Munn [7] has given a complete classification of simple regular ω-semigroups in terms of groups and group homomorphisms.

Let S be a simple regular ω-semigroup. Denote by $[\alpha, \beta]$ the sublattice of all congruences on S which contain α and are contained in β. Let $\mathscr{H}$ be the usual Green's relation and σ the minimal group congruence on S. In this paper we determine all congruences contained in $[i_s, \ \sigma \vee \mathscr{H}]$. Our work extends the results of [1] where all congruences contained in $[i_s, \mathscr{H}] \cup [\sigma, \sigma \vee \mathscr{H}]$ are determined.

For notation and definitions not given in this paper the reader is referred to Clifford and Preston [2] and [3].

1. Preliminary results

Following Munn [7], let d be a positive integer and let $\{G_i : i = 0, 1, \cdots, d-1\}$ be a set of pairwise disjoint groups. Let γ_{d-1} be a homomorphism of G_{d-1} into G_0 and let γ_i be a homomorphism of G_i into G_{i+1} $(i = 0, 1, \cdots, d-2)$. Thus we have a sequence

$$G_0 \xrightarrow{\gamma_0} G_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{d-2}} G_{d-1} \xrightarrow{\gamma_{d-1}} G_0 .$$

Denote by N the set of non-negative integers. For $n \in N$ denote by $n \ (\operatorname{mod} d)$ the integer equivalent to n modulo d, belonging to N, and less than d. Define $\gamma_n = \gamma_{n(\operatorname{mod} d)}$ for $n \in N$. For $(m, n) \in N \times N$ and $m < n$ write

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$$\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1},$$

and for all $n \in N$ let $\alpha_{n,n}$ denote the identity automorphism of $G_{n(\text{mod } d)}$. Let S be the set of all ordered triples (m, a_i, n) where $m, n \in N$, $0 \leqq i \leqq d-1$ and $a_i \in G_i$. Define a multiplication in S as follows:

$$(m, a_i, n) \cdot (p, b_j, q) = (m + p - p \wedge n, (a_i \alpha_{u,w})(b_j \alpha_{v,w}), q + n - p \wedge n),$$

where $p \wedge n = \min\{p, n\}$, $u = nd + i$, $v = pd + j$ and $w = \max\{u, v\}$. Denote the groupoid so formed by

$$S(d;\, G_0, \cdots, G_{d-1};\, \gamma_0, \cdots, \gamma_{d-1})$$

or, more compactly, by $S(d;\, G_i;\, \gamma_i)$. Then, as was shown in [7], $S(d;\, G_i;\, \gamma_i)$ is a simple regular ω-semigroup and any simple regular ω-semigroup is isomorphic to a semigroup $S(d;\, G_i : \gamma_i)$.

For $0 \leqq i \leqq d-1$ put

$$S_i = \{(m, a_i, n) : m, n \in N,\ a_i \in G_i\}.$$

S_i is a bisimple subsemigroup of S; further $S = \bigcup_{0 \leqq i \leqq d-1} S_i$. It is evident that $\alpha_{i,i+d}$ is an endomorphism of G_i. In the terminology of Reilly [10], $S_i = S(G_i, \alpha_{i,i+d})$.

For $n \in N$ and $i = 0, 1, \cdots, d-1$ write

$$e_i^n = (n, e_i, n)$$

where e_i is the identity of the group G_i. The elements e_i^n are the idempotents of $S(d;\, G_i;\, \gamma_i)$ and we have

$$e_0^0 > e_1^0 > \cdots > e_{d-1}^0 > e_0^1 > e_1^1 > \cdots > e_{d-1}^1 > e_0^2 > \cdots.$$

The semigroup $S(d;\, G_i;\, \gamma_i)$ is in fact an inverse semigroup with identity e_0^0. Further, $(m, a_i, n)^{-1} = (n, a_i^{-1}, m)$.

Put

$$B_d = \{(m, e_i, n) : m, n \in N, 0 \leqq i \leqq d-1\}.$$

B is a subsemigroup of $S(d;\, G_i, \gamma_i)$. We note that B is uniquely determined by the number d. When $d = 1$, B_d becomes the bicyclic semigroup.

A congruence ρ on a semigroup S is called idempotent-separating if each congruence class contains at most one idempotent of S. Lallement [4] has proved that a congruence on a regular semigroup is idempotent-separating if and only if it is contained in Green's equivalence $\mathscr{H}$. From the definition of multiplication in $S(d;\, G_i;\, \gamma_i)$ it is easy to show that the equivalence $\mathscr{H}$ on a simple regular ω-semigroup is given by

$$((m, a_i, n), (p, b_j, q)) \in \mathscr{H} \text{ if and only if } i = j,\ m = p \text{ and } n = q.$$

It was shown in [1] that $\mathscr{H}$ is in fact a congruence and that $S/\mathscr{H} \cong B_d$.

A congruence ρ on a semigroup S is called a group congruence if S/ρ is a group. The following lemma provides a characterization of the minimum group congruence σ on an inverse semigroup.

LEMMA 1.1 (Munn [5], Theorem 1.) *Let S be an inverse semigroup and let a relation σ be defined on S by the rule that $(x, y) \in \sigma$ if and only if $ex = ey$ for some idempotent e in S (or, equivalently, if and only if $xf = yf$ for some idempotent f in S). Then σ is a group conguence on S. Furthermore, a congruence ρ on S is a group congruence if and only if $\sigma \subseteq \rho$ and so S/ρ is isomorphic with some quotient group of S/σ.*

Let $\Lambda(S)$ be the lattice of all congruences on S. For α, $\beta \in \Lambda(S)$ and $\alpha \subseteq \beta$, define

$$[\alpha, \beta] = \{ \lambda \in \Lambda(S) : \alpha \subseteq \lambda \subseteq \beta\};$$

$[\alpha, \beta]$ is a sublattice of $\Lambda(S)$.

We conclude this section, with a theorem which gives the fundamental property of congruences on bisimple ω-semigroups.

THEOREM 1.2 (Munn and Reilly [8], Theorem 1.3.) *A congruence on a bisimple ω-semigroup, $S(G; \alpha)$, is either an idempotent-separating congruence or a group congruence.*

2. Uniform congruences and γ-admissible subsets

Let $S = S(d; G_i; \gamma_i)$ be a simple regular ω-semigroup. A *congruence* μ on the semigroup of idempotents E of $S(d; G_i; \gamma_i)$ is called *uniform* if

$$(e_i^n, e_j^m) \in \mu \text{ implies that } (e_i^{n+p}, e_j^{m+p}) \in \mu$$

for all integers $p \geqq -\min\{n, m\}$.

We recall that a congruence λ on an idempotent semigroup T is convex. That is, if $(e,f) \in \lambda$ and $e \leqq f (e,f \in T)$, then $(e, x) \in \lambda$ for all $x \in T$ such that $e \leqq x \leqq f$.

Let μ be a uniform congruence on the idempotents E of S. Suppose that there exist e_i^n, $e_i^{n+1} \in E$ such that $(e_i^n, e_i^{n+1}) \in \mu$. Then, since μ is uniform we have that

$$(e_i^0, e_i^1) \in \rho, (e_i^1, e_i^2) \in \rho, \cdots, (e_i^n, e_i^{n+1}) \in \rho, (e_i^{n+1}, e_i^{n+2}) \in \rho, \cdots.$$

Since μ is convex we conclude that $(e_i^0, x) \in \mu$ for all $x \leqq e_i^0, x \in E$. In particular, we have that $(e_0^1, e_0^2) \in \mu$. Arguing as above we obtain that $(e_0^0, x) \in \mu$ for all $x \leqq e_0^0$, $x \in E$. Thus μ is the universal relation ω_E on E.

LEMMA 2.1. *Let μ be a uniform congruence on the idempotents E of $S(d; G_i; \gamma_i)$. Suppose that $\mu \neq \omega_E$ and that $(e_i^n, e_j^m) \in \mu$. Then we may assume*

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without loss of generality that $e_j^n \geqq e_j^m$. Further, under the above assumption $m = n$ and $j \geqq i$ or $m = n+1$ and $j < i$.

PROOF. Clearly we may assume without loss of generality that $e_j^m \leqq e_i^n$. Suppose it is not the case that $m = n$ and $j \geqq i$ or $m = n+1$ and $j < i$. Then $(e_i^n, e_i^{n+1}) \in \mu$, since μ is convex. Thus by the above remark $\mu = \omega_E$, a contradiction. This completes the proof of Lemma 2.1.

THEOREM 2.2. *Let λ be a congruence on a simple regular ω semigroup $S(d; G_i; \gamma_i)$. Then $\lambda | E$ is a uniform congruence on E, where E is the set of idempotents of $S(d; G_i; \gamma_i)$.*

PROOF. Clearly $\lambda | E$ is a congruence on E and it remains to show that $\lambda | E$ is a uniform congruence.

Suppose that $(e_i^n, e_i^{n+1}) \in \lambda$. Then $(e_i^n, e_i^{n+1}) \in \lambda | S_i$ and it follows from Theorem 1.2 that $\lambda | S_i$ is a group congruence on S_i. Since $\lambda | E$ is convex we have that $(e_i^0, x) \in \lambda$ for all $x \leqq e_i^0, x \in E$. In particular we have that $(e_0^1, e_0^2) \in \lambda$. Arguing as above we obtain that $(e_0^0, x) \in \lambda$ for all $x \leqq e_0^0$, $x \in E$. Thus $\lambda | E = \omega_E$ and $\lambda | E$ is uniform.

Let now $(e_i^n, e_j^m) \in \lambda$ and suppose that $\lambda | E \neq \omega_E$. Arguing as in Lemma 2.1 and using the remark above we may assume without loss of generality that $n = m$ and $i \leqq j$ or $n+1 = m$ and $j < i$.

Firstly, $(e_i^n, e_j^n) \in \lambda$ implies that

$$((n+p, e_i, n)e_i^n(n, e_i, n+p), (n+p, e_i, n)e_j^n(n, e_i, n+p)) \in \lambda,$$

for all integers $p \geqq -n$. Hence $(e_i^{n+p}, e_j^{n+p}) \in \lambda$, for all integers $p \geqq -n$ and $\lambda | E$ is uniform.

The above argument applies to the second case and the proof of Theorem 2.2 is complete.

Let $S = S(d; G_i; \gamma_i)$ be a simple regular ω-semigroup. Put $G = G_0 \times G_1 \times \cdots \times G_{d-1}$, the cartesian product of G_i, $i = 0, 1, \cdots, d-1$. As in [1] a subset A of G will be called *γ-admissible* if it satisfies the following conditions,

(i) $A = A_0 \times A_1 \times \cdots \times A_{d-1}$, for some $A_i \subseteq G_i$, $i = 0, 1, \cdots, d-1$,

(ii) $A_i \Delta G_i$, for $i = 0, 1, \cdots, d-1$, and

(iii) $A_{d-1}\gamma_{d-1} \subseteq A_0$ and $A_i\gamma_i \subseteq A_{i+1}$, for $i = 0, 1, \cdots, d-2$.

Again following [1], for any congruence λ on S we define a subset A^λ of G as follows;

$$A^\lambda = A_0^\lambda \times A_1^\lambda \times \cdots \times A_{d-1}^\lambda,$$

where

$$A_i = \{a_i \in G_i : (0, a_i, 0) \in e_1^0\lambda\}, \qquad i = 0, 1, \cdots, d-1.$$

Evidently $A^\lambda = A^{\lambda \wedge \mathscr{H}}$, since the $\mathscr{H}$-class containing e_i^0 is $\{(0, a_i, 0) \in S : a_i \in G_i\}$.

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THEOREM 2.3. (Baird [1], Lemma 2.1.) *For any congruence λ on $S(d;\, G_i;\lambda_i)$ A^λ is a γ-admissible subset of G.*

3. Linked uniform congruences and γ-admissible subsets

Let μ be a uniform congruence on E, the set of idempotents of a simple regular ω-semigroup $S(d; G_i;\gamma_i)$. Further, let

$$A = A_0 \times A_1 \times \cdots \times A_{d-1}$$

be a γ-admissible subset of $G = G_0 \times G_1 \times \cdots \times G_{d-1}$, the cartesian product of G_i, $i = 0, 1, \cdots, d-1$.

We define, for $i = 0, 1, \cdots, d-1$, μ-rad A_i (the μ-radical of A_i) by:

$$\mu\text{-rad}\, A_i = \{a_i \in G_i : \alpha_i \alpha_{nd+i,md+j} \in A_j, \text{ for some } n, m, j, \text{ such that}$$
$$(e_i^n, e_j^m) \in \mu \text{ and } e_j^m \leqq e_i^n\}.$$

Using these radicals we define the *μ-Radical* of A, *μ-Rad A*, as follows:

$$\mu\text{-Rad}\, A = \mu\text{-rad}\, A_0 \times \mu\text{-rad}\, A_1 \times \cdots \times \mu\text{-rad}\, A_{d-1}.$$

LEMMA 3.1. *Let μ be a uniform congruence on E and A a γ-admissible subset of G. Then μ-Rad A is a γ-admissible subset of G.*

LEMMA 3.2. *Let μ, μ' be uniform congruences on E and A, A' γ-admissible subsets of G. Then*

(i) $A \subseteq \mu\text{-Rad}\, A$,
(ii) $A \subseteq A'$ *implies* $\mu\text{-Rad}\, A \subseteq \mu\text{-Rad}\, A'$,
(iii) $\mu \subseteq \mu'$ *implies* $\mu\text{-Rad}\, A \subseteq \mu'\text{-Rad}\, A$, *and*
(iv) $\mu\text{-Rad}(\mu\text{-Rad}\, A) = \mu\text{-Rad}\, A$.

The proofs of the above two lemmas are straightforward and are omitted.

Properties (i), (ii) and (iv) imply that μ-Rad is a closure operator on the set of γ-admissible subsets of G.

Let A be a γ-admissible subset of G. In [1] *the radical of A*, Rad A, is defined as follows:

$$\text{Rad}\, A = \text{rad}\, A_0 \times \text{rad}\, A_1 \times \cdots \times \text{rad}\, A_{d-1},$$

where

$$\text{rad}\, A_i = \{a_i \in G_i : a_i \alpha_{i,i+d}^n \in A_i, \text{ for some } n\}.$$

It is straightforward to check that $\text{rad}\, A_i = \omega_E\text{-rad}\, A_i$ and hence $\text{Rad}\, A = \omega_E\text{-Rad}\, A$.

A uniform congruence μ on E and a γ-admissible subset A of G are called linked if μ-Rad $A = A$.

THEOREM 3.3. *Let λ be a congruence on a simple regular ω-semigroup*

$S(d; G_i; \gamma_i)$. Then the γ-admissible subset A^λ of G and the uniform congruence $\lambda | E$ of E are linked.

Proof. Suppose that λ is a group congruence on S. It follows from Lemma 3.2 of [1] that $\operatorname{Rad} A^\lambda = A^\lambda$. Hence $\lambda | E$ $(= \omega_E)$ and A^λ are linked.

Suppose now that λ is not a group congruence and $a_i \in \lambda | E\text{-rad}\, A_i$. Then there exists n, m, and j such that

$$a_i \alpha_{nd+i, md+j} \in A_j^\lambda, \; (e_i^n, e_j^m) \in \lambda | E \text{ and } e_j^m \leqq e_i^n.$$

Since λ is not a group congruence, $\lambda | E \neq \omega_E$ and it follows from Lemma 2.1 that

$$n = m \text{ and } i \leqq j \text{ or } n + 1 = m \text{ and } j < i.$$

If $m = n$ and $i \leqq j$, then $a_i \alpha_{nd+i, md+j} = a_i \alpha_{i,j}$. Put $a = (0, a_i, 0)$. Now $(e_i^0, e_j^0) \in \lambda$, since $\lambda | E$ is uniform, and so

$$(e_i^0 a, e_j^0 a) \in \lambda.$$

That is,

$$(a, (0, a_i \alpha_{i,j}, 0)) \in \lambda.$$

Now

$$((0, a_i \alpha_{i,j}, 0), e_j^0) \in \lambda.$$

Hence

$$((0, a_i \alpha_{i,j}, 0), e_i^0) \in \lambda.$$

Thus $(a, e_i^0) \in \lambda$ and so $a_i \in A_i^\lambda$. Hence $\lambda | E\text{-rad}\, A_i^\lambda = A_i^\lambda$ and so $\lambda | E\text{-Rad}\, A^\lambda = A^\lambda$. We conclude that $\lambda | E$ and A^λ are linked.

If on the other hand $m = n + 1$ and $i < j$, then $a_i \alpha_{nd+i, md+j} = a_i \alpha_{i, j+d}$. Put $a = (0, a_i, 0)$. Now $(e_i^0, e_j^1) \in \lambda$, since $\lambda | E$ is uniform, and so

$$(e_i^0 a, e_j^1 a) \in \lambda.$$

That is,

$$(a, (1, a_i \alpha_{i, j+d}, 1)) \in \lambda.$$

Now

$$((0, a_i \alpha_{i, j+d}, 0), e_j^0) \in \lambda.$$

Put $b = (1, e_j, 0)$. Then

$$(b(0, a_i \alpha_{i\; j+d}, 0) b^{-1}, b e_j^0 b^{-1}) \in \lambda.$$

That is,

$$((1, a_i \alpha_{i, j+d}, 1), e_j^1) \in \lambda.$$

Thus $(a, e_j^1) \in \lambda$ and so $(a, e_i^0) \in \lambda$. Further, $a_i \in A_i^\lambda$. Hence $\lambda | E\text{-rad}\, A_i^\lambda = A_i^\lambda$ and so $\lambda | E\text{-Rad}\, A^\lambda = A^\lambda$. We conclude that $\lambda | E$ and A^λ are linked. This completes the proof of Theorem 3.3.

4. Congruences contained in $[i, \sigma \vee \mathscr{H}]$

We now fix our attention on the sublattice $[i, \sigma \vee \mathscr{H}]$. The following theorem provides a characterisation of the congruence $\sigma \vee \mathscr{H}$ on a simple regular ω-semigroup.

THEOREM 4.1. (Baird [1], Lemma 3.3). *Let $S(d; G_i, \gamma_i)$ be a simple regular ω-semigroup. Then,*

$$((m, a_i, n),(p, b_j, q)) \in \sigma \vee \mathscr{H}$$

if and only if $m - n = p - q$.

Let $A = A_0 \times A_1 \times \cdots \times A_{d-1}$ be a γ-admissible subset of G. For (m, a_i, n) and (p, b_j, q) in $S(d; G_i; \gamma)$ we shall write

$$\left\| (m, a_i, n),(p, b_j, q) \right\| \in A$$

to mean that $a_i \alpha_{u,w} b_j^{-1} \alpha_{v,w} \in A_{w (\bmod d)}$, where $u = nd + i$, $v = qd + j$ and $w = \max \{u, v\}$.

We are now in a position to state our main result.

THEOREM 4.2. *Let μ be a uniform congruence on E, the set of idempotents of a simple regular ω-semigroup $S(d; G_i; \gamma_i)$. Let A be a γ-admissible subset of G. Further, suppose that μ and A are linked. Then*

$$\tau = \{((m, a_i, n),(p, b_j, q)) \in S \times S : \left\| (m, a_i, n),(p, b_j, q) \right\| \in A,\ m - n = p - q,$$
$$\text{and } (e_i^m, e_j^p) \in \mu\}$$

is a congruence on $S(d; G_i; \gamma_i)$ contained in $[i, \sigma \vee \mathscr{H}]$ such that $A^\tau = A$ and $\tau | E = \mu$.

Conversely, if τ is a congruence on $S(d; G_i; \gamma_i)$ *contained in $[i, \sigma \vee \mathscr{H}]$, then τ is of the above form with $\mu = \tau | E$* and $A = A^\tau$.

REMARK (i). The lack of symmetry in the definition of τ is only apparent. Since $m - n = p - q$, μ uniform, and $(e_i^m, e_j^p) \in \mu$ together imply that $(e_i^n, e_j^q) \in \mu$.

REMARK (ii). Let $S = S(d; G_i; \gamma_i)$ be a simple regular ω-semigroup. Then there is a one-to-one correspondence between the congruences contained in $[i, \sigma \vee \mathscr{H}]$ and the linked pairs of uniform congruences on E and γ-admissible subsets of G.

We prove Theorem 4.2 by a series of lemmas.

LEMMA 4.3 (Munn [6], Lemma 4 (ii).). *Let ρ be a group congruence on an inverse semigroup S with identity e. Then $(x, y) \in \rho$ if and only if $xy^{-1} \in e\rho$.*

LEMMA 4.4 (Baird [1], Lemma 3.4.). *Let ρ be a congruence on $S(d; G_i; \gamma_i)$. Further, suppose that $\rho \in [\sigma, \sigma \vee \mathscr{H}]$. Then*

$$e_0^0\rho = \{(m, a_i, m) \in S : a_i \in A_i^\rho, 0 \leqq i \leqq d-1\}.$$

LEMMA 4.5. *Let ρ be a congruence on $S(d; G_i; \gamma_i)$. Further, suppose that $\rho \in [\sigma, \sigma \bigvee \mathscr{H}]$. Then*

$$\rho = \{(m, a_i, n), (p, b_j, q)) \in S \times S : \|(m, a_i, n), (p, b_j, q)\| \in A^\rho$$
$$\text{and } m - n = p - q.\}.$$

Lemma 4.5 follows immediately from 4.3 and Lemma 4.4.

LEMMA 4.6 (Baird [1], Lemma 3.5.). *Let $S = S(d; G_i; \gamma_i)$ be a simple regular ω-semigroup. Let A be a γ-admissible subset of G. Further, suppose that* Rad $A = A$. *Then*

$$\tau = \{((m, a_i, n), (p, b_j, q)) \in S \times S : \|(m, a_i, n), (p, b_j, q))\| \in A$$
$$\text{and } m - n = p - q\}$$

is a congruence on S contained in $[\sigma, \sigma \bigvee \mathscr{H}]$ such that $A^\tau = A$.

If in Theorem 4.2 $\mu = \omega_E$ the congruence τ is a group congruence. Conversely if τ is a group congruence $\tau | E = \omega_E$. Combining Lemma 4.5 and Lemma 4.6 we obtain a proof of Theorem 4.2 in the case that $\mu = \omega_E$ or conversely when τ is a group congruence.

LEMMA 4.7 (Munn [6], Lemma 1.). *Let ρ be a congruence on $S = S(d; G_i; \gamma_i)$. Further, assume that $\rho | S_i \in [i, \mathscr{H}]$. Then*

$$((m, a_i, n), (p, b_i, q)) \in \rho | S_i$$

if and only if $m = p$, $n = q$, and $a_i b_i^{-1} \in A_i^\rho$.

LEMMA 4.8. *Let ρ be a congruence on $S = S(d; G_i; \gamma_i)$. Further, assume that ρ is not a group congruence on S. Then*

$$p = \{((m, a_i, n), (p, b_j, q)) \in S \times S : (e_i^m, e_j^p) \in \rho,\ m - n = p - q$$
$$\text{and } \|(m, a_i, n), (p, b_j, q)\| \in A^\rho\}.$$

PROOF. Let $((m, a_i, n), (p, b_j, q)) \in \rho$. Since S is an inverse semigroup it follows that

$$((m, a_i, n)^{-1}, (p, b_j, q)^{-1}) \in \rho$$

(see for instance section 7.4 of [3].) That is,

$$((n, a_i^{-1}, m), (q, b_j^{-1}, p)) \in \rho.$$

And so

$$((m, a_i, n)(n, a_i^{-1}, m), (p, b_j, q)(q, b_j^{-1}, p)) \in \rho.$$

Thus $(e_i^m, e_j^p) \in \rho$. Further,

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$$((m, a_i, n)(q, b_j^{-1}, p), (p, b_j, q)(q, b_j^{-1}, p)) \in \rho.$$

Hence

$$((m + q - n \wedge q,\ a_i\alpha_{u,w}b_j^{-1}\alpha_{v,w},\ p + n - n \wedge q),\ e_j^p) \in \rho$$

where $u = nd + i$, $v = qd + j$, and $w = \max\{u, v\}$.

Suppose that $w(\mathrm{mod}\, d) = j$. Then

$$((m + q - n \wedge q,\ a_i\alpha_{u,w}b_j^{-1}\alpha_{v,w},\ p + n - n \wedge q),\ e_j^p) \in \rho \,|\, S_j.$$

Now ρ is not a group congruence and so $\rho \,|\, S_j$ is idempotent-separating, since $\rho \,|\, E$ is uniform and by Lemma 2.1. It follows from Lemma 4.7 that

$$m + q - n \wedge q = p = p + n - n \wedge q$$

that is, $m - n = p - q$. Further, $a_i\alpha_{u,w}bj^{-1}\alpha_{v,w} \in A^\rho$, that is,

$$\|(m, a_i, n), (p, b_j, q)\| \in A^\rho.$$

Suppose on the other hand that $w(\mathrm{mod}\, d) = i$. Now $(e_j^p, e_i^m) \in \rho$ and so it follows from the transitivity of ρ that

$$((m + q - n \wedge q,\ a_i\alpha_{u,w}b_j^{-1}\alpha_{v,w},\ p + n - n \wedge q),\ e_i^m) \in \rho \,|\, S_i.$$

Arguing as above we obtain $m - n = p - q$ and

$$\|(m, a_i, n), (p, b_jq)\| \in A^\rho.$$

Suppose now that $(e_i^m, e_j^p) \in \rho$, $m - n = p - q$ and

$$\|(m, a_i, n), (p, b_j, q)\| \in A^\rho.$$

Let us assume that $e_i^m \geqq e_j^p$. It follows from Lemma 4.7 that

$$((p, a_i\alpha_{nd+i\ qd+j}, q), (p, b_j, q)) \in \rho \,|\, S_j$$

and so

$$((p, a_i\alpha_{nd+i,qd+j}, q), (p, b_j, q)) \in \rho.$$

Now $(e_i^m, e_j^p) \in \mu$ and $m - n = p - q$ implies that $(e_i^n, e_j^q) \in \rho$. Hence

$$((m, a_i, n)e_i^n, (m, a_i, n)e_j^q) \in \rho.$$

Thus

$$((m, a_i, n), (m + q - q \wedge n, a_i\alpha_{nd+i,qd+j}, q + n - q \wedge n)) \in \rho.$$

Now $q \wedge n = n$ and $m + q - n = p$. Hence

$$((m, a_i, n), (p, a_i\alpha_{nd+i,qd+j}, q)) \in \rho.$$

Transitivity implies that $((m, a_i, n), (p, b_j, q)) \in \rho$. The proof for the case $e_i^m \leqq e_j^p$ is similar. This completes the proof of Lemma 4.8.

LEMMA 4.9. *Let μ be a uniform congruence on the idempotents E of*

$S(d;G_i;\gamma_i)$ and A a γ-admissible subset of G. Further, assume that $\mu \neq \omega_E$ and that μ and A are linked. Then

$$\tau = \{((m, a_i, n),(p, b_j, q)) \in S \times S: \\ \|((m, a_i, n),(p, b_j, q)\| \in A,\ m - n = p - q,$$

and

$$(e_i^m, e_j^p) \in \mu\}$$

is a congruence on S. Further, τ is not a group congruence, $\tau | E = \mu$ and $A^\tau = A$.

REMARK. Since $\mu \neq \omega_E$ it follows from Lemma 2.1 that we may assume without loss of generality that $m = p$ and $j \geqq i$ or $m + 1 = p$ and $j < i$.

The proof of Lemma 4.9 follows mutatis mutandis the proof of Lemma 4.5 in [1]. We omit the details.

We remark that if τ is not a group congruence then it follows from Lemma 4.8, 4.9 and Theorem 4.1 that $\tau \in [i, \sigma \vee \mathscr{H}]$.

Theorem 4.2 now follows from Lemmas 4.5, 4.6, 4.8 and 4.9.

5. On the sublattice $[i, \sigma \vee \mathscr{H}]$

We begin this section by determining the γ-admissible subsets determined by the meet and join of two congruences ρ and λ on $S(d; Gi; \gamma_i)$ contained in $[i, \sigma \vee \mathscr{H}]$ in terms of the γ-admissible subsets determined by ρ and λ.

THEOREM 5.1 (This result is due to Scheiblich and Reilly and is contained in [9].). *Let S be an inverse semigroup with idempotents E. Further, suppose that ρ and λ are congruences on S. Then*

(i) $\rho \vee \lambda | E = \rho | E \vee \lambda | E$, and

(ii) $\rho \wedge \lambda | E = \rho | E \wedge \lambda | E$.

THEOREM 5.2 *Let ρ and λ be congruences on $S(d; G_i; \gamma_i)$. Further, assume that $\rho, \lambda \in [i, \sigma \vee \mathscr{H}]$. Then $\rho \subseteq \lambda$ if and only if $A^\rho \subseteq A^\lambda$ and $\rho | E \subseteq \lambda | E$.*

Theorem 5.2 follows readily from Theorem 4.2. We omit the details.

If $A = A_0 \times A_1 \times \cdots \times A_{d-1}$ and $B = B_0 \times B_1 \times \cdots \times B_{d-1}$ are γ-admissible subsets of G we define,

$$A \vee B = A_0B_0 \times A_1B_1 \times \cdots \times A_{d-1}B_{d-1}, \text{ and}$$

$$A \wedge B = A_0 \cap B_0 \times A_1 \cap B_1 \times \cdots \times A_{d-1} \cap B_{d-1}.$$

We shall often write $AB = A \vee B$. It is clear that AB and $A \wedge B$ are again γ-admissible subset of G. Thus the set of all γ-admissible subsets of G form a lattice under $\vee$ and $\wedge$.

The following theorem generalizes Lemma 4.1 of [1].

THEOREM 5.3. *Let ρ and λ be congruences on $S(d;G_i;\gamma_i)$. Further, assume that ρ and λ are contained in $[i,\sigma \vee \mathscr{H}]$. Then*

(i) $A^{\lambda \vee \rho} = (\lambda | E \vee \rho | E) - \text{Rad}\, A^\lambda A^\rho$, and

(ii) $A^{\lambda \wedge \rho} = A^\lambda \wedge A^\rho$.

PROOF (i). It follows from Theorem 4.2 that there exists $\mu \in [i,\sigma \vee \mathscr{H}]$ such that

$$A^\mu = (\lambda | E \vee \rho | E) - \text{Rad}\, A^\lambda A^\rho \text{ and } \mu | E = \lambda | E \vee \rho | E.$$

Now $\lambda | E$, $\rho | E \subseteq \lambda | E \vee \rho | E$ and A^λ, $A^\rho \subseteq (\lambda | E \vee \rho | E) - \text{Rad}\, A^\lambda A^\rho$. By Theorem 5.1 $\lambda | E \vee \rho | E = \lambda \vee \rho | E$ and so it follows from Theorem 5.2 that $\lambda \vee \rho \subseteq \mu$.

Now $\lambda \subseteq \lambda \vee \rho$ and so again by Theorem 5.2 $A^\lambda \subseteq A^{\lambda \vee \rho}$. Similarly $A^\rho \subseteq A^{\lambda \vee \rho}$. Hence $A^\lambda A^\rho \subseteq A^{\lambda \vee \rho}$ and so

$$(\lambda | E \vee \rho | E) - \text{Rad}\, A^\lambda A^\rho \subseteq (\lambda | E \vee \rho | E) - \text{Rad}\, A^{\lambda \vee \rho}.$$

But $\lambda | E \vee \rho | E = \lambda \vee \rho | E$ by Theorem 5.1 and $\lambda \vee \rho | E$ and $A^{\lambda \vee \rho}$ are linked. Therefore,

$$(\lambda | E \vee \rho | E) - \text{Rad}\, A^\lambda A^\rho \subseteq (\lambda \vee \rho | E) - \text{Rad}\, A^{\lambda \vee \rho} = A^{\lambda \vee \rho}.$$

Hence $\mu \subseteq \lambda \vee \rho$. We conclude that $A^{\lambda \vee \rho} = A^\mu = (\lambda | E \vee \rho | E) - \text{Rad}\, A^\lambda A^\rho$.

(ii) Let $a_i \in G_i$ and put $x = (0, a_i, 0)$. Then $a_i \in A^{\lambda \wedge \rho}$ if and only if $(x, e^0) \in \lambda$ and $(x, e_i^0) \in \rho$. But $(x, e_i^0) \in \lambda$ and $(x, e_i^0) \in \rho$ if and only if $a_i \in A_i^\lambda \cap A_i^\rho$. Hence $A_i^{\lambda \wedge \rho} = A_i^\lambda \cap A_i$ and so $A^{\lambda \wedge \rho} = A^\lambda \wedge A^\rho$. This completes the proof of Theorem 5.3.

We conclude this paper by introducing two important sublattices of $[i,\sigma \vee \mathscr{H}]$.

THEOREM 5.4. *Let μ be a uniform congruence on the idempotents E of a simple regular ω-semigroup $S = S(d;G_i;\gamma_i)$.*

Then

$$M(\mu) = \{((m,a_i,n),(p,b_j,q)) \in S \times S : m - n = p - q \text{ and } (e_i^m, e_j^p) \in \mu\}$$

is a congruence contained in $[i,\sigma \vee \mathscr{H}]$. Further, $M(\mu) | E = \mu$ and if ρ is a congruence contained in $[i,\sigma \vee \mathscr{H}]$ such that $\rho | E = \mu$ then $\rho \subseteq M(\mu)$.

The proof of Theorem 5.4 is straightforward and is omitted.

We note that $M(\omega_E) = \sigma \vee \mathscr{H}$. Recall that if a congruence ρ on S is not a group congruence then $\rho \in [i,\sigma \vee \mathscr{H}]$. Hence if $\mu \neq \omega_E$ the congruence $M(\mu)$ is the maximum congruence on S inducing the congruence μ on E. On the other hand ω_S is the maximum congruence on S inducing the congruence ω_E on E and $\omega_S \neq \sigma \vee \mathscr{H} = M(\omega_E)$.

LEMMA 5.5. *Let μ_1 and μ_2 be uniform congruences on the idempotents E of a simple regular ω-semigroup S. Then*
(i) $A^{M(\mu_1) \vee M(\mu_2)} = A^{M(\mu_1 \vee \mu_2)}$, *and*
(ii) $A^{M(\mu_1) \wedge M(\mu_2)} = A^{M(\mu_1 \wedge \mu_2)}$.

PROOF. It suffices to observe that for all uniform congruences μ on E, $A^{M(\mu)} = G_0 \times G_1 \times \cdots \times G_{d-1}$. This completes the proof of Lemma 5.5.

It follows from Remark (ii) of Theorem 4.2, Theorem 5.1 and Lemma 5.5 that $R = \{M(\mu) : \mu$ a uniform congruence on $E\}$ is a sublattice of $[i, \sigma \bigvee \mathscr{H}]$. Moreover, R and the uniform congruences on E have the same lattice structure.

Denote the γ-admissible subset $\{(e, e_1, \cdots, e_{d-1})\}$ by 1.

THEOREM 5.6. *Let μ be a uniform congruence on the idempotents E of a simple regular ω-semigroup $S = S(d; G_i; \gamma_i)$. Then*

$$\begin{aligned} m(\mu) = \{((m, a_i, n), (p, b_j, q)) \in S \times S : m - n = p - q, \\ \|(m, a_i, n), (p, b_j, q)\| \in \mu - \mathrm{Rad}\, 1 \ \textit{and} \\ (e_i^m, e_i^p) \in \mu\} \end{aligned}$$

is a congruence on S. Further, $m(\mu) | E = \mu$ and if ρ is a congruence on S such that $\rho | E = \mu$ then $m(\mu) \subseteq \rho$.

PROOF. Put $A = \mu - \mathrm{Rad}\, 1$. Then A is a γ-admissible subset of G and A and μ are linked. Evidently A is the minimal γ-admissible subset B of G such that B and μ are linked. Theorem 5.6 now follows from Theorem 4.2. This completes the proof of Theorem 5.6.

We note that $m(\mu) \in [i, \sigma \bigvee \mathscr{H}]$ and that $m(\omega_E) = \sigma$. (See Section 3 of [1]). The congruence $m(\mu)$ is the minimal congruence on S inducing the congruence μ on E.

LEMMA 5.7. *Let μ_1 and μ_2 be uniform congruences on the idempotents E of a simple regular ω-semigroup S. Then*
(i) $A^{m(\mu_1) \vee m(\mu_2)} = A^{m(\mu_1 \vee \mu_2)}$ *and*
(ii) $A^{m(\mu_1) \wedge m(\mu_2)} = A^{m(\mu_1 \wedge \mu_2)}$.

PROOF. To prove (i) it suffices by Theorem 5.2 to prove that

$$(\mu_1 \bigvee \mu_2) - \mathrm{Rad}\, A^{m(\mu_1)} A^{m(\mu_2)} = A^{m(\mu_1 \vee \mu_2)}.$$

Now $A^{m(\mu_1)}, A^{m(\mu_2)} \subseteq A^{m(\mu_1 \vee \mu_2)}$ by the definition of $A^{m(\mu)}$ and Theorem 5.2. Hence

$$A^{m(\mu_1)} A^{m(\mu_2)} \subseteq A^{m(\mu_1 \vee \mu_2)}$$

and so

$$(\mu_1 \bigvee \mu_2) - \mathrm{Rad}\, A^{m(\mu_1)} A^{m(\mu_2)} \subseteq (\mu_1 \bigvee \mu_2) - \mathrm{Rad}\, A^{m(\mu_1 \vee \mu_2)} = A^{m(\mu_1 \vee \mu_2)}.$$

However, $A^{m(\mu_1 \vee \mu_2)}$ is the minimal γ-admissible subset of G which is linked with $\mu_1 \bigvee \mu_2$ and so

$$(\mu_1 \bigvee \mu_2) - \operatorname{Rad} A^{m(\mu_1)} A^{m(\mu_2)} = A^{m(\mu_1 \vee \mu_2)}.$$

This completes the proof of (i).

The proof of (ii) is straightforward. We omit the details. This completes the proof of Lemma 5.7.

It follows from Remark (ii) of Theorem 4.2, Theorem 5.1 and Lemma 5.7 that $L = \{m(\mu) : \mu$ a uniform congruence on $E\}$ is a subblattice of $[i, \sigma \bigvee \mathscr{H}]$. Moreover, L and the uniform congruences on E have the same lattice structure.

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