ON THE STRONG SUMMABILITY BY TRIANGULAR MEANS OF THE DERIVED FOURIER SERIES AND ITS CONJUGATE SERIES

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1.1. The triangular matrix $(\Lambda) = (\lambda_{n,k})$, where $n = 0, 1, 2, \ldots$; $k = 0, 1, 2, \ldots$; and $\lambda_{n,k} = 0$ for $k > n$ is regular (in the sense of defining a regular sequence to sequence transform) if

(i) $\lim_{n\to\infty} \lambda_{n,k} = 0$ for every fixed k;

(ii) $\sum_{k=0}^{n} |\lambda_{n,k}| \leq M$, independently of n;

(iii) $\lim_{n\to\infty} \sum_{k} \lambda_{n,k} = 1$.

A series

$$a_0 + a_1 + a_2 + \ldots + a_n + \ldots \tag{1.11}$$

is said to be strongly summable (Λ) or summable $[\Lambda]$ to the sum S, if

$$\sum_{k=0}^{n} \lambda_{n,k} |S_k - S| = o(1) \quad \text{as } n\to\infty, \tag{1.12}$$

$\{S_k\}$ being the sum of the first $(k+1)$ terms of the series (1.11).

In the following three cases:

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(a) $\lambda_{n,k} = \frac{1}{n+1}$ $(k \leq n)$;

(b) $\lambda_{n,k} = \frac{1}{(k+1) \sum_{j=0}^{n} \frac{1}{(j+1)}}$ $(k \leq n)$;

(c) $\lambda_{n,k} = \frac{1}{(n-k+1) \sum_{j=0}^{n} \frac{1}{(j+1)}}$ $(k \leq n)$;

summability $[\Lambda]$ becomes respectively summability Cesàro $[C, 1]$, a Riesz summability equivalent to $[R, \log n, 1]$, and Nörlund summability $[N, \frac{1}{n+1}]$.

Every series with bounded partial sums and summable $[C, r]$ for some $r > 0$, is also summable $[C, r']$ for any other $r' > 0$.

1.2. Let $f(t)$ be a function of bounded variation with period 2π. Let the Fourier series of $f(t)$ be

$$(1.21) \qquad \frac{1}{2} a_o + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) .$$

Then the derived Fourier series of $f(t)$ and the series conjugate to the derived Fourier series are respectively

$$(1.22) \qquad \sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt)$$

and

$$(1.23) \qquad \sum_{1}^{\infty} n(b_n \sin nt + a_n \cos nt) .$$

2. In generalization of two theorems proved by Prasad and Singh [1, 2], the following theorems are established.

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THEOREM 1. Let $(\lambda_{n,k})$ be a regular sequence to sequence triangular matrix, satisfying the additional condition

$$(2.1) \qquad \lim_{n\to\infty} \sum_{k=0}^{n} |\Delta\lambda_{n,k}| = 0 \qquad (\Delta\lambda_{n,k} = \lambda_{n,k} - \lambda_{n,k+1}).$$

Let $f(t)$ be a continuous function of bounded variation with period 2π, such that

$$(2.2) \qquad G(t) \equiv \int_0^t |dg(u)| = o\left\{\frac{t}{L(1/t)}\right\} \quad \text{as } t \to 0,$$

where

$$(2.3) \qquad g(u) \equiv g(x, u) = f(x+u) + f(x-u) - 2uf'(x),$$

$f'(x)$ being the first generalized derivative of $f(t)$ at $t = x$, and furthermore

$$(2.4) \qquad L(u) \geq \delta > 0 \quad (u > u_o), \quad \int^{\infty} \frac{du}{uL(u)} < \infty .$$

Then the series (1.22) for $t = x$ is summable $[\Lambda]$ to $f'(x)$, provided that $f'(x)$ exists.

THEOREM 2. Let the matrix $(\lambda_{n,k})$ be as in Theorem 1. Let $f(t)$ be a continuous function of bounded variation with period 2π, such that

$$\int_0^t |dh(u)| = o\left\{\frac{t}{L(1/t)}\right\} \quad \text{as } t \to 0,$$

where

$$h(u) = f(x+u) + f(x-u) - 2f(x),$$

and furthermore $L(u)$ satisfies the two conditions of Theorem 1. Then the series (1.23) for $t = x$ is summable $[\Lambda]$ to

$$H(x) = -\frac{1}{4\pi} \int_0^{\pi} h(x) \operatorname{cosec}^2 x/2 \; dx$$

provided that this integral exists at the lower limit in the usual Cauchy's sense.

In order to prove the theorems, we need the following lemma.

LEMMA. [3, theorem 6]. For a function $f(t)$ of bounded variation with period 2π, continuity is equivalent to

$$\lim_{n\to\infty} \sum_{k=0}^{n} |\lambda_{n,k}|\, k(|a_k| + |b_k|) = o$$

whenever $(\lambda_{n,k})$ is a regular sequence to sequence matrix satisfying condition (2.1).

3. Proof of theorem 1. If $S_n(x)$ denotes the sum of the first n terms of the series (1.22), at $t = x$, we get

$$S_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dx} \left\{\frac{\sin(n+1/2)(x-u)}{\sin (x-u)/2}\right\} f(u)\, du .$$

Following Prasad and Singh [1], we get

$$S_n(x) = \frac{1}{2\pi} \int_0^{\pi} \frac{\sin(n+1/2)t}{\sin t/2}\, dg(t) + f'(x) ,$$

where $f'(x)$ denotes the generalized differential coefficient of $f(t)$ at $t = x$, and g is given by (2.3).

Therefore, putting $\varepsilon_\nu = \varepsilon_\nu(x) = \pm 1$ in such a way as to make $\varepsilon_\nu \{S_\nu(x) - f'(x)\} \geq 0$ for $\nu = 1, 2, 3, \ldots,$ we have

$$\sum_{\nu=1}^{n} \lambda_{n,\nu} |S_\nu(x) - f'(x)|$$

(3.1) $$= \frac{1}{2\pi} \int_0^{\pi} \sum_{\nu=1}^{n} \varepsilon_\nu \lambda_{n,\nu} \frac{\sin(\nu+1/2)t}{\sin t/2}\, dg(t)$$

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$$= \frac{1}{2\pi} \int_0^\pi \cot t/2 \sum_{\nu=1}^{n} \lambda_{n,\nu} \varepsilon_\nu \sin \nu t \; dg(t)$$

$$+ \frac{1}{2\pi} \int_0^\pi \sum_{\nu=1}^{n} \lambda_{n,\nu} \varepsilon_\nu \cos \nu t \, dg(t)$$

$$= P + Q , \text{ say.}$$

Following Prasad and Singh [1, p.283] and [2]

$$\left| \frac{1}{\pi} \int_0^\pi \cos \nu t \, dg(t) \right| \le \nu(|a_\nu| + |b_\nu|) + o(1) \quad \text{as } \nu \to \infty ,$$

where a_ν, b_ν are the Fourier coefficients of $f(t)$, so that

$$|Q| \le \frac{1}{2} \sum_1^n \frac{1}{\pi} \left| \int_0^\pi \cos \nu t \, dg(t) \right| |\lambda_{n,\nu}|$$

$$\le \frac{1}{2} \sum_1^n |\lambda_{n,\nu}| \, \nu(|a_\nu| + |b_\nu|) + o\left(\sum_1^n |\lambda_{n,\nu}| \right) \tag{3.2}$$

$= o(1)$ as $n \to \infty$, by the lemma and by regularity of $(\lambda_{n,k})$.

Let $\varepsilon > o$ be given; then by (2.2), δ' can be chosen in $o < \delta' < \pi$ and sufficiently small that

$$G(t) \, L(\tfrac{1}{t}) < \varepsilon t \quad \text{for} \quad o < t \le \delta' . \tag{3.3}$$

To estimate P, we then write

$$P = \frac{1}{2\pi} \left(\int_o^{1/n} + \int_{1/n}^{\delta'} + \int_{\delta'}^{\pi} \right) \cot t/2 \; \lambda_n(t) \, dg(t)$$

$$\equiv \frac{1}{2\pi} (P_1 + P_2 + P_3),$$

where

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(3.4) $$\lambda_n(t) = \sum_{\nu=1}^{n} \varepsilon_\nu \lambda_{n,\nu} \sin \nu t .$$

But

$$|\lambda_n(t)| \leq \sum_{\nu=1}^{n} |\lambda_{n,\nu}| \, \nu \, |\sin t|$$

$$\leq n \, |\sin t| \sum_{\nu=1}^{n} |\lambda_{n,\nu}|$$

$$\leq Mn \, |\sin t|, \text{ by regularity of } (\lambda_{n,k}) .$$

Therefore

$$|P_1| \leq M \int_0^{1/n} n \sin t \cot t/2 \, |dg(t)|$$

$$\leq 2M \int_0^{1/n} n \, |dg(t)|$$

(3.5) $$= 2Mn \;\; G(1/n)$$

$$= o\{\frac{1}{L(n)}\} , \quad \text{by (2.2)}$$

$$= o(1) \text{ as } n \to \infty, \text{ by (2.4)} .$$

Also (3.4), and the regularity of $(\lambda_{n,k})$, gives at once

$$|\lambda_n(t)| \leq \sum_{\nu=1}^{n} |\lambda_{n,\nu}| \leq M ,$$

so that

$$|P_2| \leq \frac{M}{2\pi} \int_{1/n}^{\delta'} \cot t/2 \;\; |dg(t)| .$$

Noting that $t^2/2 \;\; \mathrm{cosec}^2 \; t/2 < 5$ for $o < t < \pi$, and using (2.4) and (3.3), integration by parts gives, for all $n > \frac{1}{\delta'}$,

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$$\int_{1/n}^{\delta'} \cot t/2 \; |dg(t)|$$

$$= [G(t) \cot t/2]_{1/n}^{\delta'} + \int_{1/n}^{\delta'} \frac{1}{2} \operatorname{cosec}^2 \frac{t}{2} \, G(t) \, dt$$

$$< G(\delta') \cot \delta'/2 - G(\frac{1}{n}) \cot \frac{1}{2n} + 5\varepsilon \int_{1/n}^{\delta'} \frac{dt}{tL(\frac{1}{t})}$$

$$< \frac{2}{\delta'} G(\delta') \frac{L(1/\delta')}{L(1/\delta')} + K\varepsilon$$

$$< \frac{2}{\delta} \varepsilon + K\varepsilon \; ;$$

thus

(3.6) $|P_2| < K_1 \varepsilon$ for fixed δ' and for any $n > 1/\delta'$.

Lastly define $\beta(t)$ to be an even function, vanishing for $o \leq t \leq \delta'$, and such that

$$P_3 = \frac{1}{2\pi} \int_{\delta'}^{\pi} \cot t/2 \; \lambda_n(t) \, dg(t) = \frac{1}{2\pi} \int_{o}^{\pi} \lambda_n(t) \, d\beta(t) \, ;$$

thus $\beta(t)$ is a continuous function of bounded variation in $[-\pi, \pi]$, and

$$|P_3| \leq \frac{1}{2\pi} \sum_{\nu=1}^{n} |\lambda_{n,\nu}| \, \left| \int_{o}^{\pi} \sin \nu t \, d\beta(t) \right|$$

$$\leq \frac{1}{2\pi} \sum_{\nu=1}^{n} |\lambda_{n,\nu}| \, \nu(|a'_\nu| + |b'_\nu|) \, ,$$

where a'_ν, b'_ν are the Fourier coefficients of $\beta(t)$. It then follows by the lemma that

(3.7) $P_3 = o(1)$ as $n \to \infty$, for a fixed δ'.

Substitution of (3.2), (3.5), (3.6), (3.7) into (3.1), and reference to the definition of strong summability in (1.12), completes the proof of theorem 1.

We omit the proof of the theorem 2.

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