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A Sharp Constant for the Bergman Projection

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Abstract. For the Bergman projection operator P we prove that

$$\|P\colon L^1(B, d\lambda) \to B_1\| = \frac{(2n+1)!}{n!}.$$

Here λ stands for the hyperbolic metric in the unit ball B of $\mathbb{C}^n$, and B_1 denotes the Besov space with an adequate semi-norm. We also consider a generalization of this result. This generalizes some recent results due to Perälä.

1 Introduction

This paper deals with the Bergman projection operator P_σ, which is an integral operator with the kernel

$$K_\sigma(z, w) = \frac{(1 - |w|^2)^\sigma}{(1 - \langle z, w\rangle)^{n+1+\sigma}},$$

i.e.,

$$P_\sigma f(z) = \int_B K_\sigma(z, w)\, f(w)\, dv(w), \quad z \in B$$

(for suitable f); here dv stands for the volume measure in $\mathbb{C}^n$ normalized in the unit ball B. We will not normalize this operator such that $P_\sigma 1 = 1$, since our aim is to consider more than the case where $\sigma > -1$.

Let λ stand for the hyperbolic metric in the unit ball; *i.e.*, let

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}.$$

Under a reasonable set of assumptions, the Besov space B_1 is the smallest Moebius invariant Banach space. In the same scale, the Bloch space is the maximal Moebius invariant space, and one often writes B_∞ for that space. For this and related results we refer to Zhu [7]; in particular, see Theorems 6.8 and 6.10. The space B_1 can be alternatively defined in terms of the semi-norm. We will consider the following semi-norm on B_1. For $f \in B_1$, let

$$\|f\|_{B_1} = \sum_{|\alpha|=n+1} \int_B \Big| \frac{\partial^{n+1} f(z)}{\partial^\alpha z} \Big|\, dv(z).$$

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In this paper we find the semi-norm $\|P_\sigma\colon L^1(B, d\lambda) \to B_1\|$ for $\sigma > -(n+1)$, *i.e.*, the smallest constant C such that

$$\|P_\sigma f\|_{B_1} \le C\|f\|_{L^1(B,d\lambda)}, \quad f \in B_1.$$

The following theorem is our main result.

Theorem 1.1 *The operator P_σ maps the space $L^1(B, d\lambda)$ continuously onto the Besov space B_1 if and only if $\sigma > -(n+1)$. In this case we have*

$$\|P_\sigma\colon L^1(B, \lambda) \to B_1\| = n!\,\frac{\Gamma(\mu + n + 1)}{\Gamma^2(\frac{\mu}{2} + \frac{n}{2} + \frac{1}{2})},$$

where $\mu = n + 1 + \sigma$.

Remark 1.2 According to this result, for the ordinary Bergman projection $P = P_0$ we therefore have

$$\|P\colon L^1(B, d\lambda) \to B_1\| = \frac{(2n+1)!}{n!}.$$

In particular, for $n = 1$ we have $\|P\| = 6$, as Perälä has recently showed [5].

Remark 1.3 Results concerning the semi-norm calculation of the operator $c_\sigma P_\sigma$ for $\sigma > -1$, where $c_\sigma = \Gamma(n + \sigma + 1)/\Gamma(\sigma + 1)\Gamma(n + 1)$ is a normalizing constant, when acting from the space $L^\infty(B)$ onto the Bloch space in the unit ball, can be found in [2, 3] and in the author's recent preprint [4].

2 Proof of Theorem 1.1

In order to prove our main theorem, we need some auxiliary results. These will be collected in the lemmas that follow. Some of the facts we will present can be found in the Rudin monograph [6]

It is well known that, up to unitary transformations, bi-holomorphic mappings of B onto itself have the form

$$\varphi_z(\omega) = \frac{z - \frac{\langle \omega, z\rangle}{|z|^2} z - (1 - |z|^2)^{1/2}(\omega - \frac{\langle \omega, z\rangle}{|z|^2} z)}{1 - \langle \omega, z\rangle}$$

for some $z \in B$.

The known identity

$$\big|1 - \langle z, \omega\rangle\big|\,\big|1 - \langle z, \varphi_z(\omega)\rangle\big| = 1 - |z|^2 \tag{2.1}$$

for $z,\ \omega \in B$, will be useful in the following lemma.

Lemma 2.1 *For every $z \in B$ we have*

$$\int_B \frac{(1 - |z|^2)^a}{|1 - \langle z, w\rangle|^{a+n+1}}\, dv(w) = \int_B \frac{dv(\omega)}{|1 - \langle z, \omega\rangle|^{n+1-a}},$$

where a is any real number.

Proof The real Jacobian of $\varphi_z(\omega)$ is given by the expression

$$(J_{\mathbb{R}}\varphi_z)(\omega) = \frac{(1-|z|^2)^{n+1}}{|1-\langle z,\omega\rangle|^{2n+2}}.$$

Denote the integral on the left side of our lemma by J. Introducing the change of variables $w = \varphi_z(\omega)$ and using the previous relation for the pull-back measure, we obtain

$$\begin{aligned} J &= \int_B \frac{(1-|z|^2)^a}{|1-\langle z,\varphi_z(\omega)\rangle|^{a+n+1}} \frac{(1-|z|^2)^{n+1}}{|1-\langle z,\omega\rangle|^{2n+2}}\,dv(\omega) \\ &= \int_B \frac{(1-|z|^2)^{a+n+1}}{|1-\langle z,\varphi_z(\omega)\rangle|^{a+n+1}|1-\langle z,\omega\rangle|^{2n+2}}\,dv(\omega) \\ &= \int_B \frac{(|1-\langle z,\omega\rangle||1-\langle z,\varphi_z(\omega)\rangle|)^{a+n+1}}{|1-\langle z,\varphi_z(\omega)\rangle|^{a+n+1}|1-\langle z,\omega\rangle|^{2n+2}}\,dv(\omega) \\ &= \int_B \frac{1}{|1-\langle z,\omega\rangle|^{n+1-a}}\,dv(\omega). \end{aligned}$$

In third equality we have used identity (2.1) ∎

For the next lemma, see [6, Proposition 1.4.10] as well as [2]. For the properties of the Gamma function and the Gauss hypergeometric functions, we refer the reader to [1].

Lemma 2.2

(i) *For $z \in B$, c real, $t > -1$, define*

$$J_{c,t}(z) = \int_B \frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}}\,dv(w).$$

When $c < 0$, then $J_{c,t}(z)$ is bounded in B.

Moreover, $J_{c,t}$ is radially symmetric and increasing in $|z|$, since it can be represented as

$$J_{c,t}(z) = \frac{\Gamma(n+1)\Gamma(t+1)}{\Gamma^2(\lambda)} \sum_{k=0}^{\infty} \frac{\Gamma^2(\lambda+k)}{\Gamma(k+1)\Gamma(n+1+t+k)}|z|^{2k},$$

where $\lambda = \frac{n}{2}+\frac{1}{2}+\frac{t}{2}+\frac{c}{2}$.

(ii) *Furthermore, we can write $J_{c,t}(z)$ in the closed form as*

$$J_{c,t}(z) = \frac{\Gamma(n+1)\Gamma(t+1)}{\Gamma(n+1+t)}\,{}_2F_1(\lambda,\lambda,n+1+t,|z|^2),$$

where ${}_2F_1$ is the Gauss hypergeometric function. In particular, if $c < 0$, then

$$J_{c,t}(e_1) = \frac{\Gamma(n+1)\Gamma(t+1)\Gamma(-c)}{\Gamma^2(\frac{n}{2}+\frac{1}{2}+\frac{t}{2}-\frac{c}{2})},$$

where $e_1 = (1,0,\dots,0) \in \mathbb{C}^n$.

Proof of Theorem 1.1 Since for every $\alpha \in \mathbb{N}^{n+1}$ that satisfies $|\alpha| = n+1$ we have

$$\frac{\partial^{n+1}P_\sigma f(z)}{\partial^\alpha z} = \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)} \int_B \frac{\overline{w}^\alpha(1-|w|^2)^\sigma}{(1-\langle z,w\rangle)^{\mu+n+1}} f(w)\,dv(w),$$

we should consider the operators given by

$$\widetilde{P}_{\sigma,\alpha} f(z) = \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)} \int_B \frac{\overline{w}^{\alpha}(1-|w|^2)^{\sigma}}{(1-\langle z,w\rangle)^{\mu+n+1}} f(w)\,d\nu(w).$$

Let

$$\widetilde{P}_{\sigma} = (\underbrace{\dots,\widetilde{P}_{\sigma,\alpha},\dots}_{d}),$$

where $d = \binom{2n}{n+1}$.

One readily sees that

$$\begin{aligned}
\|P_\sigma\colon L^1(B,\lambda)\to B_1\| &= \|\widetilde{P}_\sigma\colon L^1(B,\lambda)\to \bigotimes_{k=1}^{d} L^1(B)\| \\
&= \|\widetilde{P}_\sigma^{*}\colon \bigotimes_{k=1}^{d} L^\infty(B)\to L^\infty(B)\| \\
&= \max_{|\alpha|=n+1} \|\widetilde{P}_{\sigma,\alpha}^{*}\colon L^\infty(B)\to L^\infty(B)\|.
\end{aligned} \tag{2.2}$$

We will therefore find the conjugate operator $\widetilde{P}_{\sigma,\alpha}^{*}\colon L^\infty(B)\to L^\infty(B)$.

The conjugate operator is

$$\widetilde{P}_{\sigma,\alpha}^{*} g(z) = \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)}\, z^{\alpha} \int_B \frac{(1-|z|^2)^{\mu}}{(1-\langle z,w\rangle)^{\mu+n+1}} g(w)\,d\nu(w). \tag{2.3}$$

To see that (2.3) holds, let $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the scalar product in $L^2(B,d\nu)$ and let $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ denote the scalar product in $L^2(B,d\lambda)$. Then for $f\in L^1(B,d\lambda)$ and $g\in L^\infty(B)$ we have

$$\begin{aligned}
\langle \widetilde{P}_{\sigma,\alpha} f, g\rangle &= \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)} \int_B d\nu(z)\,\overline{g(z)} \int_B \overline{w}^{\alpha} \frac{(1-|w|)^{\sigma}}{(1-\langle z,w\rangle)^{\mu+n+1}} f(w)\,d\nu(w) \\
&= \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)} \int_B f(w)\,\overline{w}^{\alpha}(1-|w|)^{\sigma}\,d\nu(w) \int_B \frac{\overline{g(z)}\,d\nu(z)}{(1-\langle z,w\rangle)^{\mu+n+1}} \\
&= \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)} \int_B f(w)\,\overline{w}^{\alpha}\,(1-|w|)^{\mu}\,d\lambda(w) \int_B \frac{\overline{g(z)}\,d\nu(z)}{(1-\langle z,w\rangle)^{\mu+n+1}} \\
&= \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)} \int_B f(w)\,\overline{w^{\alpha}(1-|w|)^{\mu} \int_B \frac{g(z)\,d\nu(z)}{(1-\langle w,z\rangle)^{\mu+n+1}}}\,d\lambda(w) \\
&= \langle f, \widetilde{P}_{\sigma,\alpha}^{*} g\rangle_\lambda,
\end{aligned}$$

where

$$\widetilde{P}_{\sigma,\alpha}^{*} g(w) = \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)}\, w^{\alpha} \int_B \frac{(1-|w|^2)^{\mu}}{(1-\langle w,z\rangle)^{\mu+n+1}} g(z)\,d\nu(z).$$

Let us now find $\|\widetilde{P}_{\sigma,\alpha}^{*}\colon L^\infty(B)\to L^\infty(B)\|$.

For fixed z (and α) the maximum of the integral expression in (2.3) (regarding $g\in L^\infty(B)$, $\|g\|_\infty = 1$) is attained for $g_z\in L^\infty(B)$ given by

$$g_z(w) = \frac{|1-\langle z,w\rangle|^{\mu+n+1}}{(1-\overline{\langle z,w\rangle})^{\mu+n+1}}.$$

Note that $\|g_z\|_\infty = 1$.

Therefore, we have (for fixed z and α)

$$|\widetilde{P}^*_{\sigma,\alpha} g(z)| \le \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)}\, |z|^\alpha \int_B \frac{(1-|z|^2)^\mu}{|1-\langle z,w\rangle|^{\mu+n+1}}\, dv(w)$$

for all $g \in L^\infty(B)$, $\|g\|_\infty \le 1$.

It follows that

$$\|\widetilde{P}^*_{\sigma,\alpha}\colon L^\infty(B) \to L^\infty(B)\| \le \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)} \sup_{z\in B} \int_B \frac{(1-|z|^2)^\mu}{|1-\langle z,w\rangle|^{\mu+n+1}}\, dv(w)$$

for all $\alpha \in \mathbb{N}^n$, $|\alpha| = n+1$.

We transform the last integral as

$$\int_B \frac{(1-|z|^2)^\mu}{|1-\langle z,w\rangle|^{\mu+n+1}}\, dv(w) = \int_B \frac{dv(\omega)}{|1-\langle z,\omega\rangle|^{n+1-\mu}} = J_{-\mu,0}(z)$$

(see Lemma 2.1).

Regarding the first part of Lemma 2.2 the number $\sup_{z\in B} J_{-\mu,0}(z)$ is finite if $-\mu < 0$, *i.e.*, $\sigma > -(n+1)$. In this case, by the second part of this lemma, we can write

$$\sup_{z\in B} J_{-\mu,0}(z) = \frac{\Gamma(n+1)\Gamma(\mu)}{\Gamma^2(\frac{\mu}{2}+\frac{n}{2}+\frac{1}{2})}.$$

Therefore, we have

$$\begin{aligned}\|\widetilde{P}^*_{\sigma,\alpha}\colon L^\infty(B) \to L^\infty(B)\| &\le \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)} \frac{\Gamma(n+1)\Gamma(\mu)}{\Gamma^2(\frac{\mu}{2}+\frac{n}{2}+\frac{1}{2})}\\ &= n!\, \frac{\Gamma(\mu+n+1)}{\Gamma^2(\frac{\mu}{2}+\frac{n}{2}+\frac{1}{2})}.\end{aligned}$$

Regarding the relation (2.2) for $\sigma > -(n+1)$, we obtain

$$\|P_\sigma\colon L^1(B,\lambda) \to B_1\| \le n!\, \frac{\Gamma(\mu+n+1)}{\Gamma^2(\frac{\mu}{2}+\frac{n}{2}+\frac{1}{2})},$$

which gives one part of our theorem.

We will now prove the reverse inequality and that the condition $\sigma > -(n+1)$ is necessary for boundedness of P_σ on $L^1(B, d\lambda)$. We also use relation (2.2).

For $\varepsilon \in (0,1)$ denote

$$g_\varepsilon(w) = \frac{|1-\langle \varepsilon e_1, w\rangle|^{\mu+n+1}}{(1-\overline{\langle \varepsilon e_1, w\rangle})^{\mu+n+1}}.$$

Then $g_\varepsilon \in L^\infty(B)$, $\|g_\varepsilon\|_\infty = 1$ and for any $|\alpha| = n+1$, we have

$$\begin{aligned}\widetilde{P}^*_{\sigma,\alpha} g_\varepsilon(\varepsilon e_1) &= \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)}\, \varepsilon^{n+1} \int_B \frac{(1-\varepsilon^2)^\mu}{|1-\langle \varepsilon e_1, w\rangle|^{\mu+n+1}}\, dv(w)\\ &= \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)}\, \varepsilon^{n+1} \int_B \frac{dv(w)}{|1-\langle \varepsilon e_1, w\rangle|^{n+1-\mu}}\\ &= \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)}\, \varepsilon^{n+1}\, J_{-\mu,0}(\varepsilon e_1).\end{aligned}$$

It follows that

$$\begin{aligned}
\|\widetilde{P}^*_{\sigma,\alpha}\colon L^\infty(B)\to L^\infty(B)\| &\ge \sup_{z\in B}|\widetilde{P}^*_{\sigma,\alpha}g_\varepsilon(z)| \ge \limsup_{\varepsilon\to 1}|\widetilde{P}^*_{\sigma}g_\varepsilon(\varepsilon e_1)|\\
&= \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)}\lim_{\varepsilon\to 1} J_{-\mu,0}(\varepsilon e_1)\\
&= \frac{\Gamma(\mu+n+1)}{\Gamma(\mu)}\frac{\Gamma(n+1)\Gamma(\mu)}{\Gamma^2(\frac{\mu}{2}+\frac{n}{2}+\frac{1}{2})} = n!\,\frac{\Gamma(\mu+n+1)}{\Gamma^2(\frac{\mu}{2}+\frac{n}{2}+\frac{1}{2})},
\end{aligned}$$

only for $\sigma > -(n+1)$.

Thus, in view of (2.2) we have

$$\|P_\sigma\colon L^1(B,\lambda)\to B_1\| \ge n!\,\frac{\Gamma(\mu+n+1)}{\Gamma^2(\frac{\mu}{2}+\frac{n}{2}+\frac{1}{2})}$$

for $\sigma > -(n+1)$. ■

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