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DIVISIBLE SEMIPLANES, ARCS, AND RELATIVE DIFFERENCE SETS

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0. Introduction. In this paper we shall be concerned with arcs of divisible semiplanes. With one exception, all known divisible semiplanes **D** (also called "elliptic" semiplanes) arise by omitting the empty set or a Baer subset from a projective plane Π, i.e., $\mathbf{D} = \Pi \backslash S$, where S is one of the following:

(i) S is the empty set.

(ii) S consists of a line L with all its points and a point p with all the lines through it.

(iii) S is a Baer subplane of Π.

We will introduce a definition of "arc" in divisible semiplanes; in the examples just mentioned, arcs of **D** will be arcs of Π that interact in a prescribed manner with the Baer subset S omitted. The precise definition (to be given in Section 2) is chosen in such a way that divisible semiplanes admitting an abelian Singer group (i.e., a group acting regularly on both points and lines) and then a relative difference set D will always contain a large collection of arcs related to D (to be precise, $-D$ and all its translates will be arcs). This seems worth studying since the use of such arcs has already provided interesting non-existence results for ordinary difference sets [**17**] and for relative difference sets corresponding to divisible semiplanes defined by Baer subsets of type (ii) above ([**14**] and [**15**]). Thus the author expected the study of arcs in divisible semiplanes in general to be interesting per se, and to provide a common setting for the results mentioned above. After reviewing some basic definitions and results in Section 1, we will define arcs in Section 2; first properties will be investigated leading to a characterization of those divisible semiplanes admitting "hyperovals". In Section 3, we will discuss the connection to relative difference sets. After giving examples of maximal arcs in divisible semiplanes arising from a Baer subset of type (ii) in Section 4, we shall concentrate on type (iii) semiplanes in the remaining five sections. As will become clear, the existence problem for large arcs in such semiplanes poses interesting questions regarding the interaction of Baer subplanes and ovals (resp. conics) in both Desarguesian and non-Desarguesian planes. It also leads to a possible geometric attack on a

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non-existence proof for the corresponding relative difference sets, a problem studied by algebraic means by Ganley and Spence [**8**]. Specifically we will give a conjecture (and some evidence for its truth) that would imply the non-existence for relative difference sets corresponding to type (iii), with the exception of $\Pi = PG(2, 4)$.

1. Divisible semiplanes and relative difference sets. Let **D** be an incidence structure consisting of mn points and mn lines, split into m point classes (resp. parallel classes) of n points (resp. lines) each. If any two points p, q in the same point class (written $p \sim q$) are not joined at all, whereas any two points in distinct classes are joined by precisely one line, if each point is on exactly k lines, and if the dual conditions likewise hold, we call **D** a *divisible semiplane* with parameters n, m, k. (Many authors use the term "elliptic semiplane" instead. Note, however, that divisible semiplanes are a special case of divisible designs in general (see e.g. [**13**]), which is the reason for our terminology.) An easy counting argument shows that $k(k - 1) = n(m - 1)$.

With the exception of an isolated example of Baker [**0**] (with parameters $n = 3$, $m = 15$ and $k = 7$), all known divisible semiplanes arise in the following way [**3**]: Let Π be a projective plane of order q, and omit from Π a set S of one of the following types:

(i) S is the empty set.

(ii) S consists of a line L together with all its points, and a point p together with all the lines through p.

(iii) S is a Baer subplane of Π, i.e., a subplane of order $\sqrt{q}$.

The resulting structure **D** then is a divisible semiplane. Of course, type (i) just yields $\mathbf{D} = \Pi$ again, i.e., the projective plane Π itself is a divisible semiplane with parameters $n = 1$, $m = q^2 + q + 1$, $k = q + 1$. We will not study this case in the present paper. In case (ii), we have to distinguish between the subcases $p \in \mathrm{L}$ and $p \notin \mathrm{L}$, respectively. If $p \in \mathrm{L}$, we obtain a *symmetric net* of order q, i.e., a divisible semiplane with parameters $n = m = k = q$. If $p \notin \mathrm{L}$, we obtain a *biaffine plane* of order q, i.e., a divisible semiplane with parameters $n = q - 1$, $m = q + 1$, $k = q$. Finally, in case (iii) **D** is a *Baer semiplane* of order q, i.e., a divisible semiplane with parameters $n = q - \sqrt{q}$, $m = q + \sqrt{q} + 1$ and $k = q$. It is well-known that Π may be reconstructed uniquely from **D** in all of these cases (see [**3**]).

We mention in passing that Dembowski also proved that any divisible semiplane **D** which does not arise from a projective plane necessarily satisfies $n < k - \sqrt{k}$; also, a necessary condition for the existence of such an example is the existence of a symmetric design (see e.g. [**1**] for background on designs) with parameters (m, k, n) (defined on the point classes and parallel classes). For example, Baker's semiplane [**0**] corresponds to a symmetric (15, 7, 3)-design.

We will be interested in divisible semiplanes **D** admitting a *Singer group*, i.e., a group G acting regularly (= sharply 1-transitively) on both the point and line set of **D**. It is well-known that such a semiplane may be represented as the *development*

$$\text{dev } D = (G, \{D + x : x \in G\}, \in)$$

of a relative difference set (RDS) D in G. This is a k-subset of G for which the $k(k - 1)$ differences $d - d'$ ($d, d' \in D$, $d \neq d'$) are pairwise distinct and cover all elements of G excepting those in a certain subgroup N of order n. We will restrict ourselves to the case of abelian RDS's in this paper. (There is a more general notion of RDS, corresponding to divisible designs in general. See, e.g., [**6**] and [**13**].) Then the point classes of **D** are the cosets $N + x (x \in G)$ of N. For proofs, see e.g. [**13**].

Finally, let us mention the known examples of relative difference sets for divisible semiplanes. As Baker's example does not admit such a representation, we only encounter examples corresponding to semiplanes arising from projective planes.

(i) Projective planes. The Desarguesian plane $PG(2, q)$ admits a cyclic Singer group; indeed, this classical result of Singer [**24**] has provided the reason for present day terminology.

(iia) Symmetric nets. Each projective plane over a commutative semifield (= division ring in the terminology of [**12**]) gives rise to a symmetric net **D** admitting a commutative Singer group G. If **D** has odd order, G is elementary abelian; otherwise G is isomorphic to a direct sum of groups Z_4. These results are due to Hughes [**10**]; cf. also [**13**] for simple proofs.

(iib) Biaffine planes. The Desarguesian plane $PG(2, q)$ yields the biaffine plane $BAG(2, q)$ admitting a cyclic Singer group. This is due to Bose [**2**]; see also [**13**] for a more general result.

(iii) Baer semiplanes. The only known example is described by the RDS with parameters $n = 2$, $m = 7$, $k = 4$ in Z_{14}: $D = \{0, 1, 4, 6\}$. This corresponds to $\mathbf{D} = PG(2, 4)\backslash PG(2, 2)$. (There are, however, non-abelian examples for $q = 9$ and $q = 16$, see Section 8 and [**13**, Corollary 4.10].)

We shall say more about these examples and discuss some necessary conditions later, as far as they are related to arcs in the corresponding semiplanes. But first we will have to study the basic properties of arcs in divisible semiplanes. According to a result of Lam [**20**], all cyclic relative difference sets with $n \neq 1$ and $k \leqq 50$ are of one of the types discussed above. A final remark: Relative difference sets yielding semiplanes belonging to a projective plane are equivalent to certain quasiregular collineation groups of such planes. (A group G is called *quasiregular* if, for each orbit, each group element fixes either none or all of the elements in the orbit.) See [**3**] and [**8**].

2. Arcs and ovals in divisible semiplanes. Recall that an arc in a projective plane is a subset A such that no line intersects A in more than two points. We will use a similar definition for semiplanes. Thus let **D** be a divisible semiplane, and let A be an h-subset of the points of **D**. A is called an *arc* (more precisely an *h-arc*), provided that

(A1) $|A \cap L| \leqq 2$ for each line L

and

(A2) $|A \cap P| \leqq 1$ for each point class P.

Equivalently, (A2) requires that any two points of A are joined in **D**. Condition (A1) requires no further comment, whereas (A2) is not quite as natural. Clearly, one would not want to consider the point classes arcs; so some condition is needed. If **D** belongs to a projective plane Π, the point classes correspond to lines of Π; so one might consider replacing (A2) by the weaker condition $|A \cap P| \leqq 2$. As we will see, (A2) is satisfied by the examples of arcs arising from relative difference sets, which is our motivation for requiring this stronger condition. The following lemma is similar to well-known results for projective planes, except for parts (iii) and (iv) which are immediate consequences of (A2). Thus we will leave the details of the proof to the reader.

2.1 LEMMA. *Let A be an h-arc in a divisible semiplane* **D** *with parameters n, m, k. Then the following hold*:

(i) $h \leqq k + 1$ *with equality if and only if* $|A \cap L| \in \{0, 2\}$ *for each line L.*

(ii) *The number of tangents* (*i.e., lines L with* $|A \cap L| = 1$) *to A in a point p of A is* $k - h + 1$.

(iii) *For each point* $p \notin A$, *there is at most one point* $p' \in A$ *with* $p \sim p'$.

(iv) *If* **D** *arises from a projective plane* Π, *then A is also an arc of* Π.

As usual, we will call a k-arc an *oval* and a $(k + 1)$-arc a *hyperoval* of **D**. Not surprisingly, hyperovals can only arise in very special situations:

2.2 PROPOSITION. *Let A be a hyperoval of a divisible semiplane* **D** *with parameters n, m, k. Then either*

(i) **D** *is a projective plane of even order* $k - 1$ *or*

(ii) **D** *is a biaffine plane of even order k.*

Proof. Case 1. k is odd, i.e., $k + 1$ is even. Let p be any point not on A. By Lemma 2.1 (i), all lines through p meeting A meet A in precisely two points. By 2.1 (iii) this shows that p is joined to all points of A. So each point on A has to form a point class by itself, i.e., $n = 1$, and **D** is a projective plane. A well-known result of Quist [**22**] (see also [**9**] or [**12**]) states that this can only happen if **D** has even order.

Case 2. k is even. Similar arguments as in Case 1 show that now for each point $p \notin A$ there is a unique point $p' \in A$ with $p \sim p'$; hence $n \neq 1$. Note that in this way each point class meets A in exactly one point. Thus $m = k + 1$, hence $n = k - 1$ from $k(k - 1) = n(m - 1)$, and **D** is a biaffine plane of even order k.

Generalizing another result of [**22**], we now show that each oval in a (projective or) biaffine plane of even order extends to a hyperoval. This fact was crucial in the arguments of [**15**] and [**17**] concerning ovals related to relative difference sets.

2.3 PROPOSITION. *Let A be an oval in a projective or biaffine plane* **D** *of even order. Then A is contained in a unique hyperoval of* **D**, *obtained by adding the nucleus of A (i.e., the point of intersection of all tangents of A).*

Proof. For projective planes, this is Quist's result [**22**]. Thus let **D** be a biaffine plane of even order k. Let Π be the projective plane belonging to **D**. By Lemma 2.1 (iv), A is a k-arc of Π. Also, by (A2) the point of intersection of all point classes of **D** (considered as lines of Π) may be added to A to obtain an oval A' of Π. By Quist's theorem, A' extends to a hyperoval H of Π by adjoining its nucleus p; since the infinite line L_∞ of Π (the line deleted with all its points in defining **D**) has to meet H twice or not at all and since $A' \cap L_\infty = \emptyset$, we see that $p \notin L_\infty$, i.e., $p \in$ **D**. Thus $A \cup \{p\}$ is the desired hyperoval of **D**.

As for examples of arcs and ovals in semiplanes, it is easy to construct such sets in the symmetric net or biaffine plane case by using suitable ovals of the corresponding projective plane; we shall do this in Section 4. As we will see in the later sections, the Baer semiplane case is totally different, posing considerable difficulties. But first we will describe the connection between relative difference sets and ovals.

3. Relative difference sets and ovals. The following result generalizes Lemma 2.1 of [**17**]; other special cases have been used by Jungnickel [**14**] and [**15**].

3.1 PROPOSITION. *Let D be an abelian relative difference set in G for a divisible semiplane* **D** $=$ dev D *with parameters n, m, k. Then $-D + y$ is an oval for each $y \in G$, and the tangent to $-D + y$ at $-d + y$ is the line $D - 2d + y$.*

Proof. The proof of [**17**] carries over to the present situation, giving the validity of condition (A1). Now assume $-d + y, -d' + y \in N + x$; then $d - d' \in N$ and thus $d = d'$. Hence each point class meets $-D + y$ at most once, proving the validity of condition (A2).

Proposition 3.1 can be used to obtain some non-existence results for abelian relative difference sets; see [**3**], [**7**], [**14**], [**15**], [**17**], [**18**] and [**19**] for such results in cases (i) and (ii). The implications of Proposition

3.1 for case (iii), i.e., relative difference sets describing a Baer semiplane, are discussed in the present paper.

We mention three general consequences of Proposition 3.1:

3.2 COROLLARY. *Let* **D** *be the divisible semiplane belonging to an abelian* RDS D. *Then the* mn *ovals* $-D + y$ ($y \in G$) *form a divisible semiplane isomorphic to* **D**.

3.3 COROLLARY. *Let* **D** *be the divisible semiplane belonging to an abelian* RDS D *with parameters* n, m, k. *Then* $\mathrm{dev}(D, -D)$ *is a point divisible design with parameters* n, m, k, $\lambda_1 = 0$, $\lambda_2 = 2$ *and intersection numbers* 0, 1, 2. *Such designs exist in the cases described in Section* 1.

3.4 COROLLARY. -1 *is never a multiplier of an abelian* RDS.

Note that 3.4 generalizes a well-known result of Johnsen: -1 is never a multiplier for a planar abelian difference set. The relative difference sets described in Section 1 thus furnish us with a large supply of ovals. More examples will be constructed in the following sections. But first let us show how one can use Proposition 2.1 to reconstruct from the RDS $D = \{0, 1, 4, 6\} \subset Z_{14}$ the projective plane $\Pi \cong PG(2, 4)$ and the Baer subplane $\Pi_0 \cong PG(2, 2)$ omitted from Π in defining the divisible semiplane **D** with parameters $n = 2$, $m = 7$ and $k = 4$. This is both interesting in its own right and also quite instructive when contrasted to the results on Baer semiplanes in Section 5.

3.5 *Example*. Thus let $\mathbf{D} = \mathrm{dev}\{0, 1, 4, 6\}$ mod 14. By Proposition 3.1, $-D = \{0, 8, 10, 13\}$ is an oval with tangents D, $D - 12 = \{2, 3, 6, 8\}$, $D - 8 = \{6, 7, 10, 12\}$, and $D - 2 = \{2, 4, 12, 13\}$. By Lemma 2.1, $-D$ is a 4-arc in the projective plane $\Pi \cong PG(2, 4)$. Thus we will have 4 further tangents corresponding to the point classes of **D**: these are the lines partially given by $\{0, 7\}$, $\{1, 8\}$, $\{3, 10\}$, $\{6, 13\}$. Note that the 4 tangents D, $D - 12$, $D - 8$ and $\{6, 13\}$ are concurrent in 6. So $A = \{0, 6, 8, 10, 13\}$ is a 5-arc in Π (though of course not in **D**), and this extends to a hyperoval of Π. Hence the remaining 4 tangents $D - 2$ and $\{0, 7\}$, $\{1, 8\}$, $\{3, 10\}$ have to be concurrent in Π_0, say in a point x_0. Applying the elements in $G = Z_{14}$ to the line $\{2, 4, 12, 13, x_0\}$ of Π, we obtain the 14 lines of Π not contained in Π_0; here $x_0 + i = x_i$ (indices modulo 7). Also, we know that x_0 is on the lines $\{0, 7\}$, $\{1, 8\}$, and $\{3, 10\}$. Again applying G, we see that $\{0, 7\}$ contains the points x_0, x_4 and x_6. So the 7 lines of Π in Π_0 are just the images of $\{0, 7, x_0, x_4, x_6\}$ under Z_{14}. Note that this corresponds to the ordinary difference set $\{x_0, x_4, x_6\}$ mod 7 for $PG(2, 2)$. In fact, one may obtain G and D by using the Singer group Z_7 for $PG(2, 2)$ and extending it by its multiplier group $\langle 2 \rangle$; see [**13**, Proposition 4.9].

We shall consider a non-abelian RDS for a Baer semiplane of order 9 in Section 8; in particular, we will see that Proposition 3.1 does not hold for non-abelian groups.

4. Some examples of maximal arcs. In this section, we shall study some examples of maximal arcs in symmetric nets and in biaffine planes. The reader is referred to [**9**] for arcs in Desarguesian projective planes. Let us first consider the case of symmetric nets. Denote by $S(q)$ the symmetric net belonging to $PG(2, q)$. Note that this makes sense, as $PGL(3, q)$ is transitive on non-incident point-line pairs; thus all symmetric nets derived from $PG(2, q)$ are isomorphic. We already know that $S(q)$ contains an oval, by Proposition 3.1. Now choose any oval H of $PG(2, q)$, and a line L with $|L \cap H| = 2$. Using this line L and one of the points in $L \cap H$ for defining $S(q)$, it is easily seen that $A = H \backslash L$ is a $(q - 1)$-arc of $S(q)$. Now let q be odd. If A is not maximal in $S(q)$, we may choose a point $p \in S(q)$ such that $A' = A \cup \{p\}$ is an oval in $S(q)$. By Lemma 2.1, A' is also a q-arc of $PG(2, q)$. But A' intersects the oval H of $PG(2, q)$ in $q - 1$ points. For $q \geqq 7$, we have $q - 1 > (q + 3)/2$; by a result of [**22**] (see also [**21**, Theorem 16]), this implies $A' \subset H$, a contradiction. Thus A is indeed a maximal $(q - 1)$-arc of $S(q)$ for all odd values $q \geqq 7$. Now let B be any maximal $(q - 1)$-arc of $S(q)$; then B is a $(q - 1)$-arc of $PG(2, q)$ too. If $q \geqq 123$, then

$$q - 1 > q - \frac{1}{4}\sqrt{q} + \frac{7}{4};$$

then B extends to an oval H of $PG(2, q)$ by a famous result of Segre (see [**9**, Theorem 10.4.4]). Thus for $q \geqq 123$, the only maximal $(q - 1)$-arcs of $S(q)$ are those arising from an oval of $PG(2, q)$, as described above. Thus we have:

4.1 THEOREM. *$S(q)$ contains a maximal $(q - 1)$-arc A for all odd $q \geqq 7$. For $q \geqq 123$, any such arc extends to an oval H of $PG(2, q)$; one has $|H \cap L| = 2$, where L is the line of $PG(2, q)$ used in defining $S(q)$.*

Note that the first part of the result remains true for any projective plane of order q containing an oval H; of course, one has to use a line L with $|H \cap L| = 2$. Note further that the resulting symmetric net will, in general, depend on the choice of L. We also remark that the analogous construction for even q only yields ovals in the symmetric net. In fact, there is a deeper reason for this: By another result of Segre (see [**9**, Theorem 10.3.3]), each h-arc of $PG(2, q)$ (q even) with $h > q - \sqrt{q} + 1$ is contained in an oval, and thus in a hyperoval of $PG(2, q)$. By Proposition 2.1 (iv), each h-arc A of $S(q)$ (q even) with $h > q - \sqrt{q} + 1$ thus is contained in a hyperoval H of $PG(2, q)$. Because of Proposition 2.1 (i), the special line L used in defining $S(q)$ has to meet H. But then $|L \cap H| = 2$. Since A is an arc of $S(q)$, condition (A2) implies that the special point p is one of the two points in $L \cap H$. Hence $H \backslash L$ is an oval of $S(2, q)$ containing A. Thus we have:

4.2 THEOREM. *Let q be even. Then every h-arc of $S(q)$ with $h > q - \sqrt{q} + 1$ is contained in an oval of $S(q)$.*

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Note that the same type of argument does not work for odd q.

Similarly, we now consider $BAG(2, q)$, i.e., the biaffine plane derived from $\Pi = PG(2, q)$. First let q be odd and choose a conic C in Π. Choose the special line L_∞ and the special point ∞ for defining $\mathbf{D} = BAG(2, q)$ such that $|C \cap L_\infty| = 2$ and $\infty \in C$. Then $A = C \backslash L_\infty$ is a $(q - 2)$-arc in $\mathbf{D}$. If $q - 2 > (q + 3)/2$, A is a maximal arc of D; this again follows from the result of Quist [**22**], as in the symmetric net case considered previously. Thus A is maximal for $q \geqq 9$. Similarly, if we choose L_∞ to be a tangent of C and again $\infty \in C$, we obtain a $(q - 1)$-arc A' in $BAG(2, q)$; by the same argument as before, A' will be maximal for $q \geqq 7$. Finally, let q be even and consider a hyperoval H instead of C. Choose L_∞ with $|L_\infty \cap H| = 2$ and $\infty \in H$. This time we will obtain a $(q - 1)$-arc A in $BAG(2, q)$. If $q - 2 > (q + 2)/2$, A will be maximal; here one uses the same type of argument as before, only quoting [**21**, (Theorem 20)] instead of Quist. So A is maximal for $q \geqq 8$. Thus we have shown:

4.3 THEOREM. *$BAG(2, q)$ contains a maximal $(q - 1)$-arc whenever $q \geqq 7$; it also contains a maximal $(q - 2)$-arc if $q \geqq 9$ is odd.*

Of course, $BAG(2, q)$ also always contains an oval resp. hyperoval (for even q); this follows from Proposition 3.1 or by a direct argument as above, using for L_∞ a line which misses C respectively H. By the results of Segre already mentioned (see [**9**, Theorems 10.3.3 and 10.4.4]), we see that the construction given above is the only possible one if q is large enough. More specifically:

4.4 PROPOSITION. *If $q \geqq 8$ is even or $q \geqq 123$ is odd, then each maximal $(q - 1)$-arc of $BAG(2, q)$ is constructed as described above. If $q \geqq 227$ is odd, this also holds for maximal $(q - 2)$-arcs of $BAG(2, q)$.*

Note that Theorem 4.3 remains valid for any projective plane Π containing an oval or hyperoval, respectively; of course, for odd orders, the biaffine planes derived from Π and containing maximal $(q - 1)$- and $(q - 2)$-arcs, respectively, do not have to be isomorphic.

We have now seen how to use ovals of $PG(2, q)$ to construct large maximal arcs in $S(q)$ and $BAG(2, q)$, respectively. Next, let us consider how maximal arcs of $PG(2, q)$ (which are not ovals or hyperovals) may be used. Thus let A be a proper maximal arc of $PG(2, q)$, i.e., $|A| \leqq q$; then in fact $|A| < q$ by well-known results of Segre and Talini, see [**9**, Theorems 8.6.10 and 8.7.2]. Let $\mathbf{D}$ denote either $BAG(2, q)$ or $S(q)$. Even if A is contained in the point set of $\mathbf{D}$, A cannot be an arc of $\mathbf{D}$; for otherwise we could adjoint the special point p used in defining $\mathbf{D}$ to A to obtain a larger arc of $PG(2, q)$ by (A2). But if we choose p as one of the points of A, and if we choose the special line L as a tangent of A in p (for the symmetric net case) or as an exterior line of A (for the biaffine case), then $A \backslash \{p\}$ will always be a maximal arc of $\mathbf{D}$. Thus we have

4.5 PROPOSITION. *Let* $\mathbf{D} = BAG(2, q)$ *or* $\mathbf{D} = S(q)$ *and assume the existence of a maximal h-arc of* $PG(2, q)$. *Then D contains a maximal* $(h - 1)$-*arc.*

We refer the reader to **[9]** for examples of proper maximal arcs in $PG(2, q)$.

5. Arcs in Baer semiplanes: Upper bounds. In the remainder of this paper, we shall consider arcs in Baer semiplanes. This turns out to be considerably more difficult (and thus more interesting) than the study of arcs in symmetric nets or biaffine planes. Whereas we have seen in Section 4 that in these cases there is a fairly straightforward connection to the arcs and ovals of the corresponding projective plane, no comparable result exists for Baer semiplanes. In the present section we will show that any oval in a projective plane Π of order q^2 ($q \geqq 7$) can only contain h-arcs of a corresponding Baer semiplane $\mathbf{D}$ with $h < q^2 - q$. In particular, we will never obtain an oval of $\mathbf{D}$ in this way (unless $q = 2$); moreover no Desarguesian Baer semiplane of order $q^2 > 4$ contains any oval. In fact, for $q \neq 2$, no example of an oval in any Baer semiplane is known to the author. We begin with a simple but fundamental observation.

5.1 LEMMA. *Let* Π *be a projective plane of order* q^2 *and* Π_0 *a Baer subplane of* Π. *For each oval A of* Π, *at most* $q + 1$ *tangents of A belong to* Π_0.

Proof. Assume that $q + 2$ tangents $G_1, \ldots, G_{q+2}$ of A are in Π_0. First let q be even. Then all tangents pass through a common point, the nucleus p of A. Hence p is in Π_0; but no point of Π_0 is on $q + 2$ lines of Π_0, a contradiction. Now let q be odd. Then the dual Π_0^d of Π_0 contains $q + 2$ points (corresponding to the tangents $G_1, \ldots, G_{q+2}$) no three of which are collinear, since any exterior point of A in Π_0 is on precisely two tangents to A. Thus Π_0^d would contain a $(q + 2)$-arc, again a contradiction.

In the Desarguesian case, we will prove a stronger result in Section 7.

5.2 PROPOSITION. *Let* Π *be a projective plane of order* q^2, Π_0 *a Baer subplane of* Π *and* $\mathbf{D}$ *the corresponding Baer semiplane. Let A be an h-arc of* $\mathbf{D}$ *with* $h \geqq q^2 - q$, *and assume* $q \geqq 7$. *Then there is no oval of* Π *containing A.*

Proof. We may assume $h = q^2 - q$. If A is contained in an oval of Π, pick such an oval A'. Then $A'' = A' \setminus A$ is a $(q + 1)$-arc of Π which may be (partially) contained in Π_0. Since A is in $\Pi \setminus \Pi_0$, each point $x \in A$ is on a unique "special" line $G_x \in \Pi_0$. By condition (A2), none of these special lines can meet A in more than one point. Thus each special line is either a tangent of A' or meets both A and A''. By Proposition 5.1, at most $q + 1$ special lines can be tangents which leaves at least $q^2 - 2q - 1$ "special secants" meeting both A and A''. Clearly each point p of A'' can be on at most $q + 1$ special secants (since $p \in \Pi_0$ if p is on at least two such lines). Thus at least $q - 2$ of the points in A'' belong to Π_0. But now each point in $A'' \cap \Pi_0$ is on at least $q - 3$ lines in Π_0 joining it to the remaining points in $A'' \cap \Pi_0$. Hence each point in A'' is on at most 4 special secants. But for $q \geqq 7$, we have $4(q + 1) < q^2 - 2q - 1$, a final contradiction.

Before applying Proposition 5.2, we consider the remaining cases for q:

5.3 PROPOSITION. *With the notation of Proposition* 5.2, *one has* $h \leqq 20$ *for* $q = 5$; $h \leqq 12$ *for* $q = 4$; $h \leqq 8$ *for* $q = 3$; *and* $h \leqq 4$ *for* $q = 2$.

Proof. Again, let A be an h-arc of $\mathbf{D}$ contained in an oval A' of Π and write $A'' = A' \backslash A$. We will once more consider the number c of points in $\Pi_0 \cap A''$. Then each of the remaining $q + 1 - c$ points $x \in A \backslash \Pi_0$ is on a unique "special" line G_x of Π_0. We now have to use case distinction. First let $q = 5$ and assume $c = 1$. The unique point $p \in A'' \cap \Pi_0$ then is on at most 6 special lines; furthermore, by Proposition 5.1 we may also have at most 6 special tangents. These special lines G_x account for at most 12 points $x \in A$. The remaining 13 points of A' have to be on special lines G_x which are secants not containing p. So there are in fact at most 5 special tangents (since 13 is odd), and at least 7 secants G_x each of which can contain at most one point of A, by condition (A2). Hence

$$h = |A| \leqq 6 + 5 + 7 = 18.$$

Similarly, for $c \neq 1$ one obtains the following bounds:

c	$h \leqq$
	$6 + 10 = 16$
2	$2.5 + 6 + 4 = 20$
3	$3.4 + 5 + 3 = 20$
4	$4.3 + 6 + 2 = 20$
5	$5.2 + 5 + 3 = 18$
6	$6.1 + 6 + 4 = 16$

The case $q = 3$ is handled similarly. Finally, for the cases $q = 4$ and $q = 2$ one considers a hyperoval A' containing A (instead of an oval); then tangents cannot occur anymore, and a similar (but simpler) case distinction gives the desired results.

Examples of arcs of $\mathbf{D}$ contained in an oval of Π will be considered in Sections 6 and 7; we will see that the bounds of Proposition 5.3 are reasonably good. For the Desarguesian case, the bound of Proposition 5.2 will be improved in Section 7. First we will apply Propositions 5.2 and 5.3 to prove the results already mentioned in the introduction to this section.

5.4 THEOREM. *Let* Π *be a projective plane of order* $q^2 \neq 4$, Π_0 *a Baer subplane and* $\mathbf{D}$ *the corresponding Baer semiplane. Then any oval of* $\mathbf{D}$ *is a maximal* q^2*-arc of* Π.

This is immediate from 5.2 and 5.3. By well-known results of Segre and Tallini (see [**9**, Theorems 8.6.10 and 8.7.2]), no Desarguesian plane of order n contains a maximal n-arc. Thus we have:

5.5 COROLLARY. *Let* **D** *be the Baer semiplane* $B(q^2)$ *of order* q^2 *derived from* $PG(2, q^2)$. *Then* **D** *does not contain an oval, unless* $q = 2$.

As we have seen in Example 3.5, the case $q = 2$ is indeed an exception to Corollary 5.5. With the notation of 5.2 and 5.3, one has $|A'' \cap \Pi_0| = 1$ in this example, and 3 of the special lines for A are concurrent in the point $p = A'' \cap \Pi_0$, whereas the fourth special line is a tangent. As another application of 5.4, we combine this result with Proposition 3.1 and obtain:

5.6 PROPOSITION. *Let* D *be a relative difference set for a Baer semiplane* **D** *of order* q^2 *in an abelian group* G, *and let* Π *be the corresponding projective plane. Then the ovals* $-D + y$ $(y \in G)$ *of* **D** *are maximal* q^2*-arcs in* Π *unless* $q = 2$.

5.7 COROLLARY. *The Baer semiplane* $B(q^2)$ *derived from* $PG(2, q^2)$ *does not admit a representation by an abelian relative difference set, unless* $q = 2$.

Corollary 5.7 is, of course, well-known; by [**8**] one even has the stronger result that an abelian relative difference set for a Baer semiplane of prime power order q^2 exists only for $q = 2$. We close this section by observing that Corollary 5.5 may be strengthened if q is sufficiently large:

5.8 THEOREM. *Let* A *be an* h*-arc of the Baer semiplane* $B(q^2)$ *derived from* $PG(2, q^2)$. *If* $q \neq 2$ *is even, one has*

$$h \leqq q^2 - q + 1;$$

if q *is odd, one has*

$$h \leqq q^2 - \frac{q}{4} + \frac{7}{4}.$$

Proof. First let q be even. If A has more than $q^2 - q + 1$ points, then A is contained in a hyperoval A' of $PG(2, q)$ by a result of Segre (see [**9**, Theorem 10.3.3]). This contradicts Propositions 5.2 (for $q \geqq 8$) and 5.3 (for $q = 4$), respectively. Now let q be odd; if

$$h > q^2 - \frac{q}{4} + \frac{7}{4},$$

then A is contained in an oval A' of $PG(2, q^2)$, see [**9**, Theorem 10.4.4]. Again, this contradicts Propositions 5.2 and 5.3.

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6. Arcs in Baer semiplanes: Some general constructions. In this section, we will give some constructions of relatively large maximal arcs in Baer semiplanes, which of course still fall far short from being ovals. The first construction will work for any projective plane of order q^2 admitting an abelian Singer group.

6.1 THEOREM. *Let* Π *be a projective plane of order* q^2 *admitting an abelian Singer group* G. *Then* Π *contains a Baer subplane* Π_1 *for which the Baer semiplane* $\mathbf{D} = \Pi \backslash \Pi_1$ *contains a maximal* $(q^2 + q + 2)/2$-*arc.*

Proof. By the proof of Theorem 4.4 of [**17**] we may represent Π by a difference set D in G containing a sub-difference set D_0 (in the subgroup H of elements of G fixed under the multiplier q^3) such that dev D_0 is a Baer subplane Π_0 of Π. Then the translates Π_x of Π_0 ($x \in G$, $\Pi_x = \Pi_0 + x$) form a partition of Π into Baer subplanes. By Proposition 3.1, $-D$ is an oval of Π and $-D_0$ is an oval of Π_0. We claim that each of the remaining $q^2 - q$ Baer subplanes $\Pi_x (x \notin H)$ contains at most one (and thus exactly one) point of $-D$. For assume $-d$, $-d' \in \Pi_x$; then $-d = h + x$, $-d' = h' + x$ for suitable $h, h' \in H$. Thus $d - d' = h' - h \in H$, so $d, d' \in D_0 \subset H$, since D_0 is a difference set for H. But then $x \in H$ and $\Pi_x = \Pi_0$.

Similarly, we show that each Π_x ($x \notin H$) contains at most one (and then exactly one) tangent of $-D$. Recall that the tangents of $-D$ are the lines $D - 2d$ ($d \in D$), see Proposition 3.1. Assume $D - 2d$, $D - 2d' \in \Pi_x$. Then, as above, $2d - 2d' \in H$, i.e., $2(d - d') \in H$. As multiplication by 2 is an automorphism for H, we see $d - d' \in H$, hence $d, d' \in H$ and $\Pi_x = \Pi_0$.

Now let Π_1 be one of the Baer subplanes Π_x ($x \notin H$). Thus $S = -D \backslash \Pi_1$ consists of q^2 points of the semiplane $\mathbf{D} = \Pi \backslash \Pi_1$. Write $p = (-D) \cap \Pi_1$. We claim that the $q + 1$ lines of Π_1 through p are secants of $-D$. In fact, they are the lines $D - d - d_0$, where $p = -d$ and $d_0 \in D_0$: Clearly, $-d \in H + x$, so $-d - d_0 \in H + x$ for all $d_0 \in D_0$. So each line $D - d - d_0$ is in $\Pi_1 + x$; finally, this line contains both $p = -d$ and $-d_0$. Thus $-D_0$ is a $(q + 1)$-arc in $\mathbf{D}$. Now consider the remaining $q^2 - q - 1$ points $y \in S \backslash (-D_0)$. Each such y is on a unique line $G_y \in \Pi_1$. As we have seen, exactly one of these lines is a tangent of $-D$, say the line G_z. So $-D \cup \{z\}$ is a $(q + 2)$-arc of $\mathbf{D}$. Finally, each of the remaining lines G_y is a secant of $-D$ for which both points of $G_y \cap (-D)$ are in S. Then we may select just one point on each of these $(q^2 - q - 2)/2$ lines. This then gives, together with $-D_0$ and z, a $(q^2 + q + 2)/2$-arc A of $\mathbf{D}$.

Finally, assume that A is not a maximal arc of $\mathbf{D}$. Then there exists a point $u \notin -D$ such that $A' = A \cup \{u\}$ is an arc. Note that (if q is even) u is also not the nucleus w of $-D$, since then z and w are on the common line $G_z \in \Pi_1$. Now A' is also an arc of Π, and A' has $(q^2 + q + 2)/2$

points in common with the oval $-D$ (resp. the hyperoval $-D \cup \{w\}$, if q is even) of Π. Since

$$\frac{q^2 + q + 2}{2} > \frac{q^2 + 3}{2},$$

we obtain $A' \subset -D$ resp. $A' \subset -D \cup \{w\}$, by the results of [**22**] and [**21**] already quoted in Section 4. So $z \in -D$ or $z = w$, a contradiction.

6.2 COROLLARY. *The Baer semiplane $B(q^2)$ derived from $PG(2, q^2)$ contains a maximal $(q^2 + q + 2)/2$-arc.*

We shall now construct other examples of maximal arcs in $B(q^2)$. First, we need an auxiliary result.

6.3 LEMMA. *Let $\{x, y, z\}$ be any triangle in $PG(2, q^2)$. Then there are precisely $q^2 + 2q + 1$ Baer subplanes containing $\{x, y, z\}$.*

Proof. As is well-known, each quadrangle of $PG(2, q^2)$ is in a unique Baer subplane. Thus each point u which is not contained in any of the lines $\overline{xy}$, $\overline{xz}$ and $\overline{yz}$ defines a unique Baer subplane Π_u. Now there are $q^4 - 2q^2 + 1$ choices for u, but each Π_u contains $q^2 - 2q + 1$ points u' with $\Pi_u = \Pi_{u'}$. Hence we obtain

$$(q^4 - 2q^2 + 1)/(q^2 - 2q + 1) = q^2 + 2q + 1$$

distinct Baer subplanes through $\{x, y, z\}$.

This simple result already is sufficient to construct further families of maximal arcs in $B(q^2)$, provided q is even.

6.4 THEOREM. *Let q be a power of 2. Then $B(q^2)$ contains both maximal $(q^2 + 2q)/2$-arcs and maximal $(q^2 + 3q - 4)/2$-arcs.*

Proof. Let H be a hyperoval of $PG(2, q^2)$ and choose three points x, y, z on H. Each of the remaining $q^2 - 1$ further points u on H defines together with x, y, z a unique Baer subplane Π_u. By Lemma 6.3, we see that there exists a Baer subplane Π_0 of $PG(2, q^2)$ for which $\Pi_0 \cap H = \{x, y, z\}$. Use Π_0 in defining the Baer semiplane

$$\mathbf{D} = \Pi \backslash \Pi_0 \cong B(q^2).$$

The $3(q - 1)$ lines of Π_0 containing exactly one of the points x, y, z each meet H in a point of $\mathbf{D}$; these $3(q - 1)$ points form a $(3q - 3)$-arc A of $\mathbf{D}$. As usual, we need the fact that each point of $H \backslash \Pi_0$ is on exactly one line in Π_0. The points of $H \backslash A$ in $\mathbf{D}$ are then each on a secant in Π_0 which has both points of intersection with H in $H \backslash (A \cup \{x, y, z\})$. We can choose just one point on each of these $(q^2 - 3q + 2)/2$ lines and add it to A. Thus we obtain a $(q^2 + 3q - 4)/2$-arc A' in $\mathbf{D}$. Using Martin's result as in the proof of 6.1, we see that A' is maximal.

To obtain also maximal $(q^2 + 2q)/2$-arcs, consider next three points x, y, z such that $x, y \in H$ and $z \notin H$. The same type of argument will now yield a maximal arc with

$$2q + \frac{q^2 - 2q}{2} = \frac{q^2 + 2q}{2}$$

elements.

Note that the case $q = 4$ yields a maximal 12-arc in $B(16)$, showing that the bound for $q = 4$ in Proposition 5.3 is best possible.

Further results on $B(q^2)$ (in particular for odd q) require more effort; this will be the topic of the following section.

7. Arcs in Baer semiplanes: The Desarguesian case. In this section, we will construct further families of maximal arcs in the Baer semiplanes $B(q^2)$; moreover, the bound given in Proposition 5.2 can be improved in the Desarguesian case. We begin with an auxiliary result which probably is known, though we could not locate any reference for it.

7.1 PROPOSITION. *Let C be a conic in $\Pi = PG(2, q^2)$ and let Π_0 be a Baer subplane of Π. Then both the number of points of C in Π_0 and the number of tangents of C in Π_0 is one of* $\{0, 1, 2, 3, 4, q + 1\}$. *If Π_0 contains $q + 1$ points of C, then $C_0 = \Pi_0 \cap C$ is a conic of Π_0.*

Proof. Write $C_0 = C \cap \Pi_0$ and assume $|C_0| \geqq 5$. Coordinatize Π by using 4 points u, v, w, x in C_0 as the quadrangle of reference. Now there is a unique conic C_0' in Π_0 containing u, v, w, x and a given further point y of C_0 (see e.g. [**9**, p. 141]); let the equation of this conic be $f(z) = 0$. Thus f is a second-degree polynomial over $GF(q)$. But f also defines a conic C' in Π. Since any 5 points in general position determine a unique conic of Π, we obtain $C' = C$ and $C_0' = C_0$. Hence $|C_0| = q + 1$ in this case, and C_0 is a sub-conic of C.

Now consider the tangents of C which belong to Π_0. First let q be even, and let p be the nucleus of C. If at least two tangents belong to Π_0, we have $p \in \Pi_0$. But then all $q + 1$ lines of Π_0 through p are tangents of C. Finally, let q be odd. By dualizing, the tangents of C form an oval of Π^d (whose tangents are the points of C), since any point $x \notin C$ is on either none or exactly two tangents in this case. Note that C is again a conic (this follows, for instance, from the theorem of Segre [**23**]); thus the desired conclusion follows from the first part of the proof.

7.2 THEOREM. *Let q be an odd prime power. Then $B(q^2)$ contains a maximal $(((q^2 + 3q - 4)/2) + \epsilon)$-arc, where ϵ takes at least one of the values* 0 *and* 1.

Proof. Choose a conic C of $\Pi = PG(2, q^2)$ and a subplane Π_0 with $|C_0| = q + 1$, where $C_0 = \Pi_0 \cap C$. This can be done as in the proof of 6.1, and is also implicit in the proof of 7.1. Choose three points x, y, z in C_0. By Lemma 6.3, there are exactly $q^2 + 2q$ Baer subplanes $\neq \Pi_0$ containing $\{x, y, z\}$. Any point u in $\Pi\backslash\Pi_0$ will determine a unique such Baer subplane Π_u. Since $C\backslash C_0$ contains only $q^2 - q$ points, there are at least $3q$ Baer subplanes Π_u with $\Pi_u \cap C = \{x, y, z\}$. Now consider the tangent to C in one of x, y, z, say the tangent T_x in x. Choosing any point w in $T_x\backslash\Pi_0$, we will obtain a unique Baer subplane Π_w containing x, y, z and T_x. But any Baer subplane containing x, y, z and a further point on T_x contains q points $\neq x$ of T_x. Thus there are precisely q Baer subplanes containing x, y, z and T_x. One of these is Π_0: if $C = -D$ and $x = -d_0$, then $T_x = D - 2d_0$ belongs to Π_0 (using the notation of the proof of Theorem 6.1). So exactly $q - 1$ of the $3q$ subplanes Π_u mentioned above contain T_x. A similar assertion holds for the tangents T_y and T_z in y and z respectively. Thus there are at least 3 Baer subplanes Π_u with $\Pi_u \cap C = \{x, y, z\}$ which contain none of T_x, T_y and T_z. Let Π_1 be such a Baer subplane and define

$$\mathbf{D} = \Pi_0\backslash\Pi_1 \cong B(q^2).$$

As usual, each point $b \in C\backslash C_0$ is on a unique "special" line $G_b \in \Pi_1$. By our choice of Π_1, each of x, y and z is on $q - 1$ of the special lines. These special lines induce a $(3q - 3)$-arc A of $\mathbf{D}$. The remaining $q^2 - 3q + 1$ points in

$$C' = C\backslash(C_0 \cup \{x, y, z\})$$

yield special lines which either meet C' a second time or are tangents of C belonging to Π_1. Since $q^2 - 3q + 1$ is odd, the number of special tangents is odd. Since $q + 1$ is even, Proposition 7.1 implies that the number of special tangents has to be 1 or 3. Assume first that there is a unique special tangent G_a. We may then adjoin a and one of the points on each of the $(q^2 - 3q)/2$ special secants to A; this yields a $(q^2 + 3q - 4)/2$-arc A' in $\mathbf{D}$. Similar arguments as before show that A' is maximal. Finally, if there are 3 special tangents, we obtain a $(q^2 + 3q - 2)/2$-arc in the same way.

At this point, it is clear that the difficulties in handling the case $B(q^2)$, q odd, stem from the presence of tangents in $PG(2, q^2)$ (which can be ignored using hyperovals in the case of q even). It would thus be nice to know the number of tangents of a conic C in $PG(2, q^2)$ which belong to a given Baer subplane Π_0. The following conjecture would seem reasonable, at least in the Desarguesian case: The number of tangents of C in Π_0 equals the number of points of C in Π_0. This was true in the examples considered in the proof of Theorem 6.1. However, it is not true in general. Consider the case $q = 3$ of Theorem 7.2. There we found at least 3 Baer subplanes Π_u with $\Pi_u \cap C = \{x, y, z\}$ and $T_x, T_y, T_z \notin \Pi_u$. Here the 6-arc A belonging to x, y, z (relative to one of these Baer subplanes chosen) leaves only one point in C' which then is on a tangent. So for all three choices of Π_1, we have $|\Pi_1 \cap C| = 3$, but only one tangent of C belongs to Π_1. These remarks also show that for $q = 3$ we get a maximal 7-arc from Theorem 7.2 (and not an 8-arc).

It is our next aim to construct a further class of maximal arcs in $B(q^2)$, q even, from hyperovals on $PG(2, q^2)$ and to show that they have the largest possible size for such examples. This requires a further auxiliary result similar to Proposition 7.1.

7.3 PROPOSITION. *Let H be a regular hyperoval of $\Pi = PG(2, q^2)$, q even (so H consists of a conic C and its nucleus p), and let Π_0 be a Baer subplane of Π. Then*

$$|H \cap \Pi_0| \in \{0, 1, 2, 3, 4, q + 2\};$$

if $|H \cap \Pi_0| = q + 2$, then $H_0 = H \cap \Pi_0$ is a regular hyperoval of Π_0, and p is the nucleus of H_0.

Proof. Put $H_0 = H \cap \Pi_0$. First assume $p \in H_0$ and without loss of generality $|H_0| \geqq 4$. Then $C_0 = C \cap \Pi_0$ is a conic in Π_0 with nucleus p, by the results of **[16]**. Thus we may now assume that $p \notin H_0$. We have to show that $|H_0| = |\Pi \cap C| \leqq 4$ in this case. Assume the contrary. Then $|\Pi \cap C| = q + 1$ and $\Pi \cap C$ is a conic C_0 in Π_0, by Proposition 7.1. But then again by **[16]**, C_0 has nucleus p and thus $p \in H_0$, a contradiction.

It would be interesting to know if results analogous to 7.1 and 7.3 hold in non-Desarguesian planes or for non-regular hyperovals in Desarguesian planes.

7.4 THEOREM. *Let q be a power of 2. Then $B(q^2)$ contains a maximal $(q^2 + 4q - 10)/2$-arc.*

Proof. Choose a regular hyperoval $H = C \cup \{p\}$ in $PG(2, q^2)$, and select three points x, y, z in C. Each further point $u \in C$ defines a unique Baer subplane Π_u containing x, y, z, u. If Π_u contains a fifth point in C, it also contains p by Proposition 7.3. Since the Baer subplane through the 4 points x, y, z, p is unique, there is at most one such Π_x (in fact, there is such a Π_x by **[16]**). So there are choices for u with $|\Pi_u \cap H| = 4$. Let Π_0 be one such Baer subplane and define $\mathbf{D} = \Pi \backslash \Pi_0 \cong B(q^2)$. The $4(q - 2)$ lines in Π_0 joining one of x, y, z, u to a point in $H \backslash \Pi_0$ determine a $4(q - 2)$-arc A of $\mathbf{D}$. The remaining $q^2 - 4q + 6$ points b in

$$H' = H \backslash (A \cup \{x, y, z, u\})$$

are each on a unique line $G_b \in \Pi_0$, and G_b meets H' in a second point. Adjoining one of the two points in $G_b \cap H'$ for each of these $(q^2 - 4q + 6)/2$ secants to A yields the desired $(q^2 + 4q - 10)/2$-arc A'. The maximality of A' follows from Martin's **[21]** result in the usual way.

Note that we thus obtain a maximal 11-arc in $B(16)$, whereas Theorem 6.4 yields a maximal 12-arc. But for $q \geqq 8$, Theorem 7.4 is best possible:

7.5 THEOREM. *Let $q = 2^a$ with $a \geqq 3$, and let A be an h-arc of $B(q^2)$ which is contained in a regular hyperoval H of the corresponding plane $PG(2, q^2)$. Then*

$$h \leqq \frac{q^2 + 4q - 10}{2}$$

with equality if and only if H is constructed as in the proof of Theorem 7.4.

Proof. Assume that A is any $(q^2 + 4q - 10)/2$-arc of $B(q)$ contained in a hyperoval H of $\Pi = PG(2, q^2)$. Let Π_0 be the Baer subplane of Π used in defining $B(q)$. Each point $x \in A$ is on a unique "special" line $G_x \in \Pi_0$. This yields $q^2/2 + 2q - 5$ special lines by (A2). But each special line G_x intersects H a second time; this second point of intersection is always in $H\backslash A$. Any point in $H\backslash A$ which is not in Π_0 occurs on at most special line, and any point in $\Pi_0 \cap H$ on at most $q + 2 - c$ special lines, where $c = |\Pi_0 \cap H|$. We claim that $H\backslash A$ contains exactly 4 points in Π_0. Note that

$$|H\backslash A| = \frac{q^2}{2} - 2q + 7.$$

We prove the claim by counting the maximum possible number M of special lines, depending on c:

c	0	1	2	3	4
M	$\frac{q^2}{2} - 2q + 7$	$\frac{q^2}{2} - q + 7$	$\frac{q^2}{2} + 5$	$\frac{q^2}{2} + q + 1$	$\frac{q^2}{2} + 2q - 5$

Thus indeed $M \geqq 4$. If we have $M > 4$, we obtain $M = q + 2$ by Proposition 7.3; in this case $H_0 = \Pi \cap H$ is a regular hyperoval of Π_0. But then no special line can meet H_0, and we obtain at most $q^2/2 - 3q + 5$ special lines, a contradiction. We have now seen that A is necessarily a maximal arc (so $h \leqq |A|$ in general), that $|\Pi_0 \cap H| = 4$ and that we obtain enough special lines if and only if each of the four points in $\Pi_0 \cap H$ yields $q - 2$ special lines and each of the remaining $q^2/2 - 2q + 3$ points in $H\backslash A$ yields one special line. But then A is constructed from H as described in the proof of Theorem 7.4.

We leave it to the reader to prove by a similar argument (using Proposition 7.1) the following analogue of Theorem 7.5 for odd prime powers. Note that any oval is a conic in this case, by Segre's theorem **[23]**.

7.6 THEOREM. *Let A be an h-arc of $B(q^2)$ ($q \geqq 5$ odd) which is contained in a conic of $PG(2, q^2)$. Then $h \leqq (q^2 + 5q - 10)/2$.*

Theorems 7.5 and 7.6 thus considerably improve Proposition 5.2 for the Desarguesian case. Whereas we have seen in Theorem 7.4 that the bound of 7.5 is best possible, we have not been able to show this for odd values of q. All we can show here (apart from the more precise results in 6.2 and 7.2) is the following:

7.7 THEOREM. *$B(q^2)$ (q odd) contains a $(q^2 + 4q - 15)/2$-arc A which extends to a conic C of $PG(2, q^2)$.*

Proof. Select three points x, y, z of a conic C in $\Pi = PG(2, q^2)$. Each further point u of C determines a unique Baer subplane Π_u containing $\{x, y, z, u\}$. By Proposition 7.1, we have $|\Pi_u \cap C| \in \{4, q + 1\}$ for each u. If one always had $|\Pi_u \cap C| = q + 1$, one would obtain a partition of the $q^2 - 2$ points in $C\backslash\{x, y, z\}$ into $(q - 2)$-sets which is impossible. Thus we may choose a subplane Π_0 with $|\Pi_0 \cap C| = 4$, say $\Pi_0 \cap C = \{x, y, z, u\}$. Each of the 4 points in $\Pi_0 \cap C$ is on at least $q - 3$ lines of Π_0 intersecting $C\backslash\Pi_0$ (of the $q + 1$ lines through x, say, in Π_0, one may be a tangent, and three join x to y, z, u). This yields a $4(q - 3)$-arc A'. Each of the remaining $q^2 - 4q + 8$ points in $C\backslash\Pi_0$ is on a unique special line in Π_0. Since $q^2 - 4q + 3$ is odd, at least one special line is a tangent or a further line meeting $\Pi_0 \cap C$; in either case, we may augment A' to a $(4q - 11)$-arc A''. Finally, we can adjoint to A'' at least further $(q^2 - 4q + 7)/2$ points, even if all other special lines are secants. This yields the desired arc A, which of course is not necessarily maximal in this case.

We conclude this section with a table giving examples of maximal arcs for a few small values of q and collecting the general results proved in Section 5 to 7:

q	maximal h-arcs A in $B(q^2)$ for h =
2	4
3	7
4	11; 12
5	16; 18 or 19
7	29; 33 or 34
8	37; 40; 42; 43
9	46; 52 or 53
11	67; 75 or 76
13	92; 102 or 103
16	137; 144; 150; 155

q even	$\frac{q^2+q+2}{2}$; $\frac{q^2+2q}{2}$; $\frac{q^2+3q-4}{2}$; $\frac{q^2+4q-10}{2}$ general upper bound: $h \leqq q^2 - q + 1$ for $q \neq 2$ if A is contained in hyperoval of $PG(2, q^2)$: $h \leqq \frac{q^2+4q-10}{4}$
q odd	$\frac{q^2+q+2}{2}$; $\frac{q^2+3q-4}{2}$ or $\frac{q^2+3q-2}{2}$; some h with $(q^2+4q-15)/2 \leqq h \leqq (q^2+\sqrt{q}-10)/2$ general upper bound: $h \leqq \min\left\{q^2-1, q^2-\frac{q}{4}+\frac{7}{2}\right\}$ if A is contained in conic of $PG(2, q^2)$: $h \leqq \frac{q^2+5q-10}{2}$

8. Arcs in Baer semiplanes: The Hughes plane of order 9. We have seen that large arcs in $B(q^2)$ cannot arise from conics and regular hyperovals in $PG(2, q^2)$. Indeed, $B(q^2)$ does not contain any oval. So one has to consider non-Desarguesian planes if one wants to construct a Baer semiplane containing an oval. Clearly, this will in general require methods which are quite different from the ones used in this paper. It seems reasonable to consider at least one small example, though. We will therefore study one of the three known non-Desarguesian planes of order 9 in this section. We have selected the Hughes plane for three reasons:

(i) It admits a representation which is particularly convenient for computations related to our problem.

(ii) In [**5**] and [**21**], several arcs (including maximal 9-arcs) are listed explicitly.

(iii) The Hughes plane also admits $S_3 \times Z_{13}$ as a quasiregular collineation group which is interesting in connection with the problem whether Proposition 3.1 holds for non-abelian relative difference sets. (It does not, as we shall see.)

We shall work with the description of the Hughes plane Π of order 9 used by Hughes [**11**] and Martin [**21**]. The point set of Π is

$$\{X_i : X = A, B, C, D, E, F, G;\ i \in Z_{13}\}$$

and the lines are obtained from the action of Z_{13} on the following 7 base lines:

$$L_1 = \{A_0, A_1, A_3, A_9, B_0, C_0, D_0, E_0, F_0, G_0\};$$

$$L_2 = \{A_0, B_1, B_8, D_3, D_{11}, E_2, E_5, E_6, G_7, G_9\};$$

$$L_3 = \{A_0, C_1, C_8, E_7, E_9, F_3, F_{11}, G_2, G_5, G_6\};$$

$$L_4 = \{A_0, B_7, B_9, D_1, D_8, F_2, F_5, F_6, G_3, G_{11}\};$$

$$L_5 = \{A_0, B_2, B_5, B_6, C_3, C_{11}, E_1, E_8, F_7, F_9\};$$
$$L_6 = \{A_0, C_7, C_9, D_2, D_5, D_6, E_3, E_{11}, F_1, F_8\};$$
$$L_7 = \{A_0, B_3, B_{11}, C_2, C_5, C_6, D_7, D_9, G_1, G_8\}.$$

More precisely, the lines of Π are the 91 sets $L_i + k$, where $L_i + k$ is obtained from L_i by adding k to all subscripts ($i = 1, \dots, 7$, $k \in Z_{13}$). As $\{0, 1, 3, 9\} \subset Z_{13}$ is a difference set for $PG(2, 3)$, it is clear that the points $A_0, \dots, A_{12}$ together with the lines $L_1 + k$ ($k \in Z_{13}$) form a Baer subplane Π_0 of Π. The corresponding Baer semiplane **D** then has as its lines the orbits of Z_{13} on the base lines $L_i' = L_i \setminus \{A_0\}$ ($i = 2, \dots, 7$). Note that a subset S of **D** is an arc if and only if

(i) $|S \cap (L_i' + k)| \leqq 2$ for all $i = 2, \dots, 7$ and all $x \in Z_{13}$; and

(ii) S contains no two points with the same subscript.

Thus condition (A2) is particularly easy to check in this representation of **D**. It is then immediately seen that none of the arcs of Martin [**21**] is an arc in **D**, even after removing the points in Π_0. In fact, none of his arcs contains an h-arc of **D** with $h > 5$. It was no problem, however, to find a 7-arc in **D**: for instance, $S = \{B_3, B_{10}, C_2, D_{11}, E_6, F_4, G_0\}$ is a maximal 7-arc of **D** which extends to a maximal 8-arc of Π by adjoining the point F_0 (note that F_0G_0 is the line $L_1 \in \Pi_0$).

We now turn our attention to the RDS-representation of **D**. The Hughes plane Π admits $S_3 \times Z_{13}$ as a quasiregular collineation group, and thus there exists an RDS with parameters $m = 13$, $n = 6$ and $k = 9$ in this group. The following example is equivalent to one due to Carol Whitehead (see [**8**, p. 153]):

$$D = \{ (\text{id}, 2), (\text{id}, 5), (\text{id}, 6), (\tau, 1), (\tau, 8), (\sigma\tau, 7), (\sigma\tau, 9), (\tau\sigma, 3), (\tau\sigma, 11) \}$$

where the elements of S_3 are $\sigma = (0, 1, 2)$, $\sigma^2 = (0, 2, 1)$, $\tau = (1, 2)$, $\tau\sigma = (0, 1)$, $\sigma\tau = (0, 2)$ and id. To see the asserted equivalence, one essentially has to multiply the example of [**8**] by 4, followed by a shift of $+1$; then only notational differences remain. We have chosen this transformation, because it allows us to show the connection between the RDS-representation and the one given before. In fact, write the coordinates in Z_{13} as subscripts and identify the elements of S_3 with B, C, D, E, F, G as follows:

$$B = \tau,\ C = \sigma^2,\ D = \tau\sigma,\ E = \text{id},\ F = \sigma,\ G = \sigma\tau;$$

then $L_2' = D$, $L_3' = D\sigma\tau$, $L_4' = D\sigma$, $L_5' = D\tau$, $L_6' = D\tau\sigma$ and $L_7' = D\sigma^2$. Thus we have $\mathbf{D} = \text{dev } D$. Now the set $-D$ of inverses of elements in D is

$$-D = \{ (\text{id}, 7), (\text{id}, 8), (\text{id}, 11), (\tau, 5), (\tau, 12), (\sigma\tau, 4), (\sigma\tau, 6), (\tau\sigma, 2), (\tau\sigma, 10) \}$$

or, in the old notation

$$-D = \{E_7, E_8, E_{11}, B_5, B_{12}, G_4, G_6, D_2, D_{10}\}.$$

It is easily checked that the only lines of **D** intersecting $-D$ more than twice are $L'_2 + 4$ (containing B_5, B_{12} and D_2), $L'_2 - 3$ (containing B_5, G_4 and G_6) and $L'_2 - 1$ (containing D_2, D_{10} and G_6). Thus $-D$ is not an oval of **D** (so Proposition 3.1 does not hold in the non-abelian case). The largest arcs of **D** contained in $-D$ are 7-arcs. For instance, the subset

$$S' = \{E_7, E_8, E_{11}, G_4, D_{10}, B_{12}, G_6\}$$

is a maximal 7-arc in **D** which is also maximal as an arc of Π. We have not found any larger arc in **D**, but no exhaustive search has been done yet.

Relying on computer results of Denniston **[5]**, we can however give a partial answer. Note that the Baer subplane Π_0 is the *real* subplane of Π, i.e., the unique Baer subplane fixed by all collineations (cf. **[4]**). In **[5]**, it has been shown that there are (up to equivalence under collineations of Π) exactly 3 maximal 9-arcs in Π. Denniston's notation is different from the one used here; but it is easily checked that these arcs of Π contain no arc of **D** of size >6. As we have seen, we can obtain larger arcs of **D** by using complete 7- or 8-arcs of Π. Thus we note:

8.1 PROPOSITION. *Let Π be the Hughes plane of order* 9, *and let Π_0 be the real Baer subplane of* Π. *Then* $\mathbf{D} = \Pi \backslash \Pi_0$ *contains maximal* 7-*arcs, but does not contain* 9-*arcs* (*i.e.*, **D** *does not contain an oval*).

Note that this leaves the question of possible 8-arcs of **D**. Also, there are 3 other classes of Baer subplanes in the Hughes plane of order 9 (see **[4]**); their corresponding Baer semiplanes might conceivably contain ovals.

9. Arcs in Baer semiplanes: Concluding remarks. As we have seen, the existence of large arcs in Baer semiplanes is a rather difficult problem. We have been able to show that $B(q^2)$ never contains an oval. On the other hand, any Baer semiplane belonging to an RDS would have to contain an oval (many ovals, in fact), and these ovals would not extend in the corresponding projective plane (unless $q^2 = 4$). So it would seem interesting to investigate arcs in non-Desarguesian Baer semiplanes. We offer the following conjecture:

9.1 CONJECTURE. *The only Baer semiplane containing an oval is* $B(4)$.

By the results of Section 5, the validity of 9.1 would imply the validity of the following:

9.2 CONJECTURE. *If there exists an abelian* RDS *for a Baer semiplane of order* q^2, *then* $q = 2$.

This would then strengthen the results of Ganley and Spence **[8]** who have proved that q is never a prime power (unless $q = 2$) and that each prime divisor of q is $\equiv 1 \bmod 4$. Both conjectures seem rather difficult. Still, the study of arcs in Baer semiplanes offers a new way of attacking Conjecture 9.2. It is also an interesting geometric problem in its own right, leading to questions about the interaction of Baer subplanes and arcs, as we have seen. In particular, the study of large arcs in non-Desarguesian Baer semiplanes (and the construction of examples) seems a nice problem, even if Conjecture 9.1 can be proved.

We will close this paper by determining a few more properties of a putative Baer semiplane $\mathbf{D} = \operatorname{dev} D$ of odd order q^2 belonging to an abelian relative difference set.

9.3 LEMMA. *Let D be an abelian* RDS *in G for a Baer semiplane $\mathbf{D}$ of odd order q^2. Then $2d - 2d' \notin N$ for all $d, d' \in D$ with $d \neq d'$. Hence any two tangents of $-D$ intersect in $\mathbf{D}$.*

Proof. Note first that $2d \neq 2d'$ for $d \neq d'$, since otherwise $d - d' = d' - d$. Also, N contains all involutions of G, since otherwise, with $g = d - d'$, one would have a second difference representation $g = -g = d' - d$. Now let $d \neq d'$ and assume $2d - 2d' \in N$. Write $x = d - d'$; so $x \notin N$, $2x \in N$. Let $G = S \oplus G_1$, where S is the Sylow 2-subgroup of G. Since

$$|G| = (q^2 + q + 1)\,|N|,$$

we have $S \leqq N$; so we also have $N = S \oplus N_1$, for some $N_1 < G_1$. Now let $x = s + g$ with $g \in G_1$ and $s \in S$. Then $2x = 2s + 2g \in N$, and hence $2g \in N_1$. Since N_1 has odd order, this implies $g \in N_1$ and thus $x = s + g \in N$, a contradiction. Hence $2d - 2d' \notin N$ whenever $d \neq d'$. By Proposition 3.1, the tangents of $-D$ are the lines $D - 2d$ ($d \in D$). If two such lines (say $D - 2d$ and $D - 2d'$) do not intersect in $\mathbf{D}$, they belong to the same parallel class of $\mathbf{D}$; but then $2d - 2d' \in N$, contradicting the first part of the lemma.

9.4 PROPOSITION. *Let A be an oval in a Baer semiplane $\mathbf{D} = \Pi \backslash \Pi_0$ of odd order q^2, and assume that any two tangents of A intersect in $\mathbf{D}$. Then the points of Π_0 which are on no tangent of A form an oval H in Π_0, and the point classes of $\mathbf{D}$ not meeting A are the tangents of H. Moreover, each point in such a point class is on a unique tangent of A.*

Proof. In Π, the q^2-arc A has $2q^2$ tangents: Each point $p \in A$ is on the unique tangent of A in $\mathbf{D}$ (denoted by T_p), and also on a unique tangent in Π_0 (corresponding to the point class of p and denoted by B_p). Since q is odd, each point of Π is on an odd number of tangents of A. We now consider points $x \in \Pi_0$ and count all flags (x, G), where G is a tangent of A. Since each T_p contains a unique point $x \in \Pi_0$ and each B_p contains $q + 1$ points in Π_0, we obtain $q^2(q + 2)$ such flags. Denote by t_x the number of tangents through x; so

$$\sum_{x \in \Pi_0} t_x = q^2(q + 2).$$

Since each x is on at most one T_x, we have $t_x \leqq q + 2$ for all x; since t_x and q are both odd, $t_x \neq q + 2$ implies $t_x \leqq q$.

Denote by y the number of points $x \in \Pi_0$ with $t_x = q + 2$. Then

$$q^2(q + 2) \leqq y(q + 2) + (q^2 + q + 1 - y)q$$

and thus $y \geqq (q^2 - q)/2$. Thus at most $(q^2 + 3q + 2)/2$ points x satisfy $t_x \leqq q$. But there are $q + 1$ lines $G_1, \ldots, G_{q+1}$ of Π_0 which do not intersect A. These lines cover at least

$$(q + 1) + q + (q - 1) + \ldots + 1 = \frac{(q + 2)(q + 1)}{2}$$

points x' of Π_0; one has equality if and only if each point of Π_0 is on at most two of the lines G_i. Each point x' is on at most q tangents of A in Π_0 and thus satisfies $t_{x'} \leqq q$. By the previous argument, we have to have equality in both inequalities. Thus $y = (q^2 - q)/2$ points x satisfy $t_x = q + 2$, and the remaining points x' satisfies $t_{x'} = q$.

Since each x' with $t_{x'} = q$ is on exactly one or two of the G_i, we see that the G_i form a dual oval in Π_0; they are, in fact, the tangents of an oval H formed by the $q + 1$ points x' which are on exactly one of the G_i. Thus each point of H is on q Baer tangents B_p of A; so the points of H are the points in Π_0 that are on none of the tangents T_p. (Note that the remaining points x' with $t_{x'} = q$ are the exterior points of H, whereas the points x with $t_x = q + 2$ are the interior points of H.)

Finally, consider a point z of **D** on one of the G_i (i.e., a point in a point class that does not meet A). Since t_z is odd, $t_z \geqq 1$. Since G_i is not a tangent of A, each z is on at least one of the tangents T_p. Also, each point $x' \in G_i \cap \Pi_0$ (excepting $G_i \cap H$) is an exterior point of H; thus x is on exactly $q - 1$ Baer tangents and on one of the tangents T_p. This already yields $(q^2 - q) + q$ tangents T_p (at least one for each of the $q^2 - q$ choices for z and for each of the q choices for x'), proving the last assertion.

Results 9.3 and 9.4 together give further restrictions on the arcs belonging to a putative abelian RDS of Baer type (in the case of odd order) and thus provide some more evidence for the validity of Conjecture 9.2. In particular, Π_0 would have to contain an oval. On the other hand, these results give some indication of what type of construction could be useful if one searches for counter-examples to Conjecture 9.1.

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