

## ANALYTIC CONTRACTIVE VECTOR FIELDS IN FRÉCHET SPACES

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Using degree theory we establish the inverse mapping theorem and a Sard-Smale theorem for analytic contractive vector fields in Fréchet spaces. Applying these theorems we obtain Cronin's theorem relating topological degree to the number of zeros of analytic contractive vector fields.

### Introduction

In [3] we have studied the differential calculus for a class of functions in Fréchet space. In this paper we prove the inverse mapping theorem for analytic contractive vector fields which may be not of class  $C^1$  in the sense in [3]. Then using this theorem we prove a Sard-Smale theorem and a theorem of Cronin's type in the case of Fréchet spaces. Finally we shall correct a small gap in [3] (*cf.* Remark 1.2).

The idea of the present proofs of our Sard-Smale theorem and our Cronin theorem came from a study of Krasnoselskii and Zabreiko's elegant proofs of these theorems for analytic compact vector fields in Banach spaces. In fact we shall follow Krasnoselskii and Zabreiko's approach basing the proofs of these theorems on the inverse mapping theorem.

### 1. Inverse mapping theorem

Throughout the paper let  $E$  be a Fréchet space over the field of complex numbers  $\mathbb{C}$ ,  $V(0)$  a base of closed balanced convex neighborhoods

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of  $0$  in  $E$ ,  $D$  an open nonvoid subset of  $E$ ,  $I$  the identity map,  $L(E)$  the set of all continuous linear mappings from  $E$  into  $E$ , and  $d$  a translation invariant metric on  $E$  (compatible with the topology on  $E$ ) such that every open ball is convex and balanced. For each  $A \subset E$  we write

$$L(A) = \inf\{r > 0 : \text{there is a finite subset } B \text{ of } E \text{ such that } A \subset B + B(0, r)\},$$

where  $B(0, r) = \{x \in E : d(x, 0) < r\}$  and

$$B + B(0, r) = \{x+y : x \in B \text{ and } y \in B(0, r)\}$$

and  $\inf \phi = \infty$ .

$L$  is called the ball measure of noncompactness on  $E$  and has the properties (M.1)–(M.8) in [4]. Now let  $A \subset E$  and  $r > 0$  such that  $L(A) < r$ , then we can choose a finite subset  $B$  of  $E$  so that

$$A \subset B + B(0, r).$$

Put  $\Gamma = \{s \in \mathbb{C} : |s| = 1\}$ ; we see that

$$\{sx : x \in A, s \in \Gamma\} \subset \{sx : x \in B, s \in \Gamma\} + B(0, r).$$

Since  $\{sx : x \in B, s \in \Gamma\}$  is compact, we have

$$L(\{sx : x \in A, s \in \Gamma\}) \leq L(B(0, r)) \leq r.$$

Then

$$(M) \quad L(A^*) \leq L(A)$$

where  $A^* = \overline{\text{co}}\{sx : x \in A, s \in \Gamma\}$ , the closure of the convex hull of  $\{sx : x \in A, s \in \Gamma\}$ .

The results of this paper are valid for the generalized measures of noncompactness defined as in [4] and satisfying condition (M) and the following:

(M') there exists a positive real number  $K \in (0, 1)$  such that for any  $W \in V(0)$  and  $A \subset E$ , there are  $a_1, a_2, \dots, a_n$  in  $E$  such that  $A \subset \bigcup_j (a_j + W)$  if  $L(A) \leq K \cdot L(W)$ . //

**DEFINITION 1.1.** Let  $f$  be a continuous mapping from  $D$  into  $E$ . We say  $f$  is *analytic* on  $D$  if for each  $x$  in  $D$  there exists a

(necessarily unique)  $f'(x)$  in  $L(E)$  such that

$$(1.1) \quad \lim_{z \rightarrow 0} \frac{f(x+zh) - f(x)}{z} = f'(x)h \text{ for all } h \in E .$$

**DEFINITION 1.2.** Let  $g$  be a continuous mapping from  $D$  into  $E$ . We say  $g$  is a *contraction* on  $D$  if  $L(g(D))$  is finite (cf. [3], p. 95) and there is a positive number  $c$  in  $(0, 1)$  such that

$$(1.2) \quad L(g(A)) \leq cL(A) \text{ for all } A \subset D .$$

Let  $g$  be an analytic contraction on  $D$ ; then  $f = I - g$  is said to be an analytic contractive vector field on  $D$ .

In this paper  $f = I - g$  is a given analytic contractive vector field  $D$  with the coefficient  $c$  in  $(0, 1)$ . In this section we shall prove the inverse mapping theorem for  $f$ . At first we study some properties of  $f$  and  $g$ .

**LEMMA 1.1.** Let  $a \in D$  and  $V \in V(0)$  be such that  $a + 2V \subset D$ . Let  $K$  be a compact subset of  $a + V$ , and  $A \subset V$ . Put

$$B = \left\{ \frac{1}{2\pi i} \int_{\Gamma} \frac{g(k+sh)}{s^2} ds : k \in K \text{ and } h \in A \right\}$$

where  $\int_{\Gamma} \frac{g(k+sh)}{s^2} ds$  is the Riemann integral  $\int_0^{2\pi} ig(k+e^{it}h)e^{-it} dt$ . Then

- (i)  $L(B)$  is finite and  $L(B) \leq cL(A)$  ;
- (ii) if  $A \subset B$ , then  $A$  is relatively compact.

**Proof.** (i) We see that  $B \subset (g(K+A^*))^*$ . Then by the properties of  $L$  we have

$$(1.3) \quad L(B) \leq L(g(K+A^*)) \leq cL(K+A^*) \leq cL(A) .$$

Since  $L(g(D))$  is finite by definition,  $L(B)$  is also.

(ii) If  $A \subset B$ , we have

$$L(A) \leq L(B) \leq cL(A) .$$

Then  $L(A)$  is finite. Since  $c < 1$ , it follows that  $L(A) = 0$  or  $A$  is relatively compact. //

**PROPOSITION 1.1.** Let  $a \in D$ ,  $V \in V(0)$  such that  $a + 2V \subset D$ ; then  $L(g'(a)V)$  is finite and for any  $A \subset V$  we have

$$L(g'(a))(A) \leq cL(A) .$$

**Proof.** By the Cauchy formula for scalar analytic functions and the Hahn-Banach theorem we have

$$(1.4) \quad g'(a)h = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(a+sh)}{s^2} ds \text{ for all } h \in V .$$

Then we have the proposition by Lemma 1.1. //

REMARK 1.1. Put  $T = g'(a)$ . We can find  $W \in V(0)$  and an integer  $m$  such that  $c^m \leq K$  (the number  $K$  in condition (M')) and  $T^j(W) \subset V$  for all  $j \leq 2m$ . Put  $T_1 = T^m$ ; by (M') and the other properties of  $L$  we have  $T_1(W) \subset B + W$  where  $B$  is a finite subset of  $E$  and

$$L\left(T_1^2(W)\right) \leq KL(T_1(W)) \leq K^2L(W) .$$

Then by induction

$$L\left(T_1^{n+1}(W)\right) \leq K^n L(T_1(W)) \text{ for all integers } n .$$

Since  $L(T_1(W))$  is finite and  $K \in (0, 1)$ , we get

$$\inf_n L(T_1^n(W)) = 0 .$$

Therefore  $T$  is a pseudocontraction (cf. [4]). Let  $G(T)$  and  $F(T)$  be denoted as in Theorem 1 of [4] and  $P$  and  $Q$  be the corresponding projections of  $E$  onto  $G(T)$  and  $F(T)$ . Then we have

- (i)  $E$  is the topological direct sum of  $G(T)$  and  $F(T)$ ,
- (ii)  $G(T)$  is finite dimensional,
- (iii)  $G(T)$  and  $F(T)$  are invariant under  $T$ ,
- (iv) the restriction of  $T$  on  $G(T)$  (respectively  $F(T)$ ) has no real eigenvalues less than 1 (respectively greater than or equal to 1),
- (v)  $P$  and  $Q$  commute with  $T$ ,
- (vi)  $I - Q \circ T$  is a homeomorphism from  $E$  onto  $E$ .

Furthermore if  $I - T$  is a homeomorphism from  $E$  onto  $E$ , then the

restriction of  $T$  on  $V$  is a limit-compact mapping and the topological degree  $\text{deg}(I-T, \overset{\circ}{V}, 0)$  is defined (cf. Remark 1.1 in [3]). Now arguing as in the proof of Theorem 3.1 in [10], we have

$$\text{deg}(I-T, \overset{\circ}{V}, 0) = (-1)^{\dim G(T)}$$

where  $\overset{\circ}{V}$  is the interior of  $V$ , and  $\dim G(T)$  is the dimension of the linear subspace  $G(T)$  over the field of real numbers. But  $G(T)$  is a complex linear subspace then its dimension is even; hence we have

**PROPOSITION 1.2.** *If  $a \in D$  and  $V \in V(0)$  are such that  $a + 2V$  is contained in  $D$  and  $f'(a)$  is a homeomorphism from  $E$  onto  $E$ , then we have  $\text{deg}(f'(a), \overset{\circ}{V}, 0) = 1$ .*

Hereafter we assume in this section

- (i)  $0 \in D$  and  $f(0) = 0$ , and  $L(g'(0)(A)) \leq cL(A)$  for every  $A \subset D$ ,
- (ii)  $f'(0)$  is a homeomorphism from  $E$  onto  $E$ ,
- (iii)  $V$  is chosen in  $V(0)$  such that  $6V \subset D$ .

Now we have the following proposition.

**PROPOSITION 1.3.** *(i) Let  $a, h \in V$  be such that  $A = C \cdot h \subset V$  and  $g'(a)h = h$ . Then  $h = 0$ .*

*(ii) There is a  $W$  in  $V(0)$  such that  $f'(a)$  is a homeomorphism from  $E$  onto  $E$  for any  $a$  in  $W$ .*

**Proof.** (i) Let  $k \in A$ . We have

$$(1.5) \quad k = g'(a)k = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(a+sk)}{s^2} ds.$$

From Lemma 1.1,  $A$  is relatively compact; then  $h = 0$ .

(ii) By Proposition 5 in [4],  $f'(a) = I - g'(a)$  is a Fredholm operator of index 0 on  $E$ . Now assume by contradiction that (ii) is false; then there exist a sequence  $\{a_n\}$  converging to 0 and a sequence  $\{h_n\}$  in  $\partial V$ , the boundary of  $V$ , such that

$$(1.6) \quad h_n = g'(a_n)h_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(a_n+sh_n)}{s^2} ds.$$

Since  $K = \{\alpha_n\} \cup \{0\}$  is compact, we have, by Lemma 1.1, that  $\{h_n\}$  is relatively compact. Then we can assume it converges to  $h$  in  $\partial V$ .

Now, for any integer  $m$ ,

$$(1.7) \quad \begin{aligned} h_m - g'(0)h_m &= g'(\alpha_m)h_m - g'(0)h_m \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\alpha_m + sh_m) - g(sh_m)}{s^2} ds . \end{aligned}$$

Since the set  $(\{h_m\} \cup \{h\})^*$  is compact,  $g$  is uniformly continuous on it. Then for each  $U \in V(0)$  there is an integer  $N$  such that

$$(1.8) \quad (g(\alpha_m + sh_m) - g(sh_m)) \in U \text{ for any } m \geq N \text{ and } s \in \Gamma .$$

It follows that

$$(1.9) \quad (h_m - g'(0)h_m) \in U \text{ for all } m \geq N .$$

Therefore  $h - g'(0)h = 0$  for any  $h$  in  $\partial V$ , which is absurd, and hence we have (ii).

**PROPOSITION 1.4.** *There exist  $W$  and  $U$  in  $V(0)$  such that*

- (i) *if  $A = \text{Ch} \subset V$  and  $g(h) = h$  then  $h = 0$ ;*
- (ii) *for any  $(t, h) \in [0, 1] \times (W \setminus \{0\})$ ,*  

$$tf(h) + (1-t)f'(0)h \neq 0 ;$$
- (iii) *for any  $(t, h) \in [0, 1] \times \partial W$  and  $q \in U$ ,*  

$$tf(h) + (1-t)f'(0)h \neq q .$$

*Proof.* (i) Let  $k = zh \in A$  we have, by (1.4),

$$\begin{aligned} z^{-1}k &= g(z^{-1}k) = \int_0^1 g'(sz^{-1}k)z^{-1}k ds = z^{-1} \int_0^1 g'(sz^{-1}k)k ds \\ &= z^{-1} \int_0^1 \frac{1}{2\pi i} \int_{\Gamma} \frac{g(sh+tk)}{t^2} dt ds . \end{aligned}$$

Then

$$k = \int_0^1 \frac{1}{2\pi i} \int_{\Gamma} \frac{g(sh+tk)}{t^2} dt ds .$$

Arguing as in Lemma 1.1, we see that  $h = 0$ .

(ii) For each  $(t, h) \in [0, 1] \times D$  we put

$$g_t(h) = tg(h) + (1-t)g'(0)h .$$

Then  $g_t$  is analytic on  $D$  and for any subset  $A$  of  $D$  we have

$$g_t(A) \subset \overline{\text{co}}(g(A) \cup g'(0)(A)) .$$

Then

$$(1.10) \quad L(g_t(A)) \leq \max(L(g(A)), L(g'(0)(A))) .$$

Thus, by Proposition 1.1,  $L(g_t(A))$  is finite and

$$(1.11) \quad L(g_t(A)) \leq cL(A) \text{ for every } A \subset D .$$

Applying (i) to the case of  $g_t$  we see that  $h = 0$  when  $\mathcal{C}h \subset V$  and  $g_t(h) = h$  .

Now assume by contradiction that there exist a sequence of non-null vectors  $\{k_m\}$  converging to 0 and a sequence  $\{t_m\}$  converging to  $t$  in  $[0, 1]$  such that

$$t_m f(k_m) + (1-t_m)f'(0)k_m = k_m - (t_m g(k_m) + (1-t_m)g'(0)k_m) = 0 .$$

For each integer  $m$  there is by the preceding argument an  $r_m > 0$  such that  $h_m = r_m^{-1}k_m \in \partial V$  . Then

$$r_m h_m = \int_0^1 (t_m g'(sk_m) + (1-t_m)g'(0)) r_m h_m ds .$$

Thus

$$(1.12) \quad h_m = \frac{1}{2\pi i} \int_0^1 \int_{\Gamma} \frac{t_m g(sk_m + uh_m) + (1-t_m)g(uh_m)}{u^2} duds .$$

Put  $B = \{h_m\}$  and  $C = \overline{\text{co}}(\{k_m\} \cup \{0\})$  . We see that  $C$  is compact and  $B$  is contained in  $(g(C+B^*) \cup g(B^*))^*$  . By the properties of  $L$  it follows that

$$L(B) \leq L(g(C+B^*) \cup g(B^*)) \leq cL(B) .$$

Then  $L(B)$  is finite; hence it is equal to 0 and  $B$  is relatively compact. We can suppose that  $\{h_n\}$  converges to  $h$  in  $\partial V$ .

On the other hand we have

$$r_m h_m - g'(0)r_m h_m = t_m \int_0^1 (g'(sk_m) - g'(0))r_m h_m ds$$

or

$$(1.13) \quad h_m - g'(0)h_m = \frac{t_m}{2\pi i} \int_0^1 \int_{\Gamma} \frac{g(sk_m + uh_m) - g(uh_m)}{u^2} dudv.$$

Arguing as in the proof of Proposition 1.3, we can find a contradiction which completes our proof.

(iii) By (ii) it is sufficient to show that the following set is closed:

$$F = \{tf(h) + (1-t)f'(0)h : (t, h) \in [0, 1] \times \partial W\}.$$

Now let  $\{t_n, h_n\}$  be a sequence in  $[0, 1] \times \partial W$  such that  $\{t_n\}$  converges to  $t$  in  $[0, 1]$  and  $\{y_n\}$  converges to  $y$  in  $E$  where

$$y_n = t_n f(h_n) + (1-t_n)f'(0)h_n = h_n - (t_n g(h_n) + (1-t_n)g'(0)h_n).$$

Put  $A = \{h_n\}$ . We have

$$A \subset \{y_n\} + \overline{\text{co}}(g(A) \cup g'(0)(A)).$$

Since  $\{y_n\}$  is relatively compact we have

$$(1.14) \quad L(A) \leq \max\{L(g(A)), L(g'(0)(A))\} \leq cL(A).$$

Then  $L(A) = 0$  and  $A$  is relatively compact. We assume  $\{h_n\}$  converges to  $h$  in  $\partial W$ . It is easy to see that  $y = tf(h) + (1-t)f'(0)h$ . Therefore  $F$  is closed. //

REMARK 1.2. Lemma 4.1 in [3] is similar to the preceding proposition. There are some mistakes in the statement of this lemma, and in its proof we have used the relations (3.2) and (3.3) of [3], but the correct version of (3.2) is as follows:



$$(3.2)' \quad q(R_n f(x, k)) = \theta_{p,q}(k)(p(k))^m \text{ for all } k \text{ in } V_q$$

where  $V_q$  is an element of  $V(E)$  and is dependent on  $q$ .

Therefore (3.3) may be false. Here we shall state the correct lemma and prove it without using (3.2) and (3.3) in [3]; the notations used in this lemma are defined in [3].

LEMMA 4.1. *Let  $W$  be an open in  $E$ ,  $f = (I-g) : W \rightarrow E$  be of class  $C^1$  on  $W$ . Suppose that  $k(L, L, W, g) < k < 1$  and for every  $a$  in  $W$ ,  $f'(a) \in H(E, E)$  and  $k(L, L, W, I-\phi(Df(a))) < (1-k)/(1+k)$ . Let  $a \in W$ ; then there exist  $U, V \in V(E)$  such that  $a + V \subset W$  and for each  $(t, q)$  in  $[0, 1] \times \bar{U}$  we have*

$$(i) \quad \begin{aligned} t(f(a+x)-f(a)) + (1-t)f'(a)(x) &\neq q \text{ for all } x \in \partial V \text{ or} \\ t(f(y)-f(a)) + (1-t)f'(a)(y-a) &\neq q \text{ for all } y \in a + \partial V; \end{aligned}$$

$$(ii) \quad f(x+a) \neq f(a) \text{ for all } x \in V \setminus \{0\}.$$

Proof. We can (and shall) suppose that  $a = f(a) = 0$  and  $Df(0) = I$  (cf. [3], p. 116). By Proposition 3.1 in [3], we can choose  $W'$  in  $V(E)$  such that  $W' \subset W$  and  $Df|_{W'}$  is a continuous mapping from  $W'$  into  $B(E, E, W')$ . Let  $V \in V(E)$  such that  $\bar{V} \subset W'$  and

$$(1.15) \quad (f'(0)-f'(x)) \in B(W', \frac{1}{2}W', \frac{1}{2}) \text{ for all } x \in \bar{V}.$$

By Proposition 3.1 we have

$$g(x) = \int_0^1 (f'(0)-f'(sx))x ds \text{ for all } x \text{ in } W'.$$

Put  $F = \{tf(x)+(1-t)x = x-tg(x) : (t, x) \in [0, 1] \times \partial V\}$ ; then  $F$  is closed (cf. [3], p. 116). We shall show that  $F$  does not contain 0, which implies (i) of the lemma.

Now suppose by contradiction that there exists  $(t, x) \in [0, 1] \times \partial V$  such that

$$0 = x - tg(x) = x - t \int_0^1 (f'(0)-f'(sx))x ds.$$

Because  $x \in W'$ , by (1.15) it follows that  $x \in \frac{1}{2}W'$ . Hence by induction  $x \in 4^{-m}W'$  for all integers  $m$ . Thus  $A = \mathbb{R}x \subset W'$  and

$$y = t \int_0^1 (f'(0) - f'(sx)) y ds \quad \text{for all } y \in A .$$

Since  $s \mapsto (f'(0) - f'(sx))$  is a continuous mapping from  $[0, 1]$  into  $(B(E, E), W')$ , we see that, by Proposition 2.1 in [3],

$$L(A) \leq L \left\{ \left\{ \int_0^1 (f'(0) - f'(sx)) y ds : y \in A \right\} \right\} \leq \frac{1}{2} L(A) .$$

Hence  $L(A) = 0$ ; thus  $x = 0$ , which contradicts the condition  $x \in \partial V$ . Then  $F$  does not contain  $0$ .

Analogously we can prove (ii). //

Now we have the inverse mapping theorem for analytic contractive vector fields as follows.

**THEOREM 1.** *Let  $f : D \rightarrow E$  be an analytic contractive vector field,  $a \in D$  be such that  $f'(a)$  is a homeomorphism from  $E$  onto  $E$ . Then there exists an open neighborhood  $X$  of  $a$  such that  $X \subset D$  and  $f|X$  is a homeomorphism of  $X$  onto an open subset  $Y$  of  $E$ . Put  $f_1 = (f|X)^{-1}$ . Then  $f_1$  is analytic on  $Y$  and for every  $y \in Y$  we have*

$$f_1'(y) = (f'(f_1(y)))^{-1} .$$

*Proof.* Applying Proposition 1.2 and arguing as in the proof of Theorem 2 in [3], we can find an open neighborhood  $X$  of  $a$  in  $D$  such that  $f|X$  is a homeomorphism of  $X$  onto an open subset  $Y$  of  $E$ . We only have to show that  $f_1$  is analytic at  $f(a)$  and

$$f_1'(f(a)) = (f'(a))^{-1} .$$

We can suppose that  $a = f(a) = 0$ . For a given vector  $k$  in  $E$ , there is a positive real number  $r$  such that for each complex number  $z$ , with  $|z| \in (0, r)$ , there exists a unique  $h_z \in X \cap \frac{1}{2}V$  such that  $f(h_z) = h_z - g(h_z) = zk$ .

If  $z^{-1}h_z \in V$  we put  $v_z = 1$ , and if  $z^{-1}h_z \notin V$  we choose  $v_z$  in  $(0, 1)$  such that  $v_z z^{-1}h_z \in \partial V$ . Then

$$\begin{aligned}
 (1.16) \quad v_z k &= v_z z^{-1} h_z - \int_0^1 g'(sh_z) v_z z^{-1} h_z ds \\
 &= v_z z^{-1} h_z - \frac{1}{2\pi i} \int_0^1 \int_{\Gamma} \frac{g(sh_z + tv_z z^{-1} h_z)}{t^2} dt ds .
 \end{aligned}$$

Because  $f|X$  is a homeomorphism and  $\{zk : z \in \mathbb{C}, 0 < |z| < r\}$  is relatively compact we see that  $K = \{h_z : 0 < |z| < r\}^*$  is compact. Thus

$$A = \left\{ v_z z^{-1} h_z : 0 < |z| < r \right\} \subset [0, 1] \times \{k\} + (g(K+A^*))^* .$$

It follows that  $L(A) \leq cL(A)$ . Then  $A$  is relatively compact; hence  $A^*$  is compact also.

On the other hand we have

$$\begin{aligned}
 (1.17) \quad v_z k - v_z z^{-1} h_z + g'(0)v_z z^{-1} h_z \\
 = \frac{1}{2\pi i} \int_0^1 \int_{\Gamma} \frac{g(sh_z + tv_z z^{-1} h_z) - g(tv_z z^{-1} h_z)}{t^2} dt ds .
 \end{aligned}$$

Arguing as in the proof of Proposition 1.3, we have

$$(1.18) \quad \lim_{z \rightarrow 0} \left( v_z k - f'(0)v_z z^{-1} k \right) = 0 .$$

We shall show that there are positive numbers  $d$  and  $r'$  such that

$$(1.19) \quad v_z > d \text{ if } |z| < r' .$$

Assume by contradiction that there is a sequence  $\{z_m\}$  converging to 0 such that  $\lim_{m \rightarrow \infty} v_{z_m} = 0$ . In this case

$$\left\{ v_{z_m} z_m^{-1} h_{z_m} \right\} \subset A^* \cap \partial V .$$

Hence we can suppose that  $\left\{ v_{z_m} z_m^{-1} h_{z_m} \right\}$  converges to  $h$  in  $\partial V$ . It

follows that  $h = g'(0)$  by (1.18). This contradiction shows (1.19).

Then, from (1.18),

$$k = \lim_{z \rightarrow 0} f'(0)z^{-1} h_z \text{ or } \lim_{z \rightarrow 0} z^{-1} h_z = (f'(0))^{-1} k .$$

Now we have

$$\lim_{z \rightarrow 0} \frac{f_1(zk) - f_1(0)}{z} = \lim_{z \rightarrow 0} z^{-1} h_z = (f'(0))^{-1} k .$$

Therefore  $f_1$  is analytic at 0 and  $f_1'(0) = (f'(0))^{-1}$ . //

## 2. The Sard-Smale theorem

In this section let  $f$  be an analytic contractive vector field from  $D$  into  $E$ . We shall establish a Sard-Smale theorem for  $f$ . The following lemma is a major step in the proof of our Sard-Smale theorem.

**LEMMA 2.1.** *Let  $a \in D$ ; then there is a  $W$  in  $V(0)$  such that the following set has a void interior:*

$$S(f, a+W) = \{f(x) : x \in (a+W) \text{ and } f'(x) \text{ is not invertible}\} .$$

*Proof.* We can suppose  $a = f(a) = 0$ . Put  $T = I - f'(0)$ , and let  $G(T)$ ,  $F(T)$ ,  $P$  and  $Q$  be as in Remark 1.1. Put

$$(2.1) \quad M(x) = x - Q \circ g(x) = f(x) + P \circ g(x) \quad \text{for all } x \in D .$$

Since  $P(E)$  is a finite dimensional vector subspace of  $E$ , we can suppose that  $P \circ g(D)$  is relatively compact. Then  $M$  is an analytic contractive vector field on  $D$  because, for any  $A \subset D$ , we have

$$(2.2) \quad L(Q \circ g(A)) \leq L(g(A) - P \circ g(A)) \leq L(g(A)) + L(P \circ g(A)) = L(g(A)) .$$

On the other hand  $M'(0) = I - Q \circ g'(0)$  is a homeomorphism from  $E$  onto  $E$  by Remark 1.1. Hence by Theorem 1 there exist  $U' \in V(0)$  and an open neighborhood  $U$  of 0 in  $D$  such that  $M|U$  is a homeomorphism of  $U$  onto  $U'$  and has an analytic inverse  $R$  with derivative

$$R'(y) = (I - Q \circ g'(R(h)))^{-1} \quad \text{for all } h \text{ in } U' .$$

We choose  $W$  in  $V(0)$  such that  $W \subset U$  and

$$(2.3) \quad f(W) \cup M(W) \subset \frac{1}{2}U' .$$

We shall show  $S(f, W)$  has a void interior. Indeed suppose to the contrary that there exist  $x \in X$  and  $W' \in V(0)$  such that  $x + W'$  is contained in  $S(f, W)$ . Put  $V = \frac{1}{2}U' \cap G(T)$  and  $h : V \rightarrow G(T)$  as follows:

$$(2.4) \quad h(v) = v - P \circ g \circ R(x+v) .$$

Since  $x \in S(f, W)$ , by (2.3) we have  $x \in \frac{1}{4}U'$ ; hence  $x + v \in U'$  for all  $v \in V$ . Thus  $h$  is well defined. By the results in [8],  $h$  is Fréchet continuously differentiable on  $V$ , because it is analytic and  $G(T)$  is finite dimensional. We shall show

$$(2.5) \quad S(g, V) \supset V_1 = W' \cap G(T)$$

which would contradict Sard's lemma [7] in the finite dimensional case. Indeed let  $u \in V_1$ . We have a  $w \in W$  such that

$$(2.6) \quad f(w) = x + u$$

and such that there exists  $e \neq 0$  with the property

$$(2.7) \quad e = g'(w)e.$$

If  $Pe = 0$  then  $e = Qe$ , in which case we have by (2.7) that  $Qe = Q \circ g'(w)e$ , a contradiction because  $M'(w)$  is invertible. We have proved that

$$(2.8) \quad Pe \neq 0.$$

Now put

$$(2.9) \quad v = M(w) - x.$$

By (2.3) and the properties of  $x$  stated above, we have  $v \in \frac{1}{2}U'$ . By (2.1) and (2.6),  $v \in G(T)$ . Thus  $v \in V$  and

$$(2.10) \quad R(x+v) = w.$$

Then

$$(2.11) \quad \begin{aligned} h(v) &= v - P \circ g(w) = w - Q \circ g(w) - x - P \circ g(w) \\ &= f(w) - x = u. \end{aligned}$$

Hence

$$(2.12) \quad h(v) = u.$$

On the other hand

$$(2.13) \quad h'(v) = I - P \circ g'(R(x+v)) \circ (I - Q \circ g'(R(x+v)))^{-1}.$$

By (2.10) and by  $Pe = e - Qe = e - Q \circ g'(w)e$ , we have

$$\begin{aligned} h'(v)Pe &= Pe - P \circ g'(w) \circ (I - Q \circ g'(w))^{-1} \circ (I - Q \circ g'(w))e \\ &= Pe - P \circ g'(w)e = Pe - Pe = 0. \end{aligned}$$

Hence

$$(2.14) \quad h'(v)Pe = 0 .$$

From (2.8), (2.12) and (2.14), we infer that  $u \in S(h, V)$ . We have just proved (2.5), which completes the proof of the lemma. //

From this lemma we have

**PROPOSITION 2.1.** *Let  $M$  be a closed subset of  $E$  contained in  $D$ ; then  $f(M)$  is closed and  $S(f, M)$  is a closed subset with the void interior.*

*Proof.* Let  $\{x_n\}$  be a sequence in  $M$  such that  $\{f(x_n)\}$  converges to  $y$  in  $E$ . Arguing as in the proof of (iii) of Proposition 1.4, we can suppose that  $\{x_n\}$  converges to  $x$  in  $M$ . Then  $y = f(x)$  belongs to  $f(M)$ ; hence  $f(M)$  is closed. Now if  $f'(x_n)$  is not invertible for all integers  $n$ , by Proposition 1.3,  $f'(x)$  cannot be invertible. Therefore  $S(f, M)$  is closed.

Now assume that there exist  $V \in V(0)$  and  $y \in E$  such that  $y + V$  is contained in  $S(f, M)$ . As above we see that  $f^{-1}(\{y\}) \cap M$  is compact. By Lemma 2.1 there exists a finite family of open sets  $\{W_1, \dots, W_m\}$  in  $D$  such that

$$f^{-1}(\{y\}) \cap M \subset \bigcup_j W_j = W$$

and the interior of  $S(f, W_j)$  is empty for any  $j = 1, \dots, m$ .

On the other hand  $f(M \setminus W)$  is a closed set and does not contain  $y$ , where  $M \setminus W = \{x \in M : x \notin W\}$ . Then there exists  $U \in V(0)$  such that  $U \subset V$  and

$$(y+U) \cap f(M \setminus W) = \emptyset .$$

Therefore

$$f^{-1}(y+U) \cap M \subset W .$$

Thus by definition

$$y + U \subset \bigcup_j S(f, W_j) .$$

This is absurd because the interior of  $\bigcup_j S(f, W_j)$  is empty. This contradiction completes the proof of the proposition. //

The following is a partial generalization to Frechet spaces of the Sard-Smale theorem ([9], [7]).

**THEOREM 2.** *The set*

$$S(f, D) = \{f(x) : x \in D \text{ } f'(x) \text{ is not invertible}\}$$

*has the empty interior if at least one of the following conditions is satisfied:*

- (i) *f is an analytic contractive vector field defined on an open set containing the closure of D ;*
- (ii) *E is separable and f is an analytic contractive vector field on D .*

*Proof.* (i) From Proposition 2.1 we have (i).

(ii) Let  $\mathcal{B}$  be a countable basis of the topology of  $E$  . For each  $x$  in  $D$  , let  $V_x \in \mathcal{B}$  be such that  $\bar{V}_x$  is a closed neighborhood of  $x$  in  $D$  and  $S(f, \bar{V}_x)$  has the empty interior. Since  $S(f, D)$  is a countable union of closed meager sets of form  $S(f, \bar{V}_x)$  , it follows from Baire's theorem that  $S(f, D)$  is meager. //

### 3. Application to degree theory

In this section let  $f$  be a continuous analytic vector field from  $\bar{D}$  into  $E$  , and analytic on  $D$  . Assume that  $0 \notin f(\partial D)$  . Using the Sard-Smale theorem in Section 2 we shall establish the relations between the topological degree and the number of zeros of  $f$  . At first we have the useful lemma

**LEMMA 3.1.** *Let  $p \in E \setminus f(\partial D)$  and  $V \in \mathcal{V}(0)$  . Then there is  $q$  in  $p + V$  such that  $q \in E \setminus f(\partial D)$  and  $f'(x)$  is invertible for each  $x$  in  $f^{-1}(\{q\})$  .*

*Proof.* Since  $f(\partial D)$  is closed, we can suppose  $p + V \subset E \setminus f(\partial D)$  . Now assume by contradiction that  $p + V \subset S(f, D)$  . Because  $f^{-1}(\{p\})$  is a compact subset of  $D$  , arguing as in the proof of Proposition 2.1, we can

find  $U \in V(0)$  and an open  $W$  in  $D$  such that  $S(f, W)$  has the empty interior and  $p + U \subset S(f, W)$ . This contradiction proves the lemma. //

The topological degree of  $f$  at 0 in  $D$  is defined as in [3] and denoted by  $\deg(f, D, 0)$ . We see that it is positive as follows.

**PROPOSITION 3.1.**  $\deg(f, D, 0) \geq 0$ .

**Proof.** Choose  $V \in V(0)$  and  $q$  as in the proof of the preceding lemma. For any  $(t, x) \in [0, 1] \times \bar{D}$  we write

$$(3.1) \quad h(t, x) = g(x) + tq.$$

It is clear that  $x - h(t, x) \neq 0$  if  $(t, x) \in [0, 1] \times \partial D$ , and for each  $A \subset \bar{D}$ , we have, by the properties of  $L$ ,

$$L(h([0, 1] \times A)) \leq L(g(A) + [0, 1] \times \{q\}) \leq L(g(A)).$$

Then  $L(h([0, 1] \times A))$  is finite and is less than or equal to  $cL(A)$ .

Thus, by (D.4) in [3] (p. 97), we have

$$(3.2) \quad \deg(I-g, D, 0) = \deg(I-g-q, D, 0).$$

It is sufficient to show

$$(3.3) \quad \deg(I-g-q, D, 0) \geq 0.$$

Indeed  $(f-q)^{-1}(\{0\}) = f^{-1}(\{q\})$  is a compact subset of  $D$  and  $f'(x)$  is invertible whenever  $x \in f^{-1}(\{q\})$ . By Proposition 1.4 we see that  $f^{-1}(\{q\})$  consists of isolated points. Hence  $f^{-1}(\{q\})$  is a finite set  $\{x_1, \dots, x_m\}$ . Applying Proposition 1.4 again we can find the disjoint open sets  $W_1, \dots, W_m$  in  $D$  such that  $x_i \in W_i$  for all  $i = 1, \dots, m$  and

$$\deg(f-q, W_i, 0) = \deg(f'(x_i), W_i - x_i, 0) = 1.$$

Therefore, by (D.3) in [3] (p. 97), we have

$$\deg(f-q, D, 0) = \sum_j \deg(f-q, W_j, 0) = m \geq 0.$$

Hence by (3.2) we have the desired result. //

We shall conclude this paper with the following

**THEOREM 3.** *The number of zeros of  $f$  in  $D$  is at most equal to*



$\text{deg}(f, D, 0)$  .

Proof. By Proposition 3.1,  $\text{deg}(f, D, 0) = k \geq 0$  . Now assume that there are  $k + 1$  distinct zeros  $z_1, \dots, z_{k+1}$  of  $f$  . We shall find a contradiction.

Put  $G_j = G(g'(z_j))$  as in Remark 1.1, and let  $P_j$  and  $Q_j$  denote the corresponding projections for each  $j = 1, \dots, k+1$  . Using the Hahn-Banach theorem, one can show that there exists a continuous linear functional  $h$  on  $E$  such that  $h(z_j - z_{j'}) \neq 0$  for all  $j \neq j'$  .

Because  $f^{-1}(\{0\})$  is compact, by (D.3) in [3] (p. 97) we can replace  $D$  by an open subset of itself, also denoted by  $D$  , such that  $h(\bar{D})$  and  $P_j(\bar{D})$  are relatively compact. For each  $x \in \bar{D}$  put

$$(3.4) \quad a_j(x) = \prod_{j' \neq j} (h(x - z_{j'}))^2 .$$

For  $t > 0$  put

$$(3.5) \quad f_t(x) = f(x) + t \sum_j a_j(x) P_j(x - z_j) .$$

Since the set  $\left\{ \sum_j a_j(x) P_j(x - z_j) : x \in \bar{D} \right\}$  is relatively compact, and  $f(\partial D)$  is a closed set and  $0 \notin f(\partial D)$  , then, for  $t > 0$  sufficiently small, we have, by (D.4) in [3] (p. 97),

$$(3.6) \quad f_t(x) \neq 0 \text{ for all } x \in \partial D$$

and

$$(3.7) \quad \text{deg}(f_t, D, 0) = \text{deg}(f, D, 0) = k .$$

Fix  $t$  and  $j$  and assume  $f'_t(z_j)x = 0$  for an  $x \neq 0$  ; then

$$\begin{aligned} f'_t(z_j)x &= (f'(z_j) + t a_j(z_j) P_j)x \\ &= (Q_j x - g'(z_j) Q_j x) + ((1 + t a_j(z_j) P_j)x - g'(z_j) P_j x) = 0 . \end{aligned}$$

By Remark 1.1,  $Q_j x = 0$  . Hence  $P_j x \neq 0$  and thus  $1 + t a_j(z_j)$  is an eigenvalue of  $g'(z_j)$  and belongs to the following set if  $t$  is small

enough:

$$B = \{z \in \mathbb{C} : |z| \geq \frac{1}{2}, z \text{ is an eigenvalue of } g'(z_j)\} \\ \text{for some } j = 1, \dots, k+1 \} .$$

By Theorem 1 in [4],  $B$  is a finite set. Then  $f'_t(z_j)$  is invertible for  $t > 0$  sufficiently small, and for every  $j = 1, \dots, k+1$  (we recall that  $f'_t(z_j)$  is a Fredholm operator of index 0 by Proposition 5 in [4]).

Now let  $t > 0$  be so small that  $f'_t(z_j)$  is invertible for all  $j$  and we have (3.7). Since  $f_t(z_j) = 0$  for every  $j = 1, \dots, k+1$  and  $f_t$  is an analytic contractive vector field on  $D$ . Arguing as in the proof of Proposition 3.1, we can find  $W \in V(0)$  such that  $\{x_j+W : j = 1, \dots, k+1\}$  are disjoint and

$$(3.8) \quad \deg(f_t, x_j+\overset{\circ}{W}, 0) = 1 \text{ for all } j = 1, \dots, k+1 ,$$

and

$$(3.9) \quad 0 \notin f_t(x_j+\overset{\circ}{W}) \text{ for all } j = 1, \dots, k+1 ,$$

where  $\overset{\circ}{W}$  is the interior of  $W$ .

On the other hand we have, by Proposition 3.1,

$$(3.10) \quad \deg\left(f_t, D \setminus \bigcup_j (x_j+W), 0\right) \geq 0 .$$

Now by (D.3) in [3] we get

$$\deg(f_t, D, 0) = \deg\left(f_t, D \setminus \bigcup_j (x_j+W), 0\right) + \sum_j \deg(f_t, x_j+\overset{\circ}{W}, 0) .$$

Therefore, by (3.8) and (3.10),

$$\deg(f_t, D, 0) \geq k + 1 ,$$

which contradicts (3.7). This contradiction completes our proof. //

REMARK 3.1. If  $f$  is a compact analytic vector field and  $E$  is a Banach space, Theorem 3 is due to Cronin and Schwartz ([2], [8], [5]). If  $f$  has the form  $I - T - C$ , where  $C$  is a compact analytic mapping and  $T$  is a contraction and  $E$  is a Banach space, the theorem is partially proved in [1].

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