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Universal Minimal Flows of Groups of Automorphisms of Uncountable Structures

Dana Bartošová

Abstract. It is a well-known fact that the greatest ambit for a topological group G is the Samuel compactification of G with respect to the right uniformity on G. We apply the original description by Samuel from 1948 to give a simple computation of the universal minimal flow for groups of automorphisms of uncountable structures using Fraïssé theory and Ramsey theory. This work generalizes some of the known results about countable structures.

1 Introduction

Universal minimal flows play an important role in topological dynamics. In [KPT05], the authors explicitly compute universal minimal flows of groups of automorphisms of countable structures with certain properties using Fraïssé theory and Ramsey theory. In the same spirit, we compute universal minimal flows for dense subgroups of automorphism groups of uncountable structures. However, for uncountable structures we do not have in hand a generic ordering, so we have to take a slightly different route. This work was initially inspired by a talk of Y. Gutman [Gut10] on the universal minimal flow of the group of homeomorphisms of $\omega^* = \beta\omega \setminus \omega$, where $\beta\omega$ is the Čech–Stone compactification of discrete ω, and a suggestion of Todorčević to reprove his result using the ideas of [KPT05], a project that has been also the subject of his joint work with P. Ursino several years ago. We observe that the same space serves as the universal minimal flow for all dense subgroups, hence for all normal subgroups of the group of trivial homeomorphisms of ω^* described by van Douwen in [vD90].

The structure of the paper is as follows: In the second section, we introduce the basic notions from topological dynamics and describe them for the case of groups that admit a basis of neighbourhoods of the neutral element of open subgroups. In the third section, we talk about groups of automorphisms and their relationship with the groups in the first section. In the fourth section, we introduce crucial ingredients—Fraïssé classes, Ramsey theory and linear orderings. In the fifth section, we prove the main theorem and apply it to describe some universal minimal flows. In the last section, we use the main theorem to characterize extremely amenable groups, generalizing Theorem 4.3 from [KPT05] to uncountable structures.

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2 Topological Dynamics

The central notion of topological dynamics is a continuous action $\pi\colon G \times X \to X$ of a topological group G on a compact Hausdorff space X. We call X a *G-flow*, and we omit π if it is understood and write gx instead of $\pi(g,x)$. A *homomorphism* of G-flows X and Y is a continuous map $\phi\colon X \to Y$ respecting the actions of G on X and Y, *i.e.*, $\phi(gx) = g\phi(x)$ for every $g \in G$, $x \in X$ and $y \in Y$. We say that Y is a *factor* of X, if there is a homomorphism from X onto Y. Every G-flow has a minimal subflow, a minimal closed subspace of X invariant under the action of G. Among all minimal G-flows, there is a maximal one—the *universal minimal flow* $M(G)$. It means that every other minimal G-flow is a factor of $M(G)$. In the study of universal minimal flows, a construction of the greatest ambit turns out to be useful. An *ambit* is a G-flow X with a distinguished point $x_0 \in X$ whose orbit $Gx_0 = \{gx_0 : g \in G\}$ is dense in X. Likewise for minimal flows, there is a maximal ambit—the *greatest ambit* $\bigl(S(G), e\bigr)$. It means that every other ambit (X, x_0) is a factor of $S(G)$ via a quotient mapping sending e to x_0. As we say below, the greatest ambit is a compactification of G with a structure of a right-topological semigroup and the universal minimal flow is a minimal left ideal of $S(G)$. The study of $S(G)$ shows that every homomorphism of $M(G)$ into itself is an isomorphism, which in turn gives that the universal minimal flow is unique up to an isomorphism. For an introduction to topological dynamics, see [dV93].

We will describe the greatest ambit and the universal minimal flow of groups of automorphisms as Stone spaces of certain Boolean algebras. For completeness, we introduce these notions and Stone duality between them.

Stone Representation Theorem

In 1936, M. H. Stone [Sto36] proved that every Boolean algebra $\mathcal{B}$ is isomorphic to the Boolean algebra of all clopen subsets of a compact totally disconnected Hausdorff space $\mathrm{Ult}(\mathcal{B})$. The points of $\mathrm{Ult}(\mathcal{B})$ are the ultrafilters on $\mathcal{B}$, and the sets $A^* = \{u \in \mathrm{Ult}(\mathcal{B}) : A \in u\}$ for $A \in \mathcal{B}$ form a clopen base of the topology on $\mathrm{Ult}(\mathcal{B})$. This gives a one-to-one correspondence that also extends to homomorphisms: if $f\colon \mathcal{B} \to \mathcal{C}$ is a homomorphism between two Boolean algebras, then $\mathrm{Ult}(f)\colon \mathrm{Ult}(\mathcal{C}) \to \mathrm{Ult}(\mathcal{B})$, given by

$$u \mapsto \text{“the ultrafilter on } \mathcal{B} \text{ generated by } \bigcup\{f^{-1}(A) : A \in u\}\text{”}$$

is a continuous map. If f is injective, then $\mathrm{Ult}(f)$ is surjective. In terms of category theory, Ult is a contravariant functor giving an equivalence between the category of Boolean algebras with homomorphisms and the category of compact totally disconnected Hausdorff spaces with continuous mappings.

Greatest Ambit

If G is a discrete group, then its greatest ambit is the ultrafilter dynamical system $\mathrm{Ult}(G)$. This is the space of all ultrafilters on the underlying set of G with the topology generated by clopen sets $A^* = \{u \in \mathrm{Ult}(G) : A \in u\}$. It means that $\mathrm{Ult}(G)$ is the

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Stone space $\mathrm{Ult}\big(\mathcal{P}(G)\big)$ of the power set algebra $\mathcal{P}(G)$ of G. The action of G on $\mathrm{Ult}(G)$ extends the multiplication in G: $gu = \{gA : A \in u\}$ for $g \in G$ and $u \in \mathrm{Ult}(G)$. Moreover, we can extend the multiplication to all of $\mathrm{Ult}(G)$, turning $\mathrm{Ult}(G)$ into a semigroup: for $u, v \in \mathrm{Ult}(G)$, we set $uv = \big\{A \subset G : \{g \in G : g^{-1}[A] \in v\} \in u\big\}$. In other words, $uv = u - \lim\{gv : g \in G\}$. Fixing an ultrafilter u, the right multiplication $\cdot u$: $\mathrm{Ult}(G) \to \mathrm{Ult}(G)$, $v \mapsto uv$ is continuous, hence $\mathrm{Ult}(G)$ is a *right-topological semigroup*.

If G is a topological group, then the greatest ambit of G is a factor of $\mathrm{Ult}(G)$ by definition. It was first described by Samuel in 1948 (see [Sam48]) in the setting of uniform spaces and seems to be overlooked by topological dynamists. We briefly describe his construction for topological groups with the right uniformity. Let G be a topological group with the neutral element e and $\mathcal{N}$ a basis of neighbourhoods of e giving the topology on G. The *right uniformity* on G is generated by covers $\{Va : a \in G\}$ for $V \in \mathcal{N}$. We define an equivalence relation $\sim$ on $\mathrm{Ult}(G)$ as follows. For an ultrafilter $u \in \mathrm{Ult}(G)$, we define a filter u' generated by $\{VA : A \in u, V \in \mathcal{N}\}$. Then for $u, v \in \mathrm{Ult}(G)$ we set $u \sim v$ if and only if $u' = v'$. The quotient space $S(G) = \mathrm{Ult}(G)/\sim$ with the quotient topology is called the Samuel compactification of G and it is the greatest ambit of G. The multiplication on $\mathrm{Ult}(G)$ also factors to $S(G)$, making $S(G)$ a right-topological semigroup with the multiplication extending the multiplication on G.

From now on, G will always be a topological group that possesses a basis $\mathcal{N}$ of open neighboorhoods of the neutral element e consisting of open (hence clopen) subgroups of G. It is easy to see that then $L = \{VA : A \subset G, V \in \mathcal{N}\}$ is a Boolean algebra, and we get the following description of the greatest ambit.

Lemma 1 *The greatest ambit $S(G)$ is equal to the Stone space of the Boolean algebra L as above with action defined by $gx = \{gA : A \in x\}$ for $g \in G$ and $x \in \mathrm{Ult}(L)$.*

Proof We will show that $u' \mapsto u \cap L$ for $u \in \mathrm{Ult}(G)$ is an isomorphism between $S(G)$ and the Stone space of L.

Let $u, v \in \mathrm{Ult}(G)$. Since $VV = V$ for every $V \in \mathcal{N}$, we have that $u \cap L$ is an ultrafilter on L that generates $u' \in S(G)$. This means that whenever $u \cap L = v \cap L$, then $u \sim v$.

On the other hand, if $u \cap L \neq v \cap L$, then there is $A \in L$ such that $A \in u \cap L$ and $G \backslash A \in v \cap L$, which implies that $u' \neq v'$. This gives us the sought-for correspondence between $S(G)$ and ultrafilters on L. ∎

The quotient multiplication on $\mathrm{Ult}(L)$ can be described explicitly as an analogue of the multiplication on $\mathrm{Ult}(G)$.

Lemma 2 *Let $u, v \in \mathrm{Ult}(L)$, $A \subset G$ and $V \in \mathcal{N}$. Then $VA \in uv$ if and only if $\{g \in G : g^{-1}[VA] \in v\} \in u$.*

Proof Let $u', v' \in \mathrm{Ult}(G)$ be representatives of the equivalence classes of u, v respectively. We need to show that the definition of uv in the claim corresponds to $u'v' \cap L$. By the definition of ultrafilter multiplication, $VA \in u'v' \cap L$ if and only if $U = \{g \in G : g^{-1}[VA] \in v'\} \in u'$. First, we show that we can replace u' with u. Notice that $VU = U$, since for every $s \in V$ and $g \in G$, we have that

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$sg \in U$ if and only if $(sg)^{-1}[VA] = g^{-1}s^{-1}[VA] = g^{-1}[VA] \in v$ if and only if $g \in U$. Therefore, $VA \in u'v' \cap L$ if and only if $U \in u' \cap L = u$. Second, we show that v' can be replaced by v. For every $g \in G$ and $W \in \mathcal{N}$, $g^{-1}[VA] \in v'$ implies $Wg^{-1}[VA] \in v$. Pick a $W \in \mathcal{N}$ such that $gWg^{-1} \subset V$. Then $Wg^{-1} \subset g^{-1}V$, so $Wg^{-1}VA \subset g^{-1}VVA = g^{-1}VA$. But obviously, $g^{-1}VA \subset Wg^{-1}VA$, hence $Wg^{-1}VA = g^{-1}VA$, which shows that $g^{-1}VA \in v'$ if and only if $g^{-1}VA \in v' \cap L = v$. This concludes the proof. ∎

Universal Minimal Flow

Let G be a group with a basis of neighbourhoods $\mathcal{N}$ of the neutral element e consisting of open subgroups and let L be as above. The universal minimal flow $M(G)$ for G, being a subspace of the greatest ambit $S(G)$, is itself a Stone space. Hence we can consider its Boolean algebra of all clopen subsets $B(G)$. For $m \in M(G)$ and $\varnothing \neq U \in B(G)$, denote by $\operatorname{Ret}(m, U)$ the set of elements of G that bring m into U, *i.e.*, $\operatorname{Ret}(m, U) = \{g \in G : gm \in U\}$. Since $h \operatorname{Ret}(m, U) = \operatorname{Ret}(m, hU)$ and $M(G)$ is compact, there are finitely many $g_1, g_2, \dots, g_n \in G$ such that $\bigcup_{i=1}^{n} g_i \operatorname{Ret}(m, U) = G$. Such sets are called *syndetic*. More generally, see the following lemma.

Lemma 3 *The following are equivalent for a G-flow X:*

(i) *X is minimal;*
(ii) *for every non-empty open set $O \subset X$, $\bigcup_{g \in G} gO = X$;*
(iii) *for every $x \in X$ and non-empty open set $O \subset X$, the set $\operatorname{Ret}(x, O)$ is syndetic.*

Proof (i) ⇒ (ii) Let X be a minimal G-flow, $x \in X$ and O a non-empty open subset of X. If $X \setminus \bigcup_{g \in G} gO \neq \varnothing$, then it is a non-trivial closed subflow witnessing non-minimality of X.

(ii) ⇒ (iii) Since X is compact, the cover $\{gO : g \in G\}$ has a finite subcover $\{g_1 O, g_2 O, \dots, g_n O\}$. Then $\bigcup_{i=1}^{n} g_i \operatorname{Ret}(x, O) = \bigcup_{i=1}^{n} \operatorname{Ret}(x, g_i O) = G$, so $\operatorname{Ret}(x, O)$ is syndetic.

(iii) ⇒ (i) If X is not minimal, then there is an $x \in X$ such that $O = X \setminus \overline{Gx}$ is a non-empty open set. But then $\operatorname{Ret}(x, O) = \varnothing$, hence not syndetic. ∎

A Boolean algebra of subsets of G is called a *syndetic algebra*, if it is invariant under left translations by elements from G and all of its non-empty elements are syndetic sets.

Now, we are ready to imitate a proof for discrete semigroups from [BF97] to characterize $B(G)$.

Theorem 1 *The universal minimal flow $M(G)$ is the Stone space of a maximal syndetic subalgebra of L. All maximal syndetic subalgebras of L are isomorphic.*

Proof Let $B(G)$ denote the algebra of clopen subsets of $M(G)$ and let $m \in M(G)$. Let $R_m \colon S(G) \to S(G)$ denote the right translation by m, *i.e.*, $u \mapsto um$.

Since $M(G)$ is a minimal left ideal of $S(G)$, $S(G)m = M(G)$. So R_m actually maps $S(G)$ onto $M(G)$. $S(G)$ being a right-topological semigroup, R_m is continuous. Since

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R_m is onto, the dual homomorphism between the Boolean algebras of clopen sets $\rho_m \colon B(G) \to L$ is injective. By Lemma 2, we know that for every $VA \in L$,

$$\begin{aligned}\rho_m\big((VA)^* \cap M(G)\big) &= \{u \in S(G) : VA \in um\}\\ &= \{g \in G : g^{-1}[VA] \in m\}\\ &= \operatorname{Ret}\big(m, (VA)^*\big).\end{aligned}$$

So $B(G)$ is isomorphic to

$$\mathcal{A} = \big\{ \operatorname{Ret}\big(m, (VA)^*\big) : A \subset G, V \in \mathcal{N} \big\},$$

which is a subalgebra of L consisting exclusively of syndetic sets.

Now we show that $\mathcal{A}$ is invariant under left translations by G. Let $h \in G$; then

$$\begin{aligned}h \operatorname{Ret}\big(m, (VA)^*\big) &= h\{g \in G : g^{-1}[VA] \in m\}\\ &= \{x \in G : x^{-1}[hVA] \in m\}\\ &= \operatorname{Ret}\big(m, (hVA)^*\big).\end{aligned}$$

It remains to show that $\mathcal{A}$ is a maximal syndetic algebra. Let $\mathcal{B} \supset \mathcal{A}$ be a syndetic algebra. Then $\operatorname{Ult}(\mathcal{B})$ with multiplication $gu = \{gA : A \in u\}$ for $g \in G$ and $u \in \operatorname{Ult}(\mathcal{B})$ is a minimal flow. The identity embedding of $i \colon \mathcal{A} \hookrightarrow \mathcal{B}$ induces a surjective G-homomorphism $\operatorname{Ult}(i)\colon \operatorname{Ult}(\mathcal{B}) \to \operatorname{Ult}(\mathcal{A}) \cong M(G)$. This means that also $\operatorname{Ult}(\mathcal{B}) \cong M(G)$. Since every G-homomorphism from $M(G)$ to itself is an isomorphism, so is $\operatorname{Ult}(i)$, which is only possible if $\mathcal{A} = \mathcal{B}$.

For the second part of the theorem, let us assume that $\mathcal{A}$ is a maximal syndetic subalgebra of L. To achieve the conclusion, it is enough to find an $m \in M(G)$ such that $\mathcal{A} = \{\operatorname{Ret}(m, O) : O \in B(G)\}$.

Clearly, $\operatorname{Ult}(\mathcal{A})$ with G-multiplication as above is a minimal G-flow. Hence there is a G-homomorphism $\phi\colon M(G) \to \operatorname{Ult}(\mathcal{A})$. Consider $p \in \operatorname{Ult}(\mathcal{A})$ given by $p = \{A \in \mathcal{A} : e \in A\}$ and its preimage m under ϕ. Given $A \in \mathcal{A}$, denote by A^* the clopen set $\{u \in \operatorname{Ult}(\mathcal{A}) : A \in u\}$. Then $\operatorname{Ret}(p, A^*) = \operatorname{Ret}(m, \phi^{-1}[A])$, since $\phi(gm) = g\phi(m) = gp$ for all $g \in G$. But also

$$\begin{aligned}\operatorname{Ret}(p, A^*) &= \{g \in G : gp \in A^*\} = \{g \in G : g^{-1}[A] \in p\}\\ &= \{g \in G : e \in g^{-1}[A]\} = \{g \in G : ge \in A\} = A.\end{aligned}$$

So we have that

$$\mathcal{A} = \{\operatorname{Ret}(p, A^*) : A \in \mathcal{A}\} \subset \{\operatorname{Ret}(m, O) : O \in B(G)\}.$$

By maximality of $\mathcal{A}$, we get that $\mathcal{A} = \{\operatorname{Ret}(m, O) : O \in B(G)\}$. ∎

Corollary 1 *$M(G)$ is a totally disconnected space.*

3 Automorphism Groups

Let κ be a cardinal number endowed with the discrete topology and denote by S_κ the group of all bijections on κ. In what follows, we consider S_κ as a topological group with the topology of pointwise convergence. The topology is given by a basis of neighbourhoods of the neutral element consisting of open subgroups $S_A = \{g \in S_\kappa : g(a) = a, a \in A\}$ where A is a finite subset of κ. We can observe that a subset H of S_κ is closed if and only if it contains every $g \in S_\kappa$ such that for any $A \subset \kappa$ finite there exists an $h \in H$ with $g|A = h|A$.

Let $G \leq S_\kappa$ be a subgroup and let A be a finite subset of κ. We define the pointwise stabilizer of A as

$$G_A = \{g \in G : ga = a, a \in A\}$$

and the set-wise stabilizer as

$$G_{(A)} = \{g \in G : gA = A\}.$$

Let L be a first order language and $\mathcal{A}$ an L-structure with the universe of cardinality κ. Then the group of automorphisms of $\mathcal{A}$, $\mathrm{Aut}(\mathcal{A})$, is a closed subgroup of S_κ. It is clear that the topology on $\mathrm{Aut}(\mathcal{A})$ is given by $\mathrm{Aut}(\mathcal{A})_A$ for A a finitely-generated substructure of $\mathcal{A}$. If every partial isomorphism between two finitely generated substructures of $\mathcal{A}$ can be extended to an automorphism of the whole structure $\mathcal{A}$, then we say that $\mathcal{A}$ is *ω-homogeneous*. In the case of Boolean algebras, ω-homogeneity coincides with the notion of homogeneity (see [MB89]). A Boolean algebra $\mathcal{B}$ is *homogeneous* if for every $b \in \mathcal{B}$ the relative Boolean algebra $\mathcal{B}|b = \{c \in \mathcal{B} : c \leq b\}$ is isomorphic to $\mathcal{B}$.

In what follows, when we say "a structure", we mean a structure for some first order language.

Finally, we describe the correspondence between groups possessing a neighbourhood basis of the neutral element of open subgroups, dense subgroups of groups of automorphisms of structures and subgroups of S_κ.

Theorem 2 *Let G be an infinite topological group and let κ be a cardinal number. Then the following are equivalent:*

(a) *G is a subgroup of S_κ;*
(b) *G has a basis $\mathcal{N}$ of neighbourhoods of the neutral element of cardinality $\lambda \leq \kappa$ consisting of open subgroups such that the family of all left translates of elements from $\mathcal{N}$ also has cardinality λ;*
(c) *G is a dense subgroup of a group of automorphisms of an ω-homogeneous relational structure on a set of cardinality κ;*
(d) *G is a dense subgroup of a group of automorphisms of a structure on a set of cardinality κ.*

Proof The equivalence of (a), (c) and (d) follows from [Hod97, Theorem 4.1.1]. (b) trivially follows from (a), so we only need to establish that also (b) implies (a). We proceed as in [BK96, Theorem 1.5.1]. Let $\{U_i : i \in \lambda\}$ be an enumeration of a basis for the topology on G given by all left translates of subgroups from $\mathcal{N}$. For every $g \in G$, define $\phi(g) = \pi_g \in S_\kappa$ by

$$\pi_g(i) = j \Leftrightarrow gU_i = U_j \quad \text{for } i \in \lambda \text{ and } \pi_g(i) = i \text{ otherwise.}$$

Obviously, $g \mapsto \pi_g$ is an injective homomorphism of G into S_κ. To show that it is continuous, let $A \subset \lambda$ be finite and let $S_{\kappa,A} = \{\pi \in S_\kappa : \pi(i) = i, i \in A\}$ be a basic open subgroup of S_κ. Then $\phi^{-1}(S_{\kappa,A}) = \{g \in G : gU_i = U_i, i \in A\}$. Since for every $i \in A$, $U_i = h_i V_i$ for some $h_i \in G$ and some subgroup $V_i \in \mathcal{N}$,

$$\begin{aligned}\phi^{-1}(S_{\kappa,A}) &= \bigcap_{i\in A}\{g \in G : gh_iV_i = h_iV_i\}\\ &= \bigcap_{i\in A}\{g \in G : h_i^{-1}gh_i \in V_i\}\\ &= \bigcap_{i\in A}\{g \in G : g \in h_iV_ih_i^{-1}\},\end{aligned}$$

which is an intersection of finitely many open sets, hence open. Similarly, let $H \in \mathcal{N}$ be an open subgroup of G, so $H = U_i$ for some $i \in \lambda$. Then $\phi(H) = \{\pi_h : h \in H\} = \{\pi \in S_\kappa : \pi(i) = i\} \cap \phi(G)$, hence ϕ is a homeomorphism onto its image. ∎

4 Fraïssé Classes, Ramsey Theory and Linear Orderings

Now, we give necessary definitions and facts about Fraïssé classes, the Ramsey property for finite structures and linear orderings. For a comprehensive treatment of these ingredients, see [KPT05].

Fraïssé classes

A class of finitely-generated structures $\mathcal{F}$ of a given language is called a *Fraïssé class*, if it satisfies the following conditions:

(HD) Hereditary property: if A is a finitely generated substructure of B and $B \in \mathcal{F}$, then also $A \in \mathcal{F}$.

(JEP) Joint embedding property: if $A, B \in \mathcal{F}$, then there exists a $C \in \mathcal{F}$ in which both A and B embed.

(AP) Amalgamation property: if $A, B, C \in \mathcal{F}$ and $i\colon A \to B$ and $j\colon A \to C$ are embeddings, then there exist $D \in \mathcal{F}$ and embeddings $k\colon B \to D$ and $l\colon C \to D$ such that $k \circ i = l \circ j$.

Let $\mathcal{A}$ be an ω-homogeneous structure. Then it is easily verified that $\mathrm{Age}(\mathcal{A})$, the class of all finitely-generated substructures of $\mathcal{A}$, is a Fraïssé class. In case of countable structures, there is a one-to-one correspondence between ω-homogeneous structures and Fraïssé classes, given by $\mathcal{A} \mapsto \mathrm{Age}(\mathcal{A})$, see [Fra54].

Example 1 The following are Fraïssé classes:

(a) finite sets
(b) linearly ordered finite graphs

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(c) finite Boolean algebras
(d) finite vectors spaces over a finite field
(e) finite linear orderings

In what follows, we will only be interested in Fraïssé classes consisting of finite structures.

Ramsey Theory

A class $\mathcal{K}$ of finite structures satisfies the *Ramsey property* if for every $A \leq B \in \mathcal{K}$ and natural number $k \geq 2$ there exists $C \in \mathcal{K}$ such that

$$C \to (B)^A_k,$$

which means that for every colouring of copies of A in C by k colours, there is a copy B' of B in C, such that all copies of A in B' have the same colour.

Example 2 All examples of Fraïssé classes in Example 1 satisfy the Ramsey property: (a) is the classical Ramsey theorem, (b) was proved by Abramson and Harrington [AH78] and Nešetřil and Rödl [NR77], (c) is equivalent to the so-called dual Ramsey theorem by Graham and Rothschild [GR], (d) was proved by Graham, Leeb, and Rothschild in [GLR72] and (e) is equivalent to (a).

Linear orderings

Let $\mathrm{LO}(\kappa)$ denote the space of all linear orderings on κ considered as a subspace of $2^{\kappa\times\kappa}$ with the product topology. The topology on $\mathrm{LO}(\kappa)$ is generated by clopen sets

$$(A,<)^* = \{<' \in \mathrm{LO}(\kappa) : <'|A = <\},$$

for $A \subset \kappa$ finite and $<$ a linear ordering on A. In other words, $\mathrm{LO}(\kappa)$ is the Stone space of the Boolean algebra generated by all $(A,<)^*$ for $A \subset \kappa$ finite and $<$ a linear ordering on A.

Let $\mathcal{A}$ be a structure of size κ and again let $\mathrm{Age}(\mathcal{A})$ denote the family of finitely-generated substructures of $\mathcal{A}$. Let L be the language of $\mathcal{A}$ and let $L' = L \cup \{<\}$ be an expansion of L by a binary relational symbol $<$ not in L. Let $\mathcal{K}$ be a family of structures for L' in which $<$ is a linear ordering. Suppose that $\mathrm{Age}(\mathcal{A})$ is a reduct of $\mathcal{K}$, *i.e.*, $\mathrm{Age}(\mathcal{A}) = \{K \mid L : K \in \mathcal{K}\}$. Let $\prec$ be a linear ordering on κ such that $(A, \prec |A) \in \mathcal{K}$ for every $A \in \mathrm{Age}(\mathcal{A})$. Then we call $\prec$ a normal ordering of $\mathcal{A}$ induced by $\mathcal{K}$. The space of all normal orderings of $\mathcal{A}$ induced by $\mathcal{K}$ is denoted by $\mathrm{NO}_{\mathcal{K}}(\mathcal{A})$ and it is a closed subspace of $\mathrm{LO}(\kappa)$.

Example 3 ([KPT05])

(BA) Let $\mathcal{F}$ be the class of all finite Boolean algebras. We call a linear ordering on a finite Boolean algebra *natural*, if it is induced antilexicographically by a linear ordering on its atoms. Let $\mathcal{K}$ denote the class of all naturally ordered finite Boolean algebras. If $\mathcal{B}$ is a homogeneous Boolean algebra, then $\mathrm{NO}(\mathcal{B}) \neq \varnothing$, since both $\mathcal{K}$ and $\mathrm{Age}(\mathcal{B}) = \mathcal{F}$ are Fraïssé classes. Moreover, they satisfy the Ramsey property.

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(VS) Let $\mathcal{F}$ be the class of all finite vector spaces over a given finite field $\mathcal{G}$. As in the previous example, we call a linear ordering on a finite vector space of $\mathcal{G}$ natural, if it is an antilexicographical ordering induced by a linear ordering on a basis and a fixed linear ordering of the field $\mathcal{G}$ (see [Tho86]). Let $\mathcal{K}$ be the class of all naturally ordered finite vector spaces. If $\mathcal{V}$ is an infinite vector space over $\mathcal{G}$, then $\mathrm{NO}(\mathcal{V}) \neq \varnothing$, since $\mathcal{V}$ is ω-homogeneous and both $\mathrm{Age}(\mathcal{V}) = \mathcal{F}$ and $\mathcal{K}$ are Fraïssé classes, the latter shown by Thomas in [Tho86]. Moreover, they satisfy the Ramsey property (see [KPT05, p. 144]).

Every subgroup G of S_κ has a natural action on $\mathrm{LO}(\kappa)$ given by $a\ (g{<})\ b$ if and only if $g^{-1}a < g^{-1}b$, and in the same way if $\mathcal{A}$ is a structure, $\mathcal{K}$ as above and H a subgroup of $\mathrm{Aut}(\mathcal{A})$, then H has a natural action on $\mathrm{NO}_{\mathcal{K}}(\mathcal{A})$.

5 Computations of Universal Minimal Flows

Let $\mathcal{A}$ be an ω-homogeneous structure whose finitely-generated substructures are finite and let G be a dense subgroup of the group of automorphisms of $\mathcal{A}$. Let A be a finite substructure of $\mathcal{A}$. Then we can identify the right coset space $G_{(A)}/G$ with copies of A in $\mathcal{A}$ via $G_{(A)}g \longleftrightarrow g^{-1}[A] \in \binom{\mathcal{A}}{A}$. The right cosets of G_A in $G_{(A)}$ then correspond to automorphisms of A via $gG_A = \{f \in G_{(A)} : f|A = g|A\} \mapsto g|A \in \mathrm{Aut}(A)$.

Now assume that $\mathcal{K}$ is a Fraïssé order class whose reduct is $\mathrm{Age}(\mathcal{A})$. Following [KPT05, Definition 5.5], we say that $\mathcal{K}$ is *order forgetful* whenever

(1) $$(A,<),(B,\prec) \in \mathcal{K} \text{ and } A \cong B \quad \text{imply } (A,<) \cong (B,\prec).$$

It is shown in Proposition 5.6 of [KPT05] that such an order forgetful expansion $\mathcal{K}$ of $\mathrm{Age}(\mathcal{A})$ has the Ramsey property if and only if $\mathrm{Age}(\mathcal{A})$ does.

In [KPT05], the univeral minimal flows are computed for groups of automorphisms of countable ω-homogeneous structures $\mathcal{A}$ whose Age has an order expansion $\mathcal{K}$ satisfying the *ordering property*, *i.e.*, for every $A \in \mathrm{Age}(\mathcal{A})$, there is $B \in \mathrm{Age}(\mathcal{A})$ such that whenever $\prec$ is a linear ordering on A, $\prec'$ is a linear ordering on B and $(A,\prec),(B,\prec') \in \mathcal{K}$, then $(A,\prec)$ is a substructure of $(B,\prec')$. Since order forgetful expansions of $\mathrm{Age}(\mathcal{A})$ trivially have the ordering property, the following result generalizes Theorem 7.5 (ii) to uncountable structures in this special case.

Theorem 3 *Let $\mathcal{A}$ be an ω-homogeneous structure. Suppose that finitely-generated substructures of $\mathcal{A}$ are finite and that they satisfy the Ramsey property. Suppose that $\mathcal{K}$ is a Fraïssé order class that is an order forgetful expansion of* $\mathrm{Age}(\mathcal{A})$. *Then* $\mathrm{NO}_{\mathcal{K}}(\mathcal{A})$ *induced by $\mathcal{K}$ is the universal minimal flow for every dense subgroup of* $\mathrm{Aut}(\mathcal{A})$.

Proof Let G be a dense subgroup of $\mathrm{Aut}(\mathcal{A})$. To prove minimality of $\mathrm{NO}_{\mathcal{K}}(\mathcal{A})$, we need to verify that $\mathrm{Ret}\big(<,(A,<')^*\big) = \{g \in G : g< \in (A,<')^*\}$ is syndetic for every normal ordering $<$ in $\mathrm{NO}_{\mathcal{K}}(\mathcal{A})$ and $(A,<') \in \mathcal{K}$. Let S^A be a set of representatives for right cosets of G_A in $G_{(A)}$. As $\mathcal{K}$ is order forgetful, $\mathrm{Ret}\big(<,(A,<')^*\big)$ intersects every right coset of $G_{(A)}$, so $S^A\,\mathrm{Ret}\big(<,(A,<')^*\big) = G$. Since A is finite, also S^A is finite and hence $\mathrm{Ret}\big(<,(A,<')^*\big)$ is syndetic.

To prove universality, we need to show that given a normal ordering $<$ in $\mathrm{NO}_{\mathcal{K}}(\mathcal{A})$, the syndetic algebra $\mathcal{B}$ generated by $\left\{ \mathrm{Ret}\left(<, (A, <')^*\right) : (A, <') \in \mathcal{K}\right\}$ is a maximal syndetic subalgebra of the algebra $L = \{G_A K : K \subset G, A \in \mathrm{Age}(\mathcal{A})\}$.

In order to do that, we show that if left translates of a set $H = G_A K \in L$ generate a syndetic subalgebra of L, then H intersects every right coset of $G_{(A)}$. Suppose that $\{g : H \cap G_{(A)} g = \varnothing\}$ is nonempty. Then $H' = S^A H = G_{(A)} K$ is in the algebra generated by left translates of H, and $G \setminus H' \neq \varnothing$. Consider the induced colouring $c\colon \binom{\mathcal{A}}{A} \to \{H', G \setminus H'\}$ given by $c(A') = H'$ if and only if there is an $h \in H$ such that $h^{-1}[A] = A'$. Suppose now that there are $g_1, g_2, \dots, g_n \in G$ witnessing that both H' and $G \setminus H'$ are syndetic. Let C be the substructure of $\mathcal{A}$ generated by $\bigcup_{i=1}^n g_i[A]$. Since $\mathrm{Age}(\mathcal{A})$ satisfies the Ramsey property, there is a copy C' of C in $\mathcal{A}$ which is monochromatic, say in the colour H'. Since G is dense in $\mathrm{Aut}(\mathcal{A})$ and $\mathcal{A}$ is ω-homogeneous, there is an $f \in G$ mapping C' to C. It means that $c(f^{-1} g_i A) = H'$ for all i, which shows that there is no $h \in G \setminus H'$ and no i such that $f = g_i \circ h$, a contradiction.

Now let $H = G_A K \in L$ be syndetic. We have that $A_\sigma := \sigma\, \mathrm{Ret}\left(<, (A, <)^*\right) \in \mathcal{B}$ for every $\sigma \in S^A$ and $G = \bigcup_{\sigma \in S^A} A_\sigma$, therefore $\bigcup_{\sigma \in S^A} (H \cap A_\sigma) = H$. For every $\sigma \in S^A$ either $A_\sigma \cap H = \varnothing$, or $A_\sigma \cap H = A_\sigma$, since A_σ takes exactly one right coset of G_A in every right coset of $G_{(A)}$, so either $H \in \mathcal{B}$ or $\mathcal{B} \cup \{H\}$ does not generate a syndetic algebra. It follows that $\mathcal{B}$ is maximal. ■

Group of All Bijections

Since the class of finite linear orders is an order-forgetful expansion of the class of finite sets satisfying the Ramsey property, we can generalize the result of Glasner and Weiss [GW02] about S_ω to S_κ for arbitrary infinite κ.

Theorem 4 *The universal minimal flow of S_κ is* $\mathrm{LO}(\kappa)$.

Homogeneous Boolean Algebras

ω-homogeneous Boolean algebras are usually called just homogeneous. The Age of a homogeneous Boolean algebra is the class of all finite Boolean algebras. Recall (see [KPT05, p. 145]) that a linear ordering $<$ on a finite Boolean algebra is natural if it is an antilexicographical order induced by an ordering of the atoms. Since both the class of finite Boolean algebras and the class of naturally ordered finite Boolean algebras are Fraïssé classes with the Ramsey property and they satisfy the assumptions of the theorem, we get the following result, which generalizes Theorem 8.2 (iii) in [KPT05] to uncountable homogeneous Boolean algebras.

Theorem 5 *Let $\mathcal{B}$ be a homogeneous Boolean algebra and $\mathcal{K}$ the class of naturally ordered finite Boolean algebras. Then the universal minimal flow of* $\mathrm{Aut}(\mathcal{B})$ *is the space* $\mathrm{NO}_{\mathcal{K}}(\mathcal{B})$ *of all linear orderings on $\mathcal{B}$ that are natural when restricted to a finite subalgebra.*

Homogeneous Boolean algbebras are in Stone duality (see Section 2) with h-homogeneous zero-dimensional compact Hausdorff spaces, so this is just a dual version of a result by Glasner and Gutman. Recall that a topological space X is called *h-homogeneous* if all non-empty clopen subsets of X are homeomorphic.

Theorem 6 ([GG]) *Let X be an h-homogeneous zero-dimensional compact Hausdorff topological space. Let $G = \mathrm{Homeo}(X)$ be equipped with the compact-open topology. Then $M(G) = \Phi(X)$, the space of maximal chains on X.*

The space of maximal chains was introduced by Uspenskij in [Usp00] as follows: let X be a compact space and denote by $\exp X$ the space of closed subsets of X equipped with the Vietoris topology. Then the space $\Phi(X)$ of all maximal chains of closed subsets of X is a closed subspace of $\exp\exp X$. The natural action of $\mathrm{Homeo}(X)$ on X induces an action on $\exp X$ and $\Phi(X)$, which is the action considered in the theorem above. There is of course an explicit isomorphism between these two universal minimal flows (see [KPT05, Theorem 8.3]).

Following a paper by van Douwen [vD90], if κ is a cardinal number, we denote by $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ the quotient algebra of the Boolean algebra of all subsets of κ by the ideal of sets of cardinality less than κ. It is easy to see that $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ is homogeneous for every cardinal κ.

Now we introduce two subgroups of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$: Denote by T_κ the set of all bijections between subsets $A, B \subset \kappa$ with $\mathrm{card}(\kappa \setminus A), \mathrm{card}(\kappa \setminus B) < \kappa$. With the operation of composition, T_κ is a monoid, but not a group. We can however assign to each $f \in T_\kappa$ an automorphism f^* of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$, $f^*([X]) = \big[f[X]\big]$, mapping T_κ onto a subgroup $T^*_\kappa = \{f^* : f \in T_\kappa\}$ of $\mathrm{Aut}(\mathcal{P}(\kappa)/[\kappa]^{<\kappa})$. Since the automorphisms in T^*_κ are induced by a pointwise bijection between subsets of κ, we call them *trivial*. Inside of T^*_κ we have a normal subgroup of those automorphisms induced by a true permutation of κ, let us denote it by S^*_κ. Shelah [She82] (see also [SS88]) proved that consistently every automorphism of $\mathcal{P}(\omega)/\operatorname{fin}$ is trivial. This has been extended to $\mathcal{P}(\kappa)/\operatorname{fin}$ for all cardinals κ in [Vel93]. Of course, consistently the two groups are different.

The next theorem shows that T^*_ω and S^*_ω do not coincide, hence T^*_ω (and thus consistently also $\mathrm{Aut}(\mathcal{P}(\omega)/\operatorname{fin})$) is not simple.

Theorem 7 ([vD90]) *There is a homomorphism h^* from T^*_ω onto $\mathbb{Z}$ with kernel S^*_ω. (In particular, $T^*_\omega \neq S^*_\omega$.)*

Van Douwen also identified all normal subgroups of T^*_ω.

Theorem 8 ([vD90]) *A subgroup G of T^*_ω is normal if and only if $\mathrm{card}\,(G) = 1$ or $G \in \{(h^*)^{-1}k\mathbb{Z} : k \in \mathbb{N}\}$.*

It follows that all non-trivial normal subgroups of T^*_ω are dense in $\mathrm{Aut}(\mathcal{P}(\omega)/\operatorname{fin})$. So we can apply Theorem 3 to obtain the following corollary.

Corollary 2 *The universal minimal flow for all normal subgroups of T^*_ω and of $\mathrm{Aut}(\mathcal{P}(\omega)/\operatorname{fin})$ is $\mathrm{NO}\big(\mathcal{P}(\omega)/\operatorname{fin}\big)$.*

Note that even though the algebraic structure of S^*_κ is inherited from S_κ, their topologies are radically different, therefore so are their universal minimal flows $\mathrm{NO}\big(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}\big)$ and $\mathrm{LO}(\kappa)$ respectively (even in cardinality).

In the uncountable case, the situation is slightly different.

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Theorem 9 ([vD90]) *If $\kappa > \omega$, then $T^*_\kappa = S^*_\kappa$.*

Nevertheless, T^*_κ is dense in $\operatorname{Aut}\big(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}\big)$, so Theorem 3 applies.

Corollary 3 *Let κ be a cardinal number. Then the universal minimal flow of $\operatorname{Aut}\big(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}\big)$ and T^*_κ is $\mathrm{NO}\big(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}\big)$.*

Vector Spaces Over Finite Fields

Every vector space over a finite field is ω-homogeneous and its Age is equal to all finite dimensional spaces over the given finite field. This forms a Fraïssé class satisfying the Ramsey property and so does the class of all finite dimensional naturally-ordered spaces. Moreover, these classes satisfy the condition (1) on order forgetfulness of Theorem 3, as noted in [KPT05, p. 144]. Thus we can generalize the result in [KPT05] about countable-dimensional vector spaces to those with uncountable dimension.

Theorem 10 *The universal minimal flow of $\operatorname{Aut}(\mathcal{V})$, where $\mathcal{V}$ is an infinite dimensional vector space over a finite field, is the space $\mathrm{NO}(\mathcal{V})$.*

6 Extremely Amenable Groups

Next theorem characterizes extremely amenable subgroups of S_κ. The implications (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) have identical proofs as for S_∞ given in [KPT05].

Definition 1 A topological group is called *extremely amenable* if its universal minimal flow is a singleton.

Theorem 11 *Let G be a subgroup of S_κ. The following are equivalent:*

(a) *G is extremely amenable,*
(b) (i) *for every finite $A \subset \kappa$, $G_A = G_{(A)}$, and* (ii) *for every colouring $c\colon G/G_A \to \{1, 2, \dots, k\}$ and for every finite $B \supset A$, there is $g \in G$ and $i \in \{1, 2, \dots, n\}$ such that $c(hG_A) = i$ whenever $h[A] \subset g[B]$.*
(c) (i$'$) *G preserves an ordering, and* (ii) *as above.*

Remark 1 Let $\mathcal{A}$ be an ω-homogeneous relational structure such that G is dense in its automorphism group. Since finitely generated substructures of $\mathcal{A}$ are finite, (ii) of (b) simply says that $\operatorname{Age}(\mathcal{A})$ satisfies the Ramsey property.

Proof (a) $\Rightarrow$ (c) Since G is extremely amenable, it has a fixed point under its natural action on $\mathrm{LO}(\kappa)$, hence (i$'$) holds. To prove (ii), suppose that $c\colon G/G_A \to \{1, 2, \dots, k\}$ is a colouring of left cosets of G_A by k many colours. Consider c as a point in the compact space $X = \{1, 2, \dots, k\}^{G/G_A}$ and an action of G on X given by $gx(hG_A) = x(g^{-1}hG_A)$. Let Y be the closure of the orbit of c in X. Since G is extremely amenable, the induced action of G on Y has a fixed point d. As G acts transitively on G/G_A, d must be a constant function, say with range $\{i\} \subset \{1, 2, \dots, k\}$ Let $B \supset A$ be finite and let $H = \{hG_A \in G : hA \subset B\} \subset G/G_A$. Since $d \in \overline{Gc}$, there is a $g \in G$ such that $g^{-1}c|H = d|H$. Then $c(ghG_A) = g^{-1}c(hG_A) = d(hG_A)$ for every $h \in H$.

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(c) $\Rightarrow$ (b) Let $<$ be an ordering given by (i′). It means that for every $g \in G_{(A)}$ we have that $g(A, <|A) = (A, <|A)$, hence $g(a) = a$ for all $a \in A$ and so $G_{(A)} = G_A$.

(b) $\Rightarrow$ (a) We will use the notation and result of Theorem 1. By the proof of Theorem 3 we know that left translates of a set $H = G_A K$ cannot generate a syndetic subalgebra of L if $H \cap G_{(A)} g = \varnothing$ for some $g \in G$. However, since $G_{(A)} = G_A$, it follows that H only generates a syndetic algebra if it is equal to all of G. ■

As an immediate consequence of Theorem 11, we obtain examples of extremely amenable groups as groups of automorphisms of structures.

Theorem 12 ([Pes98]) *The group of automorphisms of an ω-homogeneous linear order is extremely amenable.*

The following result generalizes Theorem 6.14 of [KPT05] to uncountable Boolean algebras.

Theorem 13 *Let $\mathcal{B}$ be a homogeneous Boolean algebra and let $<$ be a normal ordering induced by the class of naturally ordered finite Boolean algebras as in the Example 3. Then* $\operatorname{Aut}(\mathcal{B}, <)$ *is extremely amenable.*

Proof By minimality of $\mathrm{NO}(B)$ we have that $\operatorname{Age}(\mathcal{B}, <)$ is the class of all naturally ordered finite Boolean algebras satisfying the Ramsey property. Therefore (c) of Theorem 11 is satisfied. ■

The following result generalizes Theorem 6.13 of [KPT05] to uncountable dimensional vector spaces.

Theorem 14 *Let $\mathcal{V}$ be a vector space over a finite field and let $<$ be a normal ordering induced by the class of naturally ordered finite vector spaces as in Example 3. Then* $\operatorname{Aut}(\mathcal{V}, <)$ *is extremely amenable.*

Proof This is identical to the proof of Theorem 13. ■

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Department of Mathematics, University of Toronto, Bahen Center, 40 St. George St., Toronto, ON M5S 2E4

and

Faculty of Mathematics and Physics, Charles University in Prague, Ke Karlovu 3, 121 16 Prague, Czech Republic M5S 2E4
e-mail: dana.bartosova@utoronto.ca