

Connections on a Parabolic Principal Bundle, II

Indranil Biswas

Abstract.

In *Connections on a parabolic principal bundle over a curve, I* we defined connections on a parabolic principal bundle. While connections on usual principal bundles are defined as splittings of the Atiyah exact sequence, it was noted in the above article that the Atiyah exact sequence does not generalize to the parabolic principal bundles. Here we show that a twisted version of the Atiyah exact sequence generalizes to the context of parabolic principal bundles. For usual principal bundles, giving a splitting of this twisted Atiyah exact sequence is equivalent to giving a splitting of the Atiyah exact sequence. Connections on a parabolic principal bundle can be defined using the generalization of the twisted Atiyah exact sequence.

1 Introduction

Generalizing the notion of a parabolic vector bundle, the notion of a parabolic principal bundle was introduced in [BBN1, BBN2]. Let G be a connected, complex, linear algebraic group. A parabolic G -bundle over a complex smooth projective variety X is a smooth variety E_G equipped with an action of G as well as a projection to X such that E_G is a principal bundle over the complement of a simple normal crossing divisor in X . However the action of G over the divisor is allowed to have finite isotropies.

In [Bi2], connections on a parabolic principal bundle were defined. Before we describe connections on a parabolic principal bundle, we will first briefly recall the definition of a connection on an usual principal G -bundle. Let F_G be a holomorphic principal G -bundle over a complex manifold Y , and let

$$(1.1) \quad 0 \longrightarrow \text{ad}(F_G) \longrightarrow \text{At}(F_G) \longrightarrow TY \longrightarrow 0$$

be the corresponding Atiyah exact sequence over Y . A holomorphic (respectively, complex) connection on F_G is a holomorphic (respectively, C^∞) splitting of the above Atiyah exact sequence. Giving a holomorphic (respectively, C^∞) splitting of (1.1) is equivalent to giving a holomorphic (respectively, C^∞) one-form ω on F_G with values in the Lie algebra \mathfrak{g} of G and satisfying the following two conditions:

- the restriction of ω to each fiber of the projection $F_G \longrightarrow Y$ coincides with the Maurer–Cartan form, and
- the form ω is G -equivariant for the adjoint action of G on \mathfrak{g} .

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Let E_G be a parabolic G -bundle over X . Then there exists a Galois covering $f: Y \rightarrow X$, where Y is a smooth complex variety, together with a holomorphic principal G -bundle F_G over Y equipped with a lift of the action of the Galois group $\Gamma := \text{Gal}(f)$ on Y , such that $E_G = F_G/\Gamma$. It should be mentioned that there are many coverings of X satisfying the above conditions. In [Bi2] we noted that there is no Atiyah exact sequence for a general parabolic G -bundle. This means that for a choice of a covering Y of the above type, the exact sequence of vector bundles over X given by the Atiyah exact sequence for F_G on Y depends on the choice of covering. Since the Atiyah exact sequence is not available, we used the above description of a connection as a \mathfrak{g} -valued 1-form on the total space to define connections on a parabolic G -bundle; see [Bi2] for the details.

Let

$$0 \rightarrow \text{ad}(F_G) \otimes \Omega_Y^1 \rightarrow \text{At}(F_G) \otimes \Omega_Y^1 \xrightarrow{q} TY \otimes \Omega_Y^1 \rightarrow 0$$

be the exact sequence obtained by tensoring (1.1) with Ω_Y^1 . Consider the inclusion of \mathcal{O}_Y in $TY \otimes \Omega_Y^1$ obtained by sending the constant function 1 to the identity automorphism of TY . Therefore, from the above exact sequence we get the following short exact sequence of holomorphic vector bundles

$$(1.2) \quad 0 \rightarrow \text{ad}(F_G) \otimes \Omega_Y^1 \rightarrow \tilde{\text{At}}(F_G) := q^{-1}(\mathcal{O}_Y) \xrightarrow{q} \mathcal{O}_Y \rightarrow 0$$

over Y .

It is easy to see that giving a holomorphic (respectively C^∞) splitting of (1.1) is equivalent to giving a holomorphic (respectively C^∞) splitting of (1.2). Therefore, the twisted version of the Atiyah exact sequence given in (1.2) is also suitable for defining connections.

The exact sequence in (1.2) generalizes to the context of parabolic G -bundles. Connections on a parabolic G -bundle can be defined to be the splittings of the corresponding short exact sequence.

2 Preliminaries

Let X be a connected, smooth, projective variety of dimension d defined over \mathbb{C} . Let $D \subset X$ be a simple normal crossing hypersurface. This means that D is an effective and reduced divisor with each irreducible component of D being smooth, and furthermore, the irreducible components of D intersect transversally. Let

$$D = \sum_{i=1}^{\ell} D_i$$

be the decomposition of D into irreducible components. The above condition that the irreducible components of D intersect transversally means that if

$$(2.1) \quad x \in D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_k} \subset D$$

is a point where k distinct components of D meet, and $f_j, j \in [1, k]$, is the local equation of the divisor D_{i_j} around x , then $\{df_{i_j}(x)\}$ is a linearly independent subset

of the holomorphic cotangent space T_x^*X of X at x . This implies that for any choice of k integers

$$(2.2) \quad 1 \leq i_1 < i_2 < \dots < i_k \leq \ell,$$

each connected component of $D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_k}$ is a smooth subvariety of X .

Let E be an algebraic vector bundle over X . For each $i \in [1, \ell]$, let

$$(2.3) \quad E|_{D_i} = F_1^i \supseteq F_2^i \supseteq F_3^i \supseteq \dots \supseteq F_{m_i}^i \supseteq F_{m_i+1}^i = 0$$

be a filtration by subbundles of the restriction of E to D_i . In other words, each F_j^i is a subbundle of $E|_{D_i}$ and $\text{rank}(F_j^i) > \text{rank}(F_{j+1}^i)$ for $j \in [1, m_i]$.

A *quasiparabolic* structure on E over D is a filtration as above of each $E|_{D_i}$ satisfying the following extra condition. Take any $k \in [1, \ell]$, and take integers $\{i_j\}_{j=1}^k$ as in (2.2). If we fix some $F_{n_j}^{i_j}$, $n_j \in [1, m_{i_j}]$, then over each connected component S of $D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_k}$, the intersection

$$\bigcap_{j=1}^k F_{n_j}^{i_j} \subset E|_{D_{i_1} \cap \dots \cap D_{i_k}}$$

gives a subbundle of the restriction of E to S . It should be clarified that the rank of this subbundle may depend on the choice of the component $S \subset D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_k}$.

To explain the above condition we give an example. Let $S \subset D_{i_1} \cap D_{i_2}$ be a connected component, where $1 \leq i_1 < i_2 \leq \ell$, and take $F_{n_1}^{i_1}, F_{n_2}^{i_2}$, where $n_j \in [1, m_{i_j}]$, $j = 1, 2$. For any point $x \in S$, consider the subspace $(F_{n_1}^{i_1})_x \cap (F_{n_2}^{i_2})_x \subset E_x$. The above condition says that the dimension of this subspace is independent of the choice of $x \in S$. But this dimension depends on the choices of $i_j, n_j, j \in [1, 2]$, and it also depends on the choice of the connected component S in $D_{i_1} \cap D_{i_2}$. Note that the condition that $(F_{n_1}^{i_1})_x \cap (F_{n_2}^{i_2})_x$ is of constant dimension over a connected component S is equivalent to the condition that $(F_{n_1}^{i_1} \cap F_{n_2}^{i_2})|_S$ is a subbundle of $E|_S$.

For a quasiparabolic structure as above, *parabolic weights* are a collection of rational numbers $0 \leq \lambda_1^i < \lambda_2^i < \lambda_3^i < \dots < \lambda_{m_i}^i < 1$, where $i \in [1, \ell]$. The parabolic weight λ_j^i corresponds to the subbundle F_j^i in (2.3). A *parabolic structure* on E is a quasiparabolic structure with parabolic weights. A vector bundle over X equipped with a parabolic structure on it is also called a *parabolic vector bundle*.

For notational convenience, a parabolic vector bundle defined as above will be denoted by E_* . The divisor D is called the *parabolic divisor* for E_* .

Let G be a connected complex linear algebraic group. We will recall the definition of a parabolic G -bundle introduced in [BBN1].

Let $\text{Rep}(G)$ denote the category of all finite dimensional rational left representations of G . Let $\text{Vect}(X)$ denote the category of algebraic vector bundles over X . Nori showed that a principal G -bundle over X is a functor from $\text{Rep}(G)$ to $\text{Vect}(X)$ that is compatible with the operations of taking direct sum, tensor product, and dual. Given a principal G -bundle E_G over X , the corresponding functor $\text{Rep}(G) \rightarrow \text{Vect}(X)$ sends a G -module V to the vector bundle $E_G \times^G V$ over X associated to E_G for V ; see [No1, No2] for details.

Let $\text{Pvect}(X)$ denote the category of all parabolic vector bundles over X with D as the parabolic divisor. In [BBN1], parabolic G -bundles over X with D as the parabolic divisor were defined to be as functors from $\text{Rep}(G)$ to $\text{Pvect}(X)$ satisfying a list of conditions identical to the list of conditions of Nori in the above mentioned characterization of usual principal bundles as functors. It may be mentioned that the operations of taking direct sum, tensor product and dual of usual vector bundles are replaced by the operations of taking parabolic direct sum, parabolic tensor product and parabolic dual. See [BBN1] for the details.

In [BBN2], the notion of a ramified G -bundle over a curve was introduced. The main result of [BBN2] is the construction of a natural bijective correspondence between the ramified G -bundles over a Riemann surface C , with ramifications over a finite set of points $D_0 \subset C$, and the parabolic G -bundles over C with D_0 as the parabolic divisor.

We will define ramified G -bundles over the projective manifold X with ramification over the simple normal crossing divisor D .

A *ramified G -bundle* over X with ramification over D is a smooth complex variety E_G on which G acts (algebraically) on the right, that is, the map $f: E_G \times G \rightarrow E_G$ defining the action is an algebraic morphism, together with a surjective algebraic map

$$(2.4) \quad \psi: E_G \rightarrow X$$

satisfying the following five conditions:

- (1) $\psi \circ f = \psi \circ p_1$, where p_1 is the natural projection of $E_G \times G$ to E_G ;
- (2) for each point $x \in X$, the action of G on the reduced fiber $\psi^{-1}(x)_{\text{red}}$ is transitive;
- (3) the restriction of ψ to $\psi^{-1}(X \setminus D)$ makes $\psi^{-1}(X \setminus D)$ a principal G -bundle over $X \setminus D$, that is, the map ψ is smooth over $\psi^{-1}(X \setminus D)$ and the map to the fiber product

$$\psi^{-1}(X \setminus D) \times G \rightarrow \psi^{-1}(X \setminus D) \times_{X \setminus D} \psi^{-1}(X \setminus D)$$

defined by $(z, g) \mapsto (z, f(z, g))$ is an isomorphism;

- (4) for each irreducible component $D_i \subset D$, the reduced inverse image $\psi^{-1}(D_i)_{\text{red}}$ is a smooth divisor and

$$\widehat{D} := \sum_{i=1}^{\ell} \psi^{-1}(D_i)_{\text{red}}$$

is a normal crossing divisor on E_G ;

- (5) for any smooth point $z \in \widehat{D}$, the isotropy group $G_z \subset G$, for the action of G on E_G , is a finite cyclic group that acts faithfully on the quotient line $T_z E_G / T_z \psi^{-1}(D)_{\text{red}}$.

Note that since the map ψ commutes with the action of G , the isotropy subgroup G_z preserves $T_z \psi^{-1}(D)_{\text{red}} \subset T_z E_G$. Therefore, there is an induced action of G_z on the fiber $T_z E_G / T_z \psi^{-1}(D)_{\text{red}}$ of the normal bundle. Since the finite isotropy group G_z in the above condition (5) acts faithfully on the line $T_z E_G / T_z \psi^{-1}(D)_{\text{red}}$, it follows automatically that G_z is a cyclic group.

Let E_G be a ramified G -bundle over X with ramification over the divisor D . Fix a component $D_i \subset D$. Let $x \in D_i$ be a smooth point of D . The order of the finite cyclic group G_z , where z satisfies the condition $\psi(z) = x$, does not depend on the choices of x and z ; it depends only on the component D_i and E_G . Therefore, given any ramified G -bundle E_G , we have a positive integer η_i associated to each component D_i ; for any z as above, the order of G_z is η_i .

In the next section we will describe a correspondence between parabolic G -bundles and ramified G -bundles.

3 Ramified G -Bundles as Parabolic G -Bundles

The map from ramified G -bundles to parabolic G -bundles described in Section 2 of [Bi2]. Although it is assumed in [Bi2] that $\dim_{\mathbb{C}} X = 1$, the construction of a parabolic G -bundle from a given ramified G -bundle goes through.

Let E_* be a parabolic G -bundle over X with D as the parabolic divisor. In [BBN1] the following was proved:

There is a Galois covering

$$(3.1) \quad \varphi: Y \longrightarrow X$$

and a Γ -linearized principal G -bundle E_G over Y , where Γ is the Galois group for φ , such that E_* corresponds to E_G (see [BBN1, Theorem 4.3]).

We will show that E_G/Γ is a ramified G -bundle over X . All the properties except one of a ramified G -bundle for E_G/Γ have already been shown in [BBN2]. The only property that remains to be checked is that E_G/Γ is smooth. (The argument given in [BBN2] that E_G/Γ is smooth uses the assumption that $\dim_{\mathbb{C}} X = 1$.) We will show that E_G/Γ is smooth.

Let $S \subset Y$ be the subscheme where the map φ in (3.1) fails to be smooth. Since $X = Y/\Gamma$ is smooth, each component of S is a hypersurface. For any rational point $y \in S$, let $\Gamma_y \subset \Gamma$ be the isotropy subgroup for the action of the Galois group Γ on Y .

For any rational point $y \in S \setminus \varphi^{-1}(D)$ in the complement of the inverse image $\varphi^{-1}(D)$, the action of Γ_y on the fiber $(E_G)_y$ is trivial. Indeed, this follows from the fact the action of G on the fiber $(E_*)_{\varphi(y)}$ is a free action (recall the definition of a ramified G -bundle with ramifications over D).

For each irreducible component D_i of D , there is a maximal subgroup $H_i \subset \Gamma$ such that $H_i \subset \Gamma_y$ for all $y \in \varphi^{-1}(D_i)$. Furthermore, for the general point $y \in \varphi^{-1}(D_i)$, the equality $H_i = \Gamma_y$ holds, and also the group H_i is cyclic. The action of H_i on the fiber of E_G over a point $\varphi^{-1}(D_i)$ need not be free, but there is a fixed quotient group H'_i of H_i such that for all $y \in \varphi^{-1}(D_i)$, the action of H_y on the fiber $(E_G)_y$ factors through H'_i . Furthermore, the action of H'_i on $(E_G)_y$ is free. From these it follows that the quotient E_G/Γ is smooth.

4 Holomorphic Connections

We will first recall the usual Atiyah exact sequence and its equivalent formulations.

4.1 The Atiyah Exact Sequence

Let M be a complex manifold and

$$(4.1) \quad f: E_H \longrightarrow M$$

a holomorphic principal H -bundle over M , where H is a complex Lie group. Consider the sheaf \mathcal{F} on M that associates to any open subset $U \subset M$ the space of all H -invariant holomorphic vector fields on $f^{-1}(U) \subset E_H$. Therefore, $\mathcal{F}(U)$ is a \mathcal{O}_U -module, where \mathcal{O}_U is the algebra of holomorphic functions on U ; the multiplication of a holomorphic vector field τ on $f^{-1}(U)$ with a function $\phi \in \mathcal{O}_U$ is the vector field $(\phi \circ f) \cdot \tau$. Since the action of H on the fibers of f is transitive, it follows immediately that \mathcal{F} is a locally free, coherent, analytic sheaf on M .

The holomorphic vector bundle over M defined by \mathcal{F} is called the *Atiyah bundle* for E_H . The Atiyah bundle for E_H is denoted by $\text{At}(E_H)$.

Let $\text{ad}(E_H)$ be the adjoint bundle of E_H . So $\text{ad}(E_H)$ is the holomorphic vector bundle over M associated to E_H for the adjoint action of H on its Lie algebra \mathfrak{h} . We recall that \mathfrak{h} is identified with the vector fields on H invariant under the right translation action of H on itself. Using this it follows that for any open subset $U \subset M$, the space of all holomorphic sections of $\text{ad}(E_H)$ over U is identified with the space of all H -invariant holomorphic vector fields on $f^{-1}(U) \subset E_H$ that lie in the kernel of the differential $df: TE_H \rightarrow f^*TM$ of the projection f in (4.1). In other words, $\Gamma(U, \text{ad}(E_H))$ is the space of all H -invariant holomorphic vertical vector fields on $f^{-1}(U)$. Therefore, we obtain a short exact sequence of holomorphic vector bundles

$$(4.2) \quad 0 \longrightarrow \text{ad}(E_H) \longrightarrow \text{At}(E_H) \longrightarrow TM \longrightarrow 0$$

over M ; the projection $\text{At}(E_H) \longrightarrow TM$ is given by the differential df .

The following definitions are from [At].

Definition 4.1 A holomorphic connection on E_H is a holomorphic splitting of the short exact sequence in (4.2). A complex connection on E_H is a C^∞ splitting of the short exact sequence in (4.2).

Let

$$(4.3) \quad 0 \longrightarrow \text{ad}(E_H) \otimes \Omega_M^1 \longrightarrow \text{At}(E_H) \otimes \Omega_M^1 \xrightarrow{q} TM \otimes \Omega_M^1 \longrightarrow 0$$

be the short exact sequence obtained by tensoring (4.2) with the holomorphic cotangent bundle Ω_M^1 . Using the section of $TM \otimes T^*M$ given by the identity automorphism of TM , the structure sheaf \mathcal{O}_M is a subsheaf of $TM \otimes \Omega_M^1$. Therefore, from (4.3) we have the short exact sequence of holomorphic vector bundles

$$(4.4) \quad 0 \longrightarrow \text{ad}(E_H) \otimes \Omega_M^1 \longrightarrow \tilde{\text{At}}(E_H) := q^{-1}(\mathcal{O}_M) \xrightarrow{q|_{\tilde{\text{At}}(E_H)}} \mathcal{O}_M \longrightarrow 0$$

over M , where q is the projection in (4.3).

Remark 4.2. A holomorphic splitting of (4.2) gives a holomorphic splitting of (4.4) and conversely a holomorphic splitting of (4.4) gives a holomorphic splitting of (4.2). Similarly, giving a C^∞ splitting of the exact sequence in (4.2) is equivalent to giving a C^∞ splitting of (4.4). Therefore, giving a holomorphic connection on E_G is equivalent to giving a holomorphic splitting of (4.4). Similarly, giving a complex connection on E_G is equivalent to giving a C^∞ splitting of (4.4).

4.2 Connections on a Ramified Principal Bundle

Let E_G be a ramified G -bundle over X with ramification over D . As in (2.4), let ψ denote the projection of E_G to X . Consider the subbundle of the holomorphic tangent bundle $\mathcal{K} \subset TE_G$ defined by the orbits of the action of G on E_G . Since the isotropy subgroups, for the action of G on E_G , are all finite subgroups of G , it follows that \mathcal{K} is a subbundle of TE_G , and the vector bundle \mathcal{K} is identified with the trivial vector bundle over E_G with fiber \mathfrak{g} , where \mathfrak{g} is the Lie algebra of G . The differential $d\psi: TE_G \rightarrow \psi^*TX$ evidently vanishes on \mathcal{K} .

Let \mathcal{Q} denote the quotient bundle TE_G/\mathcal{K} . So we have a short exact sequence of holomorphic vector bundles

$$(4.5) \quad 0 \longrightarrow \mathcal{K} \longrightarrow TE_G \longrightarrow \mathcal{Q} \longrightarrow 0$$

over E_G . Tensoring (4.5) with \mathcal{Q}^* we get the exact sequence

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{Q}^* \longrightarrow TE_G \otimes \mathcal{Q}^* \xrightarrow{q_0} \mathcal{Q} \otimes \mathcal{Q}^* \longrightarrow 0$$

over E_G . As in (4.4), we will consider the inverse image of the trivial line subbundle of $\mathcal{Q} \otimes \mathcal{Q}^*$ generated by the identity automorphism of \mathcal{Q} . So we have the short exact sequence of holomorphic vector bundles

$$(4.6) \quad 0 \longrightarrow \mathcal{K} \otimes \mathcal{Q}^* \longrightarrow \mathcal{V}_{E_G} := q_0^{-1}(\mathcal{O}_{E_G}) \xrightarrow{q_0|_{\mathcal{V}_{E_G}}} \mathcal{O}_{E_G} \longrightarrow 0$$

over E_G .

We note that the action of G on E_G has natural lifts to all three vector bundles in the short exact sequence in (4.6). Furthermore, all the homomorphisms in (4.6) commute with the actions of G . Therefore, the direct image, on X , of any of the vector bundles in (4.6) is equipped with an action of G .

Define the quasi-coherent analytic sheaves

$$(4.7) \quad \mathcal{A}_{E_G} := (\psi_*(\mathcal{K} \otimes \mathcal{Q}^*))^G$$

and

$$(4.8) \quad \mathcal{B}_{E_G} := (\psi_*\mathcal{V}_{E_G})^G$$

on X , where ψ is the projection in (2.4), and by W^G , where W is any sheaf on X equipped with an action of G , we mean the G -invariant part of W . Since the action

of G on the fibers of ψ is transitive with finite isotropy subgroups, it follows that both \mathcal{A}_{E_G} and \mathcal{B}_{E_G} introduced in (4.7) and (4.8) are locally free coherent analytic sheaves on X . The holomorphic vector bundles over X defined by \mathcal{A}_{E_G} and \mathcal{B}_{E_G} will also be denoted by \mathcal{A}_{E_G} and \mathcal{B}_{E_G} respectively.

We note that the action of G on the sheaf \mathcal{O}_{E_G} in (4.6) is the trivial action. This means that the identity automorphism of \mathcal{Q} is preserved by the action of G on $\mathcal{Q} \otimes \mathcal{Q}^*$. Therefore, $(\psi_* \mathcal{O}_{E_G})^G = \mathcal{O}_X$.

Using the above observations, from (4.6) we have following short exact sequence of holomorphic vector bundles over X

$$(4.9) \quad 0 \longrightarrow \mathcal{A}_{E_G} \longrightarrow \mathcal{B}_{E_G} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Definition 4.3 A holomorphic connection on E_G is a holomorphic splitting of the short exact sequence in (4.9). A complex connection on E_G is a C^∞ splitting of the short exact sequence in (4.9).

When E_G is an usual principal G -bundle, the exact sequence in (4.9) clearly coincides with the exact sequence in (4.4). In view of Remark 4.2, the above definitions coincide with those in Definition 4.1 when E_G is an usual principal bundle.

In [Bi2] we defined connections on a ramified G -bundle over a curve. The following theorem shows that the above definition coincides with the one given in [Bi2].

Theorem 4.4 Let E_G be a ramified principal G -bundle over X . Giving a holomorphic connection on E_G is equivalent to giving a holomorphic connection on E_G in the sense of [Bi2]. Similarly, giving a complex connection on E_G is equivalent to giving a complex connection on E_G in the sense of [Bi2].

Proof Let $\beta: \mathcal{O}_X \longrightarrow \mathcal{B}_{E_G}$ be a holomorphic splitting of the exact sequence in (4.9). The lift on β to E_G gives a G -equivariant holomorphic splitting

$$(4.10) \quad \tilde{\beta}: \mathcal{O}_{E_G} \longrightarrow \mathcal{V}_{E_G}$$

of the exact sequence in (4.6). Therefore, $\tilde{\beta}$ gives a homomorphism of holomorphic vector bundles

$$\beta': \mathcal{Q} \longrightarrow TE_G$$

whose composition with the projection $TE_G \longrightarrow \mathcal{Q}$ in (4.5) is the identity automorphism of \mathcal{Q} . Let

$$(4.11) \quad \tilde{\gamma}: TE_G \longrightarrow \mathcal{K}$$

be the projection given by the above homomorphism β' , where \mathcal{K} is the kernel in (4.5). Therefore, the kernel of $\tilde{\gamma}$ is the image of β' , and the composition of $\tilde{\gamma}$ with the inclusion $\mathcal{K} \hookrightarrow TE_G$ in (4.5) is the identity automorphism of \mathcal{K} .

We recall that \mathcal{K} is the trivial vector bundle over E_G whose fiber is the Lie algebra \mathfrak{g} of G . Therefore, the homomorphism $\tilde{\gamma}$ in (4.11) defines a \mathfrak{g} -valued holomorphic one-form γ on E_G . Since the homomorphism $\tilde{\beta}$ in (4.10) is G -equivariant, we conclude

that the holomorphic one-form γ on E_G is also G -equivariant for the adjoint action of G on \mathfrak{g} . From the fact that the composition of $\tilde{\gamma}$ with the inclusion $\mathcal{K} \hookrightarrow TE_G$ in (4.5) is the identity automorphism of \mathcal{K} it follows immediately that the restriction of γ to each fiber of the projection ψ (see (2.4)) is the Maurer–Cartan form on the fiber. Therefore, γ defines a holomorphic connection on E_G in the sense of [Bi2].

Conversely, let γ be a \mathfrak{g} -valued holomorphic one-form on E_G defining a holomorphic connection on E_G in the sense of [Bi2]. Therefore, γ is G -equivariant and it coincides with the Maurer–Cartan form on the fibers of the projection ψ . Consequently, γ gives a G -invariant holomorphic splitting $\tilde{\gamma}: TE_G \rightarrow \mathcal{K}$ of the exact sequence in (4.5). Consider the corresponding homomorphism $\beta': \mathcal{Q} \rightarrow TE_G$. So, the image of β' is the kernel of $\tilde{\gamma}$, and the composition of β' with the projection $TE_G \rightarrow \mathcal{Q}$ in (4.5) is the identity automorphism of \mathcal{Q} . Therefore, β' gives a G -invariant holomorphic splitting $\tilde{\beta}: \mathcal{O}_{E_G} \rightarrow \mathcal{V}_{E_G}$ of the exact sequence in (4.6). Hence $\tilde{\beta}$ descends to a holomorphic splitting of the exact sequence in (4.9).

Therefore, giving a holomorphic connection on E_G is equivalent to giving a holomorphic connection on E_G in the sense of [Bi2].

Similarly, it can be shown that giving a complex connection on E_G is equivalent to giving a complex connection on E_G in the sense of [Bi2]. This completes the proof of the theorem. ■

4.3 A Construction of \mathcal{A}_{E_G}

Consider the derivation action of TX on \mathcal{O}_X . Let

$$TX(-\log D) \subset TX$$

be the subsheaf that leaves $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ invariant. So

$$TX \otimes \mathcal{O}_X(-D) \subset TX(-\log D) \subset TX,$$

and if $x \in D_i$ is a smooth point of D , then the image of the fiber $TX(-\log D)_x$ in $T_x X$ coincides with $T_x D_i$. More generally, for any point x as in (2.1), the image of the fiber $TX(-\log D)_x$ in $T_x X$ is contained in the kernel of the projection of $T_x X$ to the fiber over x of the normal bundle to $D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_k}$. The subsheaf $TX(-\log D)$ is locally free and hence it defines a holomorphic vector bundle over X . The dual vector bundle $(TX(-\log D))^*$ is denoted by $\Omega_X^1(\log D)$.

Let E_G be a ramified G -bundle over X with ramification over D . As we saw in Section 3, the ramified G -bundle E_G gives a functor from $\text{Rep}(G)$ to $\text{PVect}(X)$. Consider the G -module \mathfrak{g} equipped with the adjoint action of G . The image of this G -module \mathfrak{g} by the functor $\text{Rep}(G) \rightarrow \text{PVect}(X)$ corresponding to E_G will be denoted by $E_*^{\mathfrak{g}}$. So $E_*^{\mathfrak{g}}$ is a parabolic vector bundle over X with parabolic structure over D .

Let $E_0^{\mathfrak{g}}$ denote the underlying holomorphic vector bundle for the above defined parabolic vector bundle $E_*^{\mathfrak{g}}$. For each irreducible component D_i of D , let

$$E_0^{\mathfrak{g}}|_{D_i} = F_1^i \supset F_2^i \supset F_3^i \supset \dots \supset F_{m_i}^i \supset F_{m_i+1}^i = 0$$

be the filtration as in (2.3). We will define a subbundle V^i of $E_0^{\mathfrak{g}}|_{D_i}$. If the parabolic weight λ_1 corresponding to F_1^i is zero, then set $V^i := F_2^i$, and if $\lambda_1 \neq 0$, then set $V^i := F_1^i$. Let $\mathcal{F} \subset E_0^{\mathfrak{g}}$ be the subsheaf that fits in the following short exact sequence of coherent sheaves on X

$$(4.12) \quad 0 \longrightarrow \mathcal{F} \longrightarrow E_0^{\mathfrak{g}} \longrightarrow \bigoplus_{i=1}^{\ell} (E_0^{\mathfrak{g}}|_{D_i})/V_i \longrightarrow 0,$$

where V_i are defined above, and $\{D_i\}_{i=1}^{\ell}$ are the irreducible components of D . Therefore, \mathcal{F} is a holomorphic vector bundle over X which is identified with $E_0^{\mathfrak{g}}$ over the complement $X \setminus D$.

Let

$$(4.13) \quad \mathcal{W}_{E_G} \subset E_0^{\mathfrak{g}} \otimes \Omega_X^1(\log D)$$

be the coherent subsheaf generated by the two subsheaves $\mathcal{F} \otimes \Omega_X^1(\log D)$ and $E_0^{\mathfrak{g}} \otimes \Omega_X^1$, where \mathcal{F} is defined in (4.12) and $\Omega_X^1(\log D)$ was defined earlier. It is easy to check that \mathcal{W}_{E_G} is locally free. Hence it defines a holomorphic vector bundle over X . This vector bundle is clearly identified with $E_0^{\mathfrak{g}} \otimes \Omega_X^1$ over the complement $X \setminus D$.

Proposition 4.5 *The vector bundle \mathcal{A}_{E_G} in (4.9) is identified with the vector bundle \mathcal{W}_{E_G} in (4.13).*

Proof There is a Galois covering

$$(4.14) \quad f: Y \longrightarrow X$$

with Y a smooth complex projective variety and a holomorphic principal G -bundle F_G over Y equipped with a lift of the action of the Galois group $\Gamma := \text{Gal}(f)$ on Y such that $E_G = F_G/\Gamma$. Indeed, in [BBN1] it was shown that given a parabolic G -bundle over X , such a pair (f, F_G) exists. We noted in Section 3 that a parabolic G -bundle is same as a ramified G -bundle.

Using the actions of the Galois group Γ on Y and F_G , the vector bundle $\text{ad}(F_G) \otimes \Omega_Y^1$ over Y is equipped with an action of Γ . The corresponding parabolic vector bundle over X has the property that its underlying vector bundle is identified with the vector bundle \mathcal{W}_{E_G} defined in (4.13). (See [Bi1] for the correspondence between parabolic vector bundles over X and Γ -linearized vector bundles over Y .)

Let $\psi_0: F_G \rightarrow Y$ be the natural projection. Let $\delta: F_G \rightarrow F_G/\Gamma = E_G$ be the quotient map. Since the actions of G and Γ on F_G commute, for any Γ -linearized vector bundle W over Y , the pullback ψ_0^*W on F_G has the following property: the invariant direct image $(f_*W)^\Gamma$ on X , where f is the projection in (4.14), is identified with the invariant direct image $(\psi_* (\delta_* \psi_0^* W)^\Gamma)^\Gamma$, where ψ is the projection in (2.4). Now setting W to be $\text{ad}(F_G) \otimes \Omega_Y^1$ we conclude that the vector bundle \mathcal{W}_{E_G} defined in (4.13) is identified with the vector bundle \mathcal{A}_{E_G} in (4.9). This completes the proof of the proposition. ■

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India
e-mail: indranil@math.tifr.res.in