

THE RELAXATION METHOD FOR LINEAR INEQUALITIES

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I. STATEMENT OF PROBLEM AND MAIN RESULTS

1. The relaxation method. Let A be a closed set of points in the n -dimensional euclidean space E_n . If p and p_1 are points of E_n such that

$$(1.1) \quad |p - a| > |p_1 - a|, \text{ for every } a \in A,$$

then p_1 is said to be *point-wise closer* than p to the set A . If p is such that there is no point p_1 which is point-wise closer than p to A , then p is called a *closest point* to the set A . In 1922 Fejér **(2)** made the interesting observation that the set of closest points to A is identical with the convex hull $K(A)$ of the set A . We have mentioned this remark because it will suggest a way of dealing with our main problem, to which we now turn.

We are given a consistent system of m linear inequalities

$$(1.2) \quad \sum_{j=1}^n a_{ij}x_j + b_i \geq 0 \quad (i = 1, \dots, m).$$

The coefficients a_{ij} and b_i being given numerically, the problem is to devise a numerical procedure which will furnish a solution (x_1, \dots, x_n) of the system (1.2). In the case of a homogeneous system, i.e. when all $b_i = 0$, we add the obvious requirement that the solution (x_1, \dots, x_n) obtained be different from the trivial solution $(0, \dots, 0)$.

A natural approach to this problem will be suggested by Fejér's idea as soon as we place the problem in its customary geometric setting. Each of the inequalities (1.2) defines a closed half-space

$$(1.3) \quad H_i: \sum_{j=1}^n a_{ij}x_j + b_i \geq 0,$$

in terms of which the set of points corresponding to the solutions of (1.2) is identical with the convex polytope

$$(1.4) \quad A = \bigcap_{i=1}^m H_i,$$

which is assumed from the outset not to be void. Let $p \notin A$ be given. The following simple construction furnishes a point p_1 which is point-wise closer than p to A : Clearly $p \notin H_j$ for some j . Let p' be symmetric to p with respect to the

Received May 11, 1953. Presented to the American Mathematical Society, September, 1952. This work was performed on a National Bureau of Standards contract with the University of California, Los Angeles, and was sponsored (in part) by the Office of Scientific Research, USAF.

boundary π_j of H_j . If p_1 is on the segment joining p to p' and $p_1 \neq p, p'$, then clearly if $a \in H_j$ then

$$|p - a| > |p_1 - a|$$

holds. As $A \subset H_j$, we see that (1.1) is verified, i.e. p_1 is point-wise closer than p to A . The numerical "construction" of p_1 is easily done as follows. Let q be the projection of p on the hyperplane π_j , choose a number λ such that $0 < \lambda < 2$ and set

$$p_1 = p + \lambda(q - p).$$

In passing from p to p_1 , the point-wise approach to A would seem to be strongest if among the H_j , not containing p , we select the one which is furthest away from p . If $\lambda = 2$ then $p_1 = p'$ and then (1.1) again holds, with the exception that we have the equality sign for the points of A which are on the boundary of H_j , if such points exist.

These remarks suggest the following systematic search for a point of A : Choose a point p at will. If $p \in A$, i.e. its coordinates satisfy (1.2), our quest has ended. If $p \notin A$, let H_j be such that

$$(1.5) \quad \text{dist}(p, H_j) = \max_i \text{dist}(p, H_i),$$

where dist denotes euclidean distance. Let q be such that

$$(1.6) \quad q \in H_j, \quad |p - q| = \text{dist}(p, H_j).$$

If λ is a constant, $0 < \lambda \leq 2$, we define

$$(1.7) \quad p_1 = p + \lambda(q - p).$$

For convenience we abbreviate this construction by writing

$$(1.8) \quad p_1 = F_\lambda(p).$$

where $F_\lambda(p)$, defined for $p \notin A$, has been made single-valued by some pre-assigned rule for choosing j in case that (1.5) should not define j uniquely.¹ If $p_1 \in A$ our process has terminated. If $p_1 \notin A$ we iterate (1.8), obtaining

$$p_2 = F_\lambda(p_1),$$

and we continue in like manner deriving a sequence of points $p = p_0, p_1, p_2, \dots$, all outside A and connected by the relation

$$(1.9) \quad p_{\nu+1} = F_\lambda(p_\nu) \quad (\nu = 0, 1, \dots).$$

There are two alternatives: (1) The process terminates after N steps with a point $p_N \in A$; (2) The process continues indefinitely producing an infinite sequence $\{p_\nu\}$.

2. Statement of the main theorems. S. Agmon (1) has recently shown that if $0 < \lambda < 2$ and the sequence $\{p_\nu\}$ is infinite, then p_ν converges, as $\nu \rightarrow \infty$,

¹For instance the smallest j satisfying (1.5).

to a point on the boundary of A . We give a new proof of this result (Theorem 1, Case 1, and Theorem 2, Case 1, below). Our main contribution, however, is the investigation of the case when $\lambda = 2$. Throughout this paper we denote by r the dimensionality of the polytope A defined by (1.4). As we assume that A is not void, r may have any value from zero to n . We denote by L_r the r -flat which contains A .

THEOREM 1. *We assume that $r = n$, i.e. A is not contained in any hyperplane of E_n . Let $\{p_\nu\}$ be a sequence of points obtained by the process described in § 1. There are two cases:*

CASE 1. *If $0 < \lambda < 2$ then either $\{p_\nu\}$ terminates or else p_ν converges to a point l on the boundary of A .*

CASE 2. *If $\lambda = 2$ then the sequence $\{p_\nu\}$ always terminates.*

The formulation of our results for the case when $r < n$ requires the following remarks concerning spherical surfaces. Let L_r be a given r -flat in E_n , $0 \leq r \leq n - 1$. We are also given a point p , $p \notin L_r$. Let X be the locus of points x such that

$$(1.10) \quad |x - a| = |p - a|, \quad \text{for every } a \in L_r.$$

We claim that X is a spherical surface S_{n-r-1} of dimension $n - r - 1$. Thus if $r = 0$, then L_r reduces to a single point a and X is evidently the S_{n-1} with center at a passing through p . In the other extreme case when $r = n - 1$, L_r is a hyperplane and the locus X contains exactly two points: the point p and its symmetric image with respect to L_r . These two points form a S_0 located on the line through p which is normal to L_r . A general proof of our assertion is as follows: Let b be the orthogonal projection of p onto L_r . Erect at b the $(n - r)$ -flat L'_{n-r} which is normal to L_r . Evidently $p \in L'_{n-r}$. Then $x \in X$ if and only if $x \in L'_{n-r}$ and $|x - b| = |p - b|$. Indeed, assume that $x \in X$. By (1.10), for $a = b$, we obtain that $|x - b| = |p - b|$. This last equality and (1.10) show that for every $a \in L_r$ the two triangles xba and pba are congruent. Since $\angle pba = 90^\circ$ we conclude that $\angle xba = 90^\circ$. Hence the line joining b and x is normal to L_r and $x \in L'_{n-r}$. Conversely, if $x \in L'_{n-r}$ and $|x - b| = |p - b|$, let us show that $x \in X$. This is now clear because the two triangles pba and xba ($a \in L_r$) are right-angled at b and have equal legs respectively. This implies (1.10), hence $x \in X$. The locus X may accordingly be defined by the two conditions

$$x \in L'_{n-r}, \quad |x - b| = |p - b|,$$

and is therefore seen to be identical with the spherical surface S_{n-r-1} of L'_{n-r} having its center at b and passing through p . We shall refer to S_{n-r-1} as a spherical surface having L_r as its axis, for indeed, by (1.10), L_r is precisely the locus of points a with the property of being equidistant from all points of S_{n-r-1} .

THEOREM 2. We assume that $r < n$, $A \subset L_r$. Let $\{p_v\}$ be a sequence of points obtained by the process of § 1.

CASE 1. If $0 < \lambda < 2$, then $\{p_v\}$ either terminates or else p_v converges to a point l of A .

CASE 2. If $\lambda = 2$, then $\{p_v\}$ either terminates or else there is a number v_0 such that the points p_v , for $v \geq v_0$, are on a spherical surface S_{n-r-1} having L_r as its axis.

3. Remarks. (a) The procedure here described for finding a solution of (1.2) is called the *relaxation method*, especially if $\lambda = 1$, when it may also be called the *projection method*. We speak of *under-relaxation* or *over-relaxation* depending on whether $0 < \lambda < 1$ or $1 < \lambda < 2$. The case when $\lambda = 2$ is an extreme case of over-relaxation which may also be called the *reflexion method*.

(b) Theorem 1, Case 2, describes the main advantage of the reflexion method ($\lambda = 2$). No other value of λ , $0 < \lambda < 2$, has the property of always leading to a terminating sequence if $r = n$. If $0 < \lambda \leq 1$, this is easily shown by consideration of a triangle A in E_2 . For a λ with $1 < \lambda < 2$, an example of a non-terminating sequence in E_2 is constructed as follows (Fig. 1): Let $\angle pqO = 90^\circ$

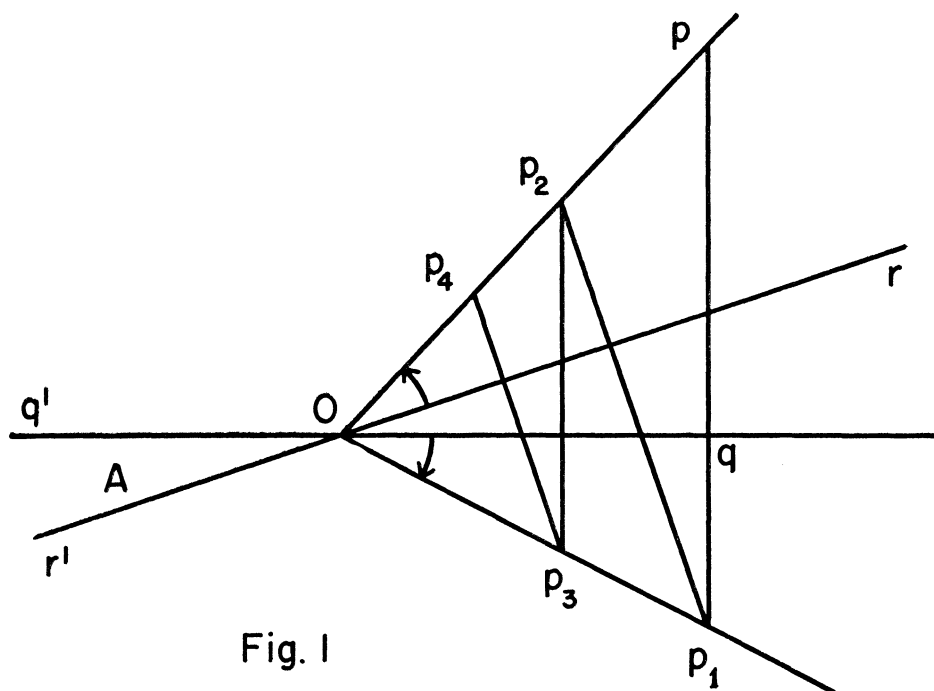


Fig. 1

and p, q, p_1 be such that $pp_1/pq = \lambda$, hence $pq > qp_1$. Draw the ray Or such that $\angle pOr = \angle qOp_1$ and produce Or into a full line $r'Or$. Let A be the intersection of the closed half-plane below $q'q$ and the closed half-plane above $r'r$.

Starting with p and iterating the process $p_1 = F_\lambda(p)$, we obtain an infinite sequence of points $\{p_\nu\}$ which oscillate between the rays Op and Op_1 , converging to O .

(c) Let $\lambda = 2$, $r < n$ and let us suppose that the sequence $\{p_\nu\}$ is infinite. By Theorem 2, Case 2, for $\nu > \nu_0$, all p_ν are on a S_{n-r-1} having L_r as its axis. Since p_ν and $p_{\nu+1}$ are both on S_{n-r-1} , the hyperplane with respect to which p_ν and $p_{\nu+1}$ are symmetric to each other must contain the axis L_r of S_{n-r-1} . We may state this result as

COROLLARY 1. *Let $r < n$ and let the reflexion process ($\lambda = 2$) lead to an infinite sequence $\{p_\nu\}$. Then there exists an integer ν_0 such that all the hyperplanes*

$$(1.11) \quad \pi_i: \sum_{j=1}^n a_{ij}x_j + b_i = 0$$

which are actually used in the reflexion process for $\nu > \nu_0$ contain the entire polytope A and therefore also the r -flat L_r which contains A . Any such hyperplane, or combination of such independent hyperplanes, may therefore be used to reduce the problem to one of a dimension less than n .

(d) When the inequalities (1.2) are all homogeneous, we wish to find a point of A distinct from its vertex o . The relaxation method may well lead to the trivial solution o , if $0 < \lambda < 2$. Thus if $1 < \lambda < 2$ and if A is the "cone" in E_2 of Fig. 1, we have the infinite sequence $\{p_\nu\}$ converging to o . By Theorem 1, Case 2, and Theorem 2, Case 2, this can never happen if $\lambda = 2$.

In II and III we prove the Theorems 1 and 2 respectively. In IV we discuss the behavior of the reflexion process for a special kind of infinite family of half-spaces, namely all half-spaces of support of a bounded and closed convex set in E_n . A study of this problem, suggested by our previous discussion, seems justified by its own geometric interest.

II. A PROOF OF THEOREM 1

4. On Fejér-monotone sequences of points. Let A be defined by (1.4) and let q_0, q_1, q_2, \dots be an infinite sequence of points outside A with the following properties:

$$(2.1) \quad q_i \neq q_{i+1},$$

$$(2.2) \quad |q_i - a| \geq |q_{i+1} - a|, \text{ for all } a \in A, i = 0, 1, \dots$$

The sequence $\{q_\nu\}$ is approaching the set A point-wise and we summarize this situation by saying that the sequence $\{q_\nu\}$ is *Fejér-monotone* with respect to A . Concerning such sequences we prove

LEMMA 1. *Let the sequence $\{q_\nu\}$ be Fejér-monotone with respect to the polytope A , assumed to be of dimension r .*

CASE 1. *If $r = n$ then the sequence $\{q_\nu\}$ converges to a point.*

CASE 2. *If $r < n$ then the sequence $\{q_\nu\}$ either converges to a point or else the set of its limit points lies on a spherical surface S_{n-r-1} whose axis is the r -flat L_r spanned by A .*

Proof. CASE 1. Let $r = n$ and consider the spherical surfaces

$$(2.3) \quad S^\nu(a) : |x - a| = |q_\nu - a| \quad (a \in A, \nu = 0, 1, \dots).$$

By (2.2) the surface $S^\nu(a)$ is non-expanding as its center a is kept fixed and ν increases. Therefore the following limits exist

$$(2.4) \quad \lim_{\nu \rightarrow \infty} |q_\nu - a| = R(a) \quad (a \in A).$$

Define the surface

$$(2.5) \quad S(a) : |x - a| = R(a) \quad (a \in A),$$

and let us consider the set

$$(2.6) \quad X = \bigcap_{a \in A} S(a).$$

(a) *Every limit point l of the sequence $\{q_\nu\}$ is in X .* Indeed by (2.4) we see that $l \in S(a)$, for every $a \in A$, hence $l \in X$, by (2.6). We conclude that X is not void because the bounded sequence $\{q_\nu\}$ has at least one limit point.

(b) *The set X contains exactly one point l .* Indeed, if $l \neq l', l \in X, l' \in X$, then let π denote the hyperplane of points equidistant from l and l' . If $a \in A$ then l, l' being both in X , are also both in $S(a)$. Hence $a \in \pi$ and we conclude that $A \subset \pi$ in contradiction to our assumption that $r = n$.

(c) *We conclude that $q_\nu \rightarrow l$.* Indeed, by (a) and (b) we see that $l = X$ is the only limit point of the sequence $\{q_\nu\}$.

CASE 2. Let $r < n, A \subset L_r$. If the sequence $\{q_\nu\}$ converges to a point then there is nothing to prove. Let us assume that

$$(2.7) \quad \text{the sequence } \{q_\nu\} \text{ does not converge to a point.}$$

In any case the bounded sequence $\{q_\nu\}$ has limit points and let p be one of them. Define as before the spheres $S^\nu(a), S(a)$ and the set X by (2.3), (2.4), (2.5) and (2.6). By (2.4) we have that $R(a) = |p - a|$ ($a \in A$). The set X is therefore identical with the set of points x such that

$$|x - a| = |p - a|, \text{ for every } a \in A.$$

Since A spans L_r we may also define X as the locus of points x such that

$$(2.8) \quad |x - a| = |p - a|, \text{ for every } a \in L_r.$$

As shown in § 2, (2.8) defines a S_{n-r-1} , provided that the locus does not reduce to a point of L_r . Since this locus contains all limit points of $\{q_\nu\}$, our assumption (2.7) excludes this possibility and (2.8) defined a spherical surface S_{n-r-1} . This completes a proof of Lemma 1.

5. Proof of Theorem 1, Case 1. Here $r = n$; assume the sequence $\{p_\nu\}$ to be infinite. As already mentioned in § 1, the sequence $\{p_\nu\}$ is Fejér-monotone with respect to A . By Lemma 1, Case 1, the sequence $\{p_\nu\}$ converges to a point l :

$$(2.9) \quad \lim p_\nu = l.$$

We have to show that $l \in A$. For this purpose we introduce the function

$$(2.10) \quad d(x) = \max_i \text{dist}(x, H_i).$$

Observe that $d(x)$ is everywhere continuous and that

$$(2.11) \quad d(x) \begin{cases} = 0, & x \in A, \\ > 0, & x \notin A. \end{cases}$$

Now (2.9) implies that $|p_{\nu-1} - p_\nu| \rightarrow 0$ and therefore also that

$$d(p_\nu) = \frac{1}{\lambda} |p_{\nu+1} - p_\nu| \rightarrow 0.$$

By the continuity of $d(x)$ we have

$$d(l) = \lim d(p_\nu) = 0$$

and therefore $l \in A$, by (2.11). The point l , being a limit of exterior points p_ν , must be on the boundary of A .

6. Proof of Theorem 1, Case 2. Let $\lambda = 2, r = n$, and let us show that the sequence $\{p_\nu\}$ must terminate. Indeed, suppose it were infinite. By Lemma 1, Case 1, we again conclude that

$$(2.12) \quad \lim p_\nu = l$$

and the argument used in the previous paragraph shows that l is on the boundary of A . Let $p_{\nu+1}$ be obtained from p_ν by reflexion in the boundary π_{j_ν} of the half-space H_{j_ν} . By (2.12) it is clear that

$$(2.13) \quad \lim_{\nu \rightarrow \infty} \text{dist}(l, \pi_{j_\nu}) = 0.$$

The given family of hyperplanes (1.11) being finite, we conclude from (2.13) that

$$\text{dist}(l, \pi_{j_\nu}) = 0,$$

provided $\nu \geq \nu_0$, hence

$$l \in \pi_{j_\nu}, \quad \nu \geq \nu_0.$$

This conclusion, however, contradicts (2.12). Indeed, every point p_ν ($\nu > \nu_0$) is obtained from the preceding point $p_{\nu-1}$ by reflexion in a hyperplane through l . All these points must therefore lie on the spherical surface

$$|x - l| = |p_{\nu_0} - l| (> 0)$$

and can therefore never converge to l , as (2.12) requires.

III. A PROOF OF THEOREM 2

7. Proof of Theorem 2, Case 1. We assume $r < n, 0 < \lambda < 2$, and that the relaxation process (1.9) furnishes an infinite sequence $\{p_\nu\}$. We are to show that

p_ν converges to a point l of A . If the sequence $\{p_\nu\}$ converges to a point l then the argument of § 5 shows that $l \in A$ and we are through. Let us now assume that

(3.1) *the sequence $\{p_\nu\}$ does not converge*

and show that we shall reach a contradiction. By Lemma 1, Case 2, and (3.1), we conclude that all the limit points of $\{p_\nu\}$ are on a spherical surface S_{n-r-1} having L_r as an axis.

(a) *Our assumptions imply that*

$$(3.2) \quad \inf |p_{\nu+1} - p_\nu| = c > 0.$$

Indeed, consider the function $d(x)$, defined by (2.10), for the points x on the surface S_{n-r-1} . Since $A \cap S_{n-r-1} = 0$, we conclude that

$$d(x) > 0, \text{ if } x \in S_{n-r-1}.$$

Since $d(x)$ is continuous and S_{n-r-1} is compact, we conclude that

$$\gamma = \min d(x) > 0, \quad x \in S_{n-r-1}.$$

Let us select γ_1 fixed such that $0 < \gamma_1 < \gamma$. Let δ be positive and let N_δ denote the set of points x defined by

$$\text{dist}(x, S_{n-r-1}) < \delta.$$

Again by the continuity of $d(x)$ we can select δ so small that

$$(3.3) \quad d(x) > \gamma_1, \quad x \in N_\delta.$$

By Lemma 1, Case 2, we have $p_\nu \in N_\delta$, provided $\nu > \nu_0$. Now (3.3) implies that

$$|p_{\nu+1} - p_\nu| = \lambda d(p_\nu) > \lambda \gamma_1,$$

provided $\nu > \nu_0$. This proves (3.2) with $c \geq \lambda \gamma_1 > 0$.

We may now easily show that the assumption (3.1) leads to a contradiction. Indeed, let the point α on S_{n-r-1} be a limit point of $\{p_\nu\}$. For an appropriate subsequence $\{\nu'\}$ of the sequence of all integers $\{\nu\}$ we have

$$p_{\nu'} \rightarrow \alpha \in S_{n-r-1}.$$

For a subsequence $\{\nu''\}$ of $\{\nu'\}$ we may also assume that

$$p_{\nu''} \rightarrow \alpha, \text{ and } p_{\nu''+1} \rightarrow \beta \in S_{n-r-1}.$$

By (3.2) we conclude that $\alpha \neq \beta$; in fact

$$|\alpha - \beta| \geq c > 0.$$

Select on the line through α and β a point η such that $\beta - \alpha = \lambda(\eta - \alpha)$, and notice, because of $0 < \lambda < 2$, that η is nearer to β than to α :

$$(3.4) \quad |\eta - \alpha| > |\eta - \beta|.$$

For the subsequence $\{\nu''\}$, the half-spaces H_j , used in obtaining $p_{\nu''+1}$ from $p_{\nu''}$, must converge to the half-space

$$(3.5) \quad H : |x - \alpha|^2 - |x - \beta|^2 \geq |\eta - \alpha|^2 - |\eta - \beta|^2;$$

in fact H_j must already be identical with H , for sufficiently large ν'' , because of the finiteness of the number of half-spaces H_i . This, however, leads to a contradiction, for on the one hand $A \subset H_j$ implies that $A \subset H$. Hence $x \in A$ implies $x \in H$ and therefore by (3.5) and (3.4)

$$|x - \alpha|^2 - |x - \beta|^2 \geq |\eta - \alpha|^2 - |\eta - \beta|^2 > 0,$$

or

$$(3.6) \quad |x - \alpha| > |x - \beta|.$$

On the other hand, A being on the axis of S_{n-r-1} , we must have in A the equality $|x - \alpha| = |x - \beta|$ in contradiction to (3.6). Thus our assumption (3.1) is untenable and the proof is completed.

8. Proof of Theorem 2, Case 2. We assume $r < n$, $\lambda = 2$, and that the reflexion process produces an infinite sequence $\{p_\nu\}$. This sequence cannot possibly converge, for its limit l would belong to A (§ 5) and would then have to terminate (§ 6). By Lemma 1, Case 2, the only alternative is that the sequence $\{p_\nu\}$ converges to a spherical surface S_{n-r-1} of axis L_r . Then (3.2) or

$$(3.7) \quad \inf |p_{\nu+1} - p_\nu| > 0$$

again holds. Out of A select $r + 1$ fixed points a_0, a_1, \dots, a_r spanning L_r and let π_{j_ν} be the reflecting hyperplane used in obtaining $p_{\nu+1}$ as the point symmetric to p_ν . Since all limit points of p_ν are on S_{n-r-1} , it is clear that

$$\lim_{\nu \rightarrow \infty} |a_k - p_\nu| = \lim_{\nu \rightarrow \infty} |a_k - p_{\nu+1}| = R(a_k) \quad (k = 0, \dots, r).$$

This, together with (3.7), shows that

$$\lim_{\nu \rightarrow \infty} \text{dist}(a_k, \pi_{j_\nu}) = 0 \quad (k = 0, \dots, r).$$

By the finiteness of our supply of reflecting hyperplanes we conclude from the last relations that

$$a_k \in \pi_{j_\nu}, \quad \nu \geq \nu_0; k = 0, \dots, r,$$

or what amounts to the same thing:

$$L_r \subset \pi_{j_\nu}, \quad \nu \geq \nu_0.$$

In other words: there is a number ν_0 such that all reflexions for $\nu \geq \nu_0$ are performed with respect to hyperplanes π_{j_ν} which contain the axis L_r of S_{n-r-1} . This, however, requires that

$$p_\nu \in S_{n-r-1}, \quad \nu \geq \nu_0.$$

Indeed, if $p_{\nu_0} \notin S_{n-r-1}$, then all p_ν ($\nu \geq \nu_0$) would lie on a surface S'_{n-r-1} of axis L_ν , passing through p_{ν_0} , and could then not converge to S_{n-r-1} , as we assumed.

IV. THE REFLEXION PROCESS WITH RESPECT TO A CONVEX DOMAIN

9. Statement of the problem. We have so far discussed the behavior of the reflexion process with respect to a finite family $\{H\}$ of half-spaces in E_n . Do the results obtained extend to infinite classes of half-spaces? We deal here only with the following special case of this problem: Let A be a given closed and bounded convex set in E_n . A closed half-space H belongs to the family F if and only if the boundary of H is a hyperplane of support of A and $A \subset H$. Let $p \notin A$ and let q be the point of A which is nearest to p . Let π_0 be the hyperplane through q which is normal to the segment joining p and q and let H_0 be the closed half-space, bounded by π_0 , which does not contain p . Evidently $H_0 \in F$; also

$$\text{dist}(p, H_0) = \max_{H \in F} \text{dist}(p, H)$$

Indeed, if there were a $H \in F$ such that $\text{dist}(p, H) > \text{dist}(p, H_0) = |p - q|$, then $q \notin H$, in contradiction to the fact that $q \in A = \bigcap H$. This shows that the reflexion process with respect to the family $F = \{H\}$ amounts to the construction of the point

$$(3.1) \quad p_1 = p + 2(q - p) = F(p) \quad (p \notin A).$$

Let us call $p_1 = F(p)$ the *image* of p with respect to A .

If $p_1 \notin A$ we may form $p_2 = F(p_1)$ and continue in like manner obtaining a sequence of points $p = p_0, p_1, p_2, \dots$ connected by the relation

$$(3.2) \quad p_{\nu+1} = F(p_\nu) \quad (\nu = 0, 1, \dots).$$

We have again the old alternative: (1) The process terminates after N steps with $p_N \in A$; (2) The process continues indefinitely, producing an infinite sequence $\{p_\nu\}$.

10. The main result. The behavior of the reflexion process with respect to A is described by the following

THEOREM 3. *Let A be a closed convex and bounded set of E_n of dimension r and let L_r be the r -flat containing A . Suppose $p_0 \notin A$ and let $\{p_\nu\}$ be the sequence obtained by the reflexion process (3.1), (3.2).*

CASE 1. *If $r = n$, then the process always terminates.*

CASE 2. *Let $r < n$. If $p_0 \in L_r$ then the process terminates. If $p_0 \notin L_r$ then the process produces an infinite sequence $\{p_\nu\}$ with the following property. There is a number ν_0 such that for all $\nu > \nu_0$ the points p_ν oscillate between two points which are symmetric with respect to L_r .*

Proof. CASE 1. Let $r = n$ and let us assume, to obtain a contradiction, that the sequence $\{p_\nu\}$ is infinite. We know that $\{p_\nu\}$ is a Fejér-monotone sequence and hence that it converges to a point a by Lemma 1, Case 1. Clearly $a \in A$, and hence a is on the boundary of A , by the argument of § 5. Let P be the projection cone of A at the point a , i.e. the intersection of all H whose boundary hyperplanes pass through a . The cone P is convex and of dimension n , since $P \supset A$. There is therefore a half-space $H \in F$, whose boundary π supports P (and also A) at the point a and whose interior normal ai , at a , is wholly interior to P , except for the point a . Let us think of π as horizontal and its normal ai as pointing vertically downward. The point i being interior to P also a certain small spherical neighborhood S of i is in P . Let us call C the slim circular cone, of vertex a and axis ai , which is circumscribed to S . This convex cone C is wholly in P .

Let us denote by Q the convex cone of vertex a which is generated by the interior normals of all H supporting P (or A) at the point a . These H support also C at a . It follows that the closed convex cone Q (called the polar cone of A at a) has only the point a in common with π , Q being below π . Q was defined as the locus of all interior normals of A at a . Let us denote by N a small neighborhood of a on the boundary of A . Let Q' be a given closed and convex cone satisfying the following conditions: (i) Every ray of Q is interior to Q' , (ii) Q' has only the point a in common with π . Clearly, a neighborhood N of a exists such that the interior normals to A at the points of N , if transferred parallel to themselves so as to start at a , will all lie in Q' . In fact otherwise we could find normals at points converging to A whose limit (which obviously being a normal at A) would be outside or on the boundary of Q' .

We now return to the sequence of points $\{p_\nu\}$ which converges to a . Let q_ν be the midpoint of the segment $p_\nu p_{\nu+1}$. It is clear by our construction that $q_\nu \in$ boundary of A and that the vectors $q_\nu p_{\nu+1}$ are interior normals to A . Also $p_\nu \rightarrow a$ implies that $q_\nu \rightarrow a$, hence $q_\nu \in N$, provided $\nu \geq \nu_0$. This leads to a contradiction. Indeed, consider the sequence of points

$$p_{\nu_0}, p_{\nu_0+1}, \dots$$

We know from what was said above that the vectors $q_\nu p_{\nu+1}$, if transferred to a , lie in Q' . That means that the vectors

$$p_\nu p_{\nu+1} \qquad (\nu \geq \nu_0)$$

have a positive component in the direction of the downward vertical vector ai . Since $p_\nu \rightarrow a$, we conclude that all points $p_\nu (\nu \geq \nu_0)$ are *above* the horizontal plane π . But this implies that also q_ν are above π . This, however, is absurd since $q_\nu \in A$ and A is below π .

CASE 2. If $r < n$ and $p_0 \in L_r$, then again the process must terminate by the previous case because we have no occasion to leave L_r in the course of our process.

We now assume that $r < n$ while $p_0 \notin L_r$. Let L_{r+1} be the $(r + 1)$ -flat containing L_r and p_0 . Note that we never leave L_{r+1} which amounts to assuming at the start that $r = n - 1$. Let, therefore, A be $(n - 1)$ -dimensional, $A \subset L_{n-1}$ and $p_0 \notin L_{n-1}$. Let p'_0 and p'_1 be the projections of p_0 and p_1 , respectively, on L_{n-1} . It should be clear that a point q_0 of A is nearest to p_0 if and only if it is nearest to p'_0 . It follows that $p_1 = F(p_0)$ implies that $p'_1 = F(p'_0)$. Consider the sequence of reflexions $\{p_\nu\}$. It is clear that the distance from p_ν to L_{n-1} has the same positive value $\text{dist}(p_0, L_{n-1})$, the points p_ν passing from one side of the plane to the other alternately. However, the sequence $\{p'_\nu\}$ of their projections on L_{n-1} has the property $p'_{\nu+1} = F(p'_\nu)$. By the previous case this sequence "terminates" with a first $p'_N \in A$. From that moment onwards the sequence p_ν must oscillate between two points on the normal to L_{n-1} at the point $p'_N = p'_{N+1} = \dots$.

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