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CHARACTERISTIC POLYNOMIALS OF THE MATRICES WITH (j,k)-ENTRY $q^{j\pm k}+t$

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Abstract

We determine the characteristic polynomials of the matrices $[q^{j-k}+t]_{1\le j,k\le n}$ and $[q^{j+k}+t]_{1\le j,k\le n}$ for any complex number $q\ne 0,1$. As an application, for complex numbers a,b,c with $b\ne 0$ and $a^2\ne 4b$, and the sequence $(w_m)_{m\in\mathbb{Z}}$ with $w_{m+1}=aw_m-bw_{m-1}$ for all $m\in\mathbb{Z}$, we determine the exact value of $\det[w_{j-k}+c\delta_{jk}]_{1\le j,k\le n}$.

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1. Introduction

For any integer $n\ge 3$, we have the determinant identity

$$\det[j-k]_{1\le j,k\le n}=0$$

since $(1-k)+(3-k)=2(2-k)$ for all $k=1,\ldots,n$. However, it is nontrivial to determine the characteristic polynomial $\det[xI_n-(j-k)]_{1\le j,k\le n}$ of the matrix $[j-k]_{1\le j,k\le n}$, where I_n is the identity matrix of order n.

For $j,k\in\mathbb{N}=\{0,1,2,\ldots\}$, the Kronecker symbol δ_{jk} takes the value 1 or 0 according to whether $j=k$ or not. In 2003, Cloitre [1] generated the sequence $\det[j-k+\delta_{jk}]_{1\le j,k\le n}$ $(n=1,2,3,\ldots)$ with the initial fifteen terms:

1, 2, 7, 21, 51, 106, 197, 337, 541, 826, 1211, 1717, 2367, 3186, 4201.

In 2013, C. Baker added a comment to [1] in which he claimed that

$$\det[j-k+\delta_{jk}]_{1\le j,k\le n}=1+\frac{n^2(n^2-1)}{12} \tag{1.1}$$

without any proof. It seems that Baker found the recurrence of the sequence using the MAPLE package `gfun`.

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Recall that the q-analogue of an integer m is given by

$$[m]_q = \frac{q^m - 1}{q - 1}.$$

Note that $\lim_{q\to 1}[m]_q = m$.

In our first theorem, we determine the characteristic polynomial of the matrix $[q^{j-k} + t]_{1\le j,k\le n}$ for any complex number $q \ne 0, 1$.

THEOREM 1.1. *Let $n \ge 2$ be an integer and let $q \ne 0, 1$ be a complex number. Then the characteristic polynomial of the matrix $P = [q^{j-k} + t]_{1\le j,k\le n}$ is*

$$\det(xI_n - P) = x^{n-2}(x^2 - n(t+1)x + t(n^2 - q^{1-n}[n]_q^2)). \tag{1.2}$$

Putting $t = -1$ and replacing x by $(q-1)x$ in Theorem 1.1, we immediately obtain the following corollary.

COROLLARY 1.2. *Let $n \ge 2$ be an integer and let $q \ne 0, 1$ be a complex number. For the matrix $P_q = [[j-k]_q]_{1\le j,k\le n}$,*

$$\det(xI_n - P_q) = x^n + \frac{q^{1-n}[n]_q^2 - n^2}{(q-1)^2} x^{n-2}.$$

REMARK 1.3. Fix an integer $n \ge 2$. Observe that

$$\begin{aligned}
\lim_{q\to 1} \frac{q^{1-n}[n]_q^2 - n^2}{(q-1)^2} &= \lim_{t\to 0} \frac{(t+1)^{1-n}(((t+1)^n - 1)/t)^2 - n^2}{t^2} \\
&= \lim_{t\to 0} \frac{(t+1)^{1-n}((\sum_{k=1}^n \binom{n}{k} t^{k-1})^2 - n^2) + ((t+1)^{1-n} - 1)n^2}{t^2} \\
&= \lim_{t\to 0} \bigg(\frac{(n + \binom{n}{2}t + \binom{n}{3}t^2 + \cdots)^2 - n^2}{(t+1)^{n-1}t^2} + n^2 \frac{1 - (t+1)^{n-1}}{(t+1)^{n-1}t^2} \bigg) \\
&= \binom{n}{2}^2 + 2n\binom{n}{3} + \lim_{t\to 0} \bigg(2n\binom{n}{2} \frac{t^{-1}}{(t+1)^{n-1}} - n^2 \frac{\sum_{k=1}^n \binom{n-1}{k} t^{k-2}}{(t+1)^{n-1}} \bigg) \\
&= \binom{n}{2}^2 + 2n\binom{n}{3} - n^2\binom{n-1}{2} = \frac{n^2(n^2-1)}{12}.
\end{aligned}$$

So, by Corollary 1.2,

$$\det[x\delta_{jk} - (j-k)]_{1\le j,k\le n} = x^n + \frac{n^2(n^2-1)}{12} x^{n-2}, \tag{1.3}$$

which indicates that when $n > 2$, the n eigenvalues of $A_n = [j-k]_{1\le j,k\le n}$ are

$$\lambda_1 = \frac{n\sqrt{n^2-1}}{2\sqrt{3}}\, i, \quad \lambda_2 = -\frac{n\sqrt{n^2-1}}{2\sqrt{3}}\, i, \quad \lambda_3 = \cdots = \lambda_n = 0.$$

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Note that (1.1) follows from (1.3) with $x=-1$. Concerning the permanent of A_n, motivated by [3, Conjecture 11.23], we conjecture that

$$\mathrm{per}(A_{p-1}) \equiv 3 \pmod p \quad \text{and} \quad \mathrm{per}(A_p) \equiv 1+4p \pmod{p^2}$$

for any odd prime p. Inspired by (1.1), Sun [4] conjectured that for any positive integers m and n,

$$\det[(j-k)^m+\delta_{jk}]_{1\le j,k\le n} = 1+n^2(n^2-1)f(n)$$

for a certain polynomial $f(x)\in\mathbb{Q}[x]$ with $\deg f=(m+1)^2-4$.

Applying Corollary 1.2 with $q=-1$, we find that

$$\det(xI_n-P_{-1}) = x^n + \frac{(-1)^{n-1}[n]_{-1}^2-n^2}{4}x^{n-2}$$

for any integer $n\ge 2$. In particular,

$$\det\Big[\frac{1-(-1)^{j-k}}{2}+\delta_{j,k}\Big]_{1\le j,k\le n} = \frac{9-(-1)^n-2n^2}{8}.$$

Applying Theorem 1.1 with $(t,x)=(-1,-2)$ and $(1,-1)$, we obtain the following result.

Corollary 1.4. *For any positive integer n,*

$$\det[2^{j-k}-1+2\delta_{jk}]_{1\le j,k\le n} = \frac{4^n-2^{n-1}n^2+1}{2}$$

and

$$\det[2^{j-k}+1+\delta_{j,k}]_{1\le j,k\le n} = (n+1)^2-2^{1-n}(2^n-1)^2.$$

In contrast to Theorem 1.1, we also establish the following result.

Theorem 1.5. *Let $n\ge 2$ be an integer and let $q\ne 0,1$ be a complex number. Then the characteristic polynomial of the matrix $Q=[q^{j+k}+t]_{0\le j,k\le n-1}$ is*

$$\det(xI_n-Q) = x^n-(nt+[n]_{q^2})x^{n-1}+(n[n]_{q^2}-[n]_q^2)tx^{n-2}. \tag{1.4}$$

The identity (1.4) with $q=2$ and $x=t=-1$ yields the following corollary.

Corollary 1.6. *For any positive integer n,*

$$\det[2^{j+k}-1+\delta_{jk}]_{0\le j,k\le n-1} = (2^n-1)^2-(n-1)\frac{4^n+2}{3}.$$

For complex numbers a and $b\ne 0$, the Lucas sequence $u_m=u_m(a,b)$ $(m\in\mathbb{Z})$ and its companion sequence $v_m=v_m(a,b)$ $(m\in\mathbb{Z})$ are defined as follows:

$$u_0=0,\ u_1=1 \quad \text{and} \quad u_{k+1}=au_k-bu_{k-1} \quad \text{for all } k\in\mathbb{Z};$$
$$v_0=2,\ v_1=a \quad \text{and} \quad v_{k+1}=av_k-bv_{k-1} \quad \text{for all } k\in\mathbb{Z}.$$

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By the Binet formula,

$$(\alpha-\beta)u_m=\alpha^m-\beta^m \quad \text{and} \quad v_m=\alpha^m+\beta^m \quad \text{for all } m\in\mathbb{Z},$$

where

$$\alpha=\frac{a+\sqrt{a^2-4b}}{2} \quad \text{and} \quad \beta=\frac{a-\sqrt{a^2-4b}}{2} \tag{1.5}$$

are the two roots of the quadratic equation $x^2-ax+b=0$. Clearly, $b^nu_{-n}=-u_n$ and $b^nv_{-n}=v_n$ for all $n\in\mathbb{N}$. For any positive integer n, it is known that

$$u_n=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1-k}{k}a^{n-1-2k}(-b)^k \quad \text{and} \quad v_n=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n}{n-k}\binom{n-k}{k}a^{n-2k}(-b)^k$$

(see [5, page 10]), which can be easily proved by induction. Note also that $u_m(2,1)=m$ for all $m\in\mathbb{Z}$.

For $P(z)=\sum_{k=0}^{n-1}a_kz^k\in\mathbb{C}[z]$, it is known (see [2, Lemma 9]) that

$$\det[P(x_j+y_k)]_{1\le j,k\le n}=a_{n-1}^n\prod_{r=0}^{n-1}\binom{n-1}{r}\times\prod_{1\le j<k\le n}(x_j-x_k)(y_k-y_j).$$

Thus, for any integer $n\ge 3$ and complex numbers a and $b\ne 0$,

$$(\alpha-\beta)^n\det[u_{j-k}(a,b)]_{1\le j,k\le n}=\det[v_{j-k}(a,b)]_{1\le j,k\le n}=0$$

(where α and β are given by (1.5)), since

$$\det[\alpha^{j-k}\pm\beta^{j-k}]_{1\le j,k\le n}=\prod_{k=1}^n\alpha^{-k}\times\prod_{j=1}^n\beta^j\times\det\Big[\Big(\frac{\alpha}{\beta}\Big)^j\pm\Big(\frac{\alpha}{\beta}\Big)^k\Big]_{1\le j,k\le n}=0.$$

As an application of Theorem 1.1, we obtain the following new result.

THEOREM 1.7. *Let a and $b\ne 0$ be complex numbers with $a^2\ne 4b$. Let $(w_m)_{m\in\mathbb{Z}}$ be a sequence of complex numbers with $w_{k+1}=aw_k-bw_{k-1}$ for all $k\in\mathbb{Z}$. For any complex number c and integer $n\ge 2$,*

$$\det[w_{j-k}+c\delta_{jk}]_{1\le j,k\le n}=c^n+c^{n-1}nw_0+c^{n-2}(w_1^2-aw_0w_1+bw_0^2)\frac{b^{1-n}u_n(a,b)^2-n^2}{a^2-4b}. \tag{1.6}$$

REMARK 1.8. It would be hard to guess the exact formula for $\det[w_{j-k}+c\delta_{jk}]_{1\le j,k\le n}$ in Theorem 1.7 by looking at various numerical examples.

COROLLARY 1.9. *Let a,b,c be complex numbers with $b\ne 0$ and $a^2\ne 4b$. For any integer $n\ge 2$,*

$$\det[u_{j-k}(a,b)+c\delta_{jk}]_{1\le j,k\le n}=c^n+c^{n-2}\frac{b^{1-n}u_n(a,b)^2-n^2}{a^2-4b}$$

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and

$$\det[v_{j-k}(a,b)+c\delta_{jk}]_{1\le j,k\le n}=c^{n-2}((n+c)^2-b^{1-n}u_n(a,b)^2).$$

For any $m\in\mathbb{Z}$, $u_m(-1,1)$ coincides with the Legendre symbol $(\frac{m}{3})$, and $v_m(1,-1)=\omega^m+\bar{\omega}^m$, where ω denotes the cube root $(-1+\sqrt{-3})/2$ of unity. Applying Corollary 1.9 with $a=-1$ and $b=1$, we get the following result.

COROLLARY 1.10. *For any integer $n\ge 2$ and complex number c,*

$$\det\left[\left(\frac{j-k}{3}\right)+c\delta_{j,k}\right]_{1\le j,k\le n}=c^n+c^{n-2}\left\lfloor\frac{n^2}{3}\right\rfloor.$$

Recall that $F_m=u_m(1,-1)$ $(m\in\mathbb{Z})$ are the well-known Fibonacci numbers and $L_m=v_m(1,-1)$ $(m\in\mathbb{Z})$ are the Lucas numbers. Corollary 1.9 with $a=1$ and $b=-1$ yields the following result.

COROLLARY 1.11. *For any integer $n\ge 2$ and complex number c,*

$$\det[F_{j-k}+c\delta_{jk}]_{1\le j,k\le n}=c^n+\frac{c^{n-2}}{5}((-1)^{n-1}F_n^2-n^2)$$

and

$$\det[L_{j-k}+c\delta_{jk}]_{1\le j,k\le n}=c^{n-2}((n+c)^2+(-1)^nF_n^2).$$

Although we have Theorem 1.5 which is similar to Theorem 1.1, it seems impossible to use Theorem 1.5 to deduce a result similar to Theorem 1.7.

2. Proof of Theorem 1.1

LEMMA 2.1. *Let n be a positive integer, and let $q\ne 0$ and t be complex numbers with $n-[n]_q+t(q^{1-n}[n]_q-n)\ne 0$. Suppose that*

$$\gamma=\frac{n(t+1)\pm\sqrt{n^2(t-1)^2+4tq^{1-n}[n]_q^2}}{2}\quad\text{and}\quad y=\frac{\gamma-[n]_q-nt}{n-[n]_q+(q^{1-n}[n]_q-n)t}.\tag{2.1}$$

Then, for any positive integer j,

$$\sum_{k=1}^{n}(q^{j-k}+t)(1+y(q^{k-n}-1))=\gamma(1+y(q^{j-n}-1)).\tag{2.2}$$

PROOF. As $\gamma^2-n(t+1)\gamma+(n^2-q^{1-n}[n]_q^2)t=0$,

$$[n]_q(n-[n]_q+(q^{1-n}[n]_q-n)t)=(\gamma-[n]_q-nt)(\gamma-n+[n]_q)$$

and hence

$$(\gamma-n+[n]_q)y=[n]_q.\tag{2.3}$$

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For $j \in \{1, 2, 3, \ldots\}$, set

$$\Delta_j = \sum_{k=1}^{n} (q^{j-k} + t)(1 + y(q^{k-n} - 1)) - \gamma(1 + y(q^{j-n} - 1)).$$

Then, by (2.3),

$$\begin{aligned}\Delta_j - t(1 + y(q^{k-n} - 1)) + \gamma(1 - y) &= q^{j-n}\Big(\sum_{k=1}^{n} q^{n-k}(1 + y(q^{k-n} - 1)) - \gamma y\Big)\\ &= q^{j-n}([n]_q(1 - y) + ny - \gamma y) = 0.\end{aligned}$$

So $\Delta_1 = \Delta_2 = \cdots$.

Next we show that $\Delta_n = 0$. Observe that

$$\begin{aligned}\sum_{k=1}^{n} (q^{n-k} + t)(1 + (q^{k-n} - 1)y) &= \sum_{k=1}^{n} (q^{n-k}(1 - y) + t(1 - y) + y + q^{k-n}ty)\\ &= [n]_q(1 - y) + nt(1 - y) + ny + q^{1-n}[n]_q ty\\ &= [n]_q + nt + y(n - [n]_q + (q^{1-n}[n]_q - n)t)\\ &= \gamma = \gamma(1 + y(q^{n-n} - 1))\end{aligned}$$

by the definition of y. So $\Delta_n = 0$.

In view of the above, $\Delta_j = 0$ for all $j = 1, 2, 3, \ldots$. This concludes the proof. □

Proof of Theorem 1.1. It is easy to verify the desired result for $n = 2$. Below we assume that $n \geq 3$.

If $n - [n]_q$ and $q^{1-n}[n]_q - n$ are both zero, then $q^{n-1} = 1$ and $n = [n]_q = 1$. As $n \geq 3$, there are infinitely many $t \in \mathbb{C}$ such that

$$n - [n]_q + t(q^{1-n}[n]_q - n) \neq 0 \quad \text{and} \quad n^2(t - 1)^2 + 4tq^{1-n}[n]_q^2 \neq 0.$$

Take such a number t, and choose γ and y as in (2.1). Then γ given in (2.1) is an eigenvalue of the matrix $P = [q^{j-k} + t]_{1 \leq j,k \leq n}$, and the column vector $v = (v_1, \ldots, v_n)^T$ with $v_k = 1 + y(q^{k-n} - 1)$ is an eigenvector of P associated with the eigenvalue γ. Note that γ given by (2.1) has two different choices since $n^2(t - 1)^2 + 4tq^{1-n}[n]_q^2 \neq 0$.

Let $s \in \{3, \ldots, n\}$. For $1 \leq k \leq n$, let us define

$$v_k^{(s)} = \begin{cases} q^{2-s}[s - 2]_q & \text{if } k = 1,\\ -q^{2-s}[s - 1]_q & \text{if } k = 2,\\ \delta_{sk} & \text{if } 3 \leq k \leq n.\end{cases}$$

It is easy to verify that

$$\sum_{k=1}^{n} v_k^{(s)} = 0 = \sum_{k=1}^{n} q^{j-k} v_k^{(s)} \quad \text{for all } j = 1, \ldots, n.$$

Thus, 0 is an eigenvalue of the matrix $P = [q^{j-k} + t]_{1 \leq j,k \leq n}$, and the column vector $v^{(s)} = (v_1^{(s)}, \ldots, v_n^{(s)})^T$ is an eigenvector of P associated with the eigenvalue 0.

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If $\sum_{s=3}^{n} c_s v^{(s)}$ is the zero column vector for some $c_3, \ldots, c_n \in \mathbb{C}$, then for each $k = 3, \ldots, n$,

$$c_k = \sum_{s=3}^{n} c_s \delta_{sk} = \sum_{s=3}^{n} c_s v_k^{(s)} = 0.$$

Thus, the $n-2$ column vectors $v^{(3)}, \ldots, v^{(n)}$ are linearly independent over $\mathbb{C}$.

By the above, the n eigenvalues of the matrix $P = [q^{j-k}+t]_{1\le j,k\le n}$ are the two values of γ given by (2.2) and $\lambda_3 = \cdots = \lambda_n = 0$. Thus, the characteristic polynomial of P is

$$\begin{aligned}\det(xI_n - P) &= \bigg(x - \frac{n(t+1)}{2} - \frac{\sqrt{n^2(t-1)^2 + 4tq^{1-n}[n]_q^2}}{2}\bigg) \\ &\quad \times \bigg(x - \frac{n(t+1)}{2} + \frac{\sqrt{n^2(t-1)^2 + 4tq^{1-n}[n]_q^2}}{2}\bigg) \prod_{s=3}^{n}(x-\lambda_s) \\ &= x^{n-2}\bigg(\bigg(x - \frac{n(t+1)}{2}\bigg)^2 - \frac{n^2(t-1)^2 + 4tq^{1-n}[n]_q^2}{4}\bigg) \\ &= x^{n-2}(x^2 - n(t+1)x + t(n^2 - q^{1-n}[n]_q^2)).\end{aligned}$$

Thus, the identity (1.2) holds for infinitely many values of t. Note that both sides of (1.2) are polynomials in t for any fixed $x \in \mathbb{C}$. Thus, if we view both sides of (1.2) as polynomials in x and t, then the identity (1.2) still holds. This completes the proof. □

3. Proof of Theorem 1.5

The following lemma is quite similar to Lemma 2.1.

LEMMA 3.1. *Let n be a positive integer, and let $q \neq 0$ and t be complex numbers with $[n]_{q^2} + (q^{1-n}t - q^{n-1})[n]_q - nt \neq 0$. Suppose that*

$$\gamma = \frac{nt + [n]_{q^2} \pm \sqrt{(nt - [n]_{q^2})^2 + 4t[n]_q^2}}{2} \quad \textit{and} \quad z = \frac{\gamma - q^{n-1}[n]_q - nt}{[n]_{q^2} + (q^{1-n}t - q^{n-1})[n]_q - nt}. \tag{3.1}$$

Then, for every $j = 0, 1, 2, \ldots,$

$$\sum_{k=0}^{n-1}(q^{j+k}+t)(1 + z(q^{k-n+1} - 1)) = \gamma(1 + z(q^{j-n+1} - 1)). \tag{3.2}$$

PROOF. Since $\gamma^2 - (nt + [n]_{q^2})\gamma + t(n[n]_{q^2} - [n]_q^2) = 0$, we have

$$(\gamma - [n]_{q^2} + q^{n-1}[n]_q)z = q^{n-1}[n]_q. \tag{3.3}$$

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For $j \in \{0, 1, 2, \ldots\}$, set

$$R_j = \sum_{k=0}^{n-1} (q^{j+k} + t)(1 + z(q^{k-n+1} - 1)) - \gamma(1 + z(q^{j-n+1} - 1)).$$

It is easy to see that

$$R_j - \sum_{k=0}^{n-1} t(1 + z(q^{k-n+1} - 1)) + \gamma(1 - z) = q^{j-n+1}(q^{n-1}[n]_q(1 - z) + z[n]_{q^2} - \gamma z) = 0$$

with the aid of (3.3). So $R_0 = R_1 = \cdots$. As

$$\sum_{k=0}^{n-1} (q^{n-1+k} + t)(1 + z(q^{k-n+1} - 1)) = \gamma = \gamma(1 + z(q^{(n-1)-n+1} - 1)),$$

we get $R_{n-1} = 0$. So the desired result follows. □

PROOF OF THEOREM 1.5. It is easy to verify the desired result for $n = 2$. Below we assume that $n \geq 3$.

If $[n]_{q^2} - q^{n-1}[n]_q$ and $q^{1-n}[n]_q - n$ are both zero, then $[n]_q \neq 0$ and

$$(q^n + 1)[n]_q = (q + 1)[n]_{q^2} = (q + 1)q^{n-1}[n]_q = (q^n + q^{n-1})[n]_q,$$

and hence $q^{n-1} = 1$ and $n = [n]_q = 1$. As $n \geq 3$, there are infinitely many $t \in \mathbb{C}$ such that

$$[n]_{q^2} + (q^{1-n}t - q^{n-1})[n]_q - nt \neq 0 \quad \text{and} \quad (nt - [n]_{q^2})^2 + 4t[n]_q^2 \neq 0.$$

Take such a number t, and choose γ and z as in (3.1). Then γ given in (3.1) is an eigenvalue of the matrix $Q = [q^{j+k} + t]_{0 \leq j,k \leq n-1}$, and the column vector $v = (v_0, \ldots, v_{n-1})^T$ with $v_k = 1 + z(q^{k-n+1} - 1)$ is an eigenvector of Q associated with the eigenvalue γ. There are two different choices for γ since $(nt - [n]_{q^2})^2 + 4t[n]_q^2 \neq 0$.

Let $s \in \{3, \ldots, n\}$. For $k \in \{0, \ldots, n - 1\}$, define

$$v_k^{(s)} = \begin{cases} q[s-2]_q & \text{if } k = 0, \\ -[s-1]_q & \text{if } k = 1, \\ \delta_{s,k+1} & \text{if } 2 \leq k \leq n-1. \end{cases}$$

It is easy to verify that

$$\sum_{k=0}^{n-1} v_k^{(s)} = 0 = \sum_{k=0}^{n-1} q^{j+k} v_k^{(s)} \quad \text{for all } j = 1, \ldots, n.$$

Thus, 0 is an eigenvalue of the matrix $Q = [q^{j+k} + t]_{0 \leq j,k \leq n-1}$, and the column vector $v^{(s)} = (v_0^{(s)}, \ldots, v_{n-1}^{(s)})^T$ is an eigenvector of Q associated with the eigenvalue 0.

If $\sum_{s=3}^{n} c_s v^{(s)}$ is the zero column vector for some $c_3,\ldots,c_n\in\mathbb{C}$, then for each $k=2,\ldots,n-1$,

$$c_{k+1}=\sum_{s=3}^{n} c_s\delta_{s,k+1}=\sum_{s=3}^{n} c_s v_k^{(s)}=0.$$

Thus, the $n-2$ column vectors $v^{(3)},\ldots,v^{(n)}$ are linearly independent over $\mathbb{C}$.

By the above, the n eigenvalues of the matrix $Q=[q^{j+k}+t]_{0\le j,k\le n-1}$ are the two values of γ given by (3.2) and $\lambda_3=\cdots=\lambda_n=0$. Thus, the characteristic polynomial of Q is

$$\begin{aligned}\det(xI_n-Q)&=\Big(x-\frac{nt+[n]_{q^2}}{2}-\frac{\sqrt{(nt-[n]_{q^2})^2+4t[n]_q^2}}{2}\Big)\\&\quad\times\Big(x-\frac{nt+[n]_{q^2}}{2}+\frac{\sqrt{(nt-[n]_{q^2})^2+4t[n]_q^2}}{2}\Big)\prod_{s=3}^{n}(x-\lambda_s)\\&=x^{n-2}\Big(\Big(x-\frac{nt+[n]_{q^2}}{2}\Big)^2-\frac{(nt-[n]_{q^2})^2+4t[n]_q^2}{4}\Big)\\&=x^n-(nt+[n]_{q^2})x^{n-1}+(n[n]_{q^2}-[n]_q^2)tx^{n-2}.\end{aligned}$$

Thus, the identity (1.4) holds for infinitely many values of t. Note that both sides of (1.4) are polynomials in t for any fixed $x\in\mathbb{C}$. If we view both sides of (1.4) as polynomials in x and t, then the identity (1.4) still holds. This concludes the proof. □

4. Proof of Theorem 1.7

PROOF OF THEOREM 1.7. If $w_0=w_1=0$ or $n=2$, then the desired result can be easily verified. Below we assume that $n\ge 3$ and $\{w_0,w_1\}\ne\{0\}$.

Let α and β be the two roots of the quadratic equation $z^2-az+b=0$. Note that $\alpha\beta=b\ne 0$. Also, $\alpha\ne\beta$ since $\Delta=a^2-4b$ is nonzero. It is well known that there are constants $c_1,c_2\in\mathbb{C}$ such that $w_m=c_1\alpha^m+c_2\beta^m$ for all $m\in\mathbb{Z}$. As $c_1+c_2=w_0$ and $c_1\alpha+c_2\beta=w_1$,

$$c_1=\frac{w_1-\beta w_0}{\alpha-\beta}\quad\text{and}\quad c_2=\frac{\alpha w_0-w_1}{\alpha-\beta}. \tag{4.1}$$

Since w_0 or w_1 is nonzero, one of c_1 and c_2 is nonzero. Without any loss of generality, we assume $c_1\ne 0$.

Let W denote the matrix $[w_{j-k}+c\delta_{jk}]_{1\le j,k\le n}$. Then

$$\begin{aligned}\det(W)&=\det[c_1\alpha^{j-k}+c_2\beta^{j-k}+c\delta_{jk}]_{1\le j,k\le n}\\&=c_1^n\prod_{j=1}^{n}\beta^j\times\prod_{k=1}^{n}\beta^{-k}\times\det\Big[\Big(\frac{\alpha}{\beta}\Big)^{j-k}+\frac{c_2+c\delta_{jk}\beta^{k-j}}{c_1}\Big]_{1\le j,k\le n}\\&=c_1^n\det[q^{j-k}+t-x\delta_{jk}]_{1\le j,k\le n}=(-c_1)^n\det[x\delta_{jk}-q^{j-k}-t]_{1\le j,k\le n},\end{aligned}$$

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where $q = \alpha/\beta \neq 0, 1$, $t = c_2/c_1$ and $x = -c/c_1$. By applying Theorem 1.1, we obtain

$$\begin{aligned}\det(W) &= (-c_1)^n x^{n-2}(x^2 - n(t+1)x + t(n^2 - q^{1-n}[n]_q^2)) \\ &= c^{n-2}\Big(c^2 + nc(c_1 + c_2) + c_1 c_2\Big(n^2 - \frac{\alpha^{1-n}}{\beta^{1-n}}\Big(\frac{(\alpha/\beta)^n - 1}{\alpha/\beta - 1}\Big)^2\Big)\Big) \\ &= c^n + nw_0 c^{n-1} + c^{n-2}c_1 c_2\Big(n^2 - (\alpha\beta)^{1-n}\Big(\frac{\alpha^n - \beta^n}{\alpha - \beta}\Big)^2\Big) \\ &= c^n + nw_0 c^{n-1} + c^{n-2}c_1 c_2(n^2 - b^{1-n}u_n(a,b)^2).\end{aligned}$$

In view of (4.1),

$$c_1 c_2 = \frac{(w_1 - \beta w_0)(\alpha w_0 - w_1)}{(\alpha - \beta)^2} = \frac{-w_1^2 + (\alpha + \beta)w_0 w_1 - \alpha\beta w_0^2}{\Delta} = -\frac{w_1^2 - a w_0 w_1 + b w_0^2}{a^2 - 4b}.$$

Therefore, the desired evaluation (1.6) follows. □

References

[1] B. Cloitre, 'Sequence A079034', *OEIS (On-Line Encyclopedia of Integer Sequences)*. Available at https://oeis.org/A079034.

[2] C. Krattenthaler, 'Advanced determinant calculus: a complement', *Linear Algebra Appl.* **411** (2005), 68–166.

[3] Z.-W. Sun, *New Conjectures in Number Theory and Combinatorics* (Harbin Institute of Technology Press, Harbin, 2021) (in Chinese).

[4] Z.-W. Sun, 'Sequences A355175 and A355326', *OEIS (On-Line Encyclopedia of Integer Sequences)*. Available at https://oeis.org.

[5] Z.-W. Sun, *Fibonacci Numbers and Hilbert's Tenth Problem* (Harbin Institute of Technology Press, Harbin, 2024) (in Chinese).

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