$\therefore \quad C\widehat{B}D = 15°,$

and $\quad \tan 15° = \dfrac{CD}{BC}.$

Now let $BC = 1$.

Then $AD = AB = 2$,

and $CD = AD - AC = 2 - \sqrt{3}$.

$$\therefore \tan 15° = \frac{2 - \sqrt{3}}{1}$$
$$= 2 - \sqrt{3}.$$

II. Let ABC be a triangle in which $\widehat{A} = \widehat{B} = 45°$, $\widehat{C} = 90°$.

With the same construction as above, we have

$$A\widehat{B}D = A\widehat{D}B = 67\tfrac{1}{2}°.$$

$$\therefore \quad C\widehat{B}D = 22\tfrac{1}{2}°.$$

Taking again $BC = 1$, we find

$$\tan 22\tfrac{1}{2}° = \frac{\sqrt{2} - 1}{1} = \sqrt{2} - 1.$$

PETER RAMSAY

The Numerically Greatest Term of a Binomial Expansion.—The problem of the greatest term of a binomial expansion is a favourite one in elementary text-books, and its solution is often difficult to a beginner. The difficulty, at least in the case where the index is negative or fractional, seems to be caused by the fact that a "formula" is provided which gives a value for r, such that the $(r+1)$th term is the greatest. Moreover, this formula is not always the same. Sometimes it is $\dfrac{(n+1)x}{x+1}$, sometimes $\dfrac{(n+1)x}{x-1}$; and unless the student has a very good memory he is sure sometimes to make mistakes. Elementary mathematics ought not to be a memory exercise. It is a platitude to say that the educational value of the teaching of mathematics lies in its training of the powers of reasoning. This element is eliminated when processes of reasoning are reduced to a rule of thumb. As well might one use "Molesworth" as a text-book of the principles of mechanics.

To return to this special example: there is really no reason for employing a formula except in discussing the general case; the saving of time is negligible, and it is much better to start from the beginning with the ratio of the $(r+1)$th term to the rth

$$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}x.$$

The values of n and x should next be substituted, and here is the point where the difficulty arises. Before proceeding further it is essential that the ratio should be made positive.

There are the usual three cases: (1) n a positive integer, (2) $n+1$ negative, (3) $n+1$ positive. The first two cases do not present any essential difficulty; the third is specially interesting, and is usually given an incomplete treatment in the text-books.

(1) *n a positive integer.* The series is finite with $n+1$ terms, so that r cannot be greater than n, and $n-r+1$ is always positive. We put then

$$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\,|\,x\,|<1,$$

and choose the least value of r consistent with the inequality. This makes the rth term the greatest; or we may get two greatest terms $T_r=T_{r+1}$.

(2) *$n+1$ negative.* Here $n-r+1$ is always negative, so that if x is positive the sign must be changed. Since the convergency-ratio tends to the value $|x|$ as r increases, if $|x|\geqq 1$ there is no greatest term, otherwise the result is similar to the first case.

(3) *$n+1$ positive.* $\dfrac{T_{r+1}}{T_r}=\left(\dfrac{n+1}{r}-1\right)x.$

The ratio has the same sign as x or the opposite sign according as $r\lessgtr n+1$.

Hence if x is positive, the terms are all $+$ up to the sth, where s is the integer next greater than $n+1$; thereafter they are alternately $-$ and $+$.

If x is negative, the terms are alternately $+$ and $-$ up to the sth, thereafter they are all $+$ or all $-$ according as s is odd or even.

There is nothing *a priori* to show whether the maximum term will occur before or after the sth.

(a) Suppose it occurs before the sth, so that $r < n+1$. Then $\left(\frac{n+1}{r} - 1\right)|x|$ is positive, and if we put $\frac{T_{r+1}}{T_r} > 1$ this inequality holds so long as $r < (n+1)\frac{|x|}{|x|+1}$. This may give a single greatest term or two equal greatest terms before the sth.

(b) Taking $r > n+1$; $\left(1 - \frac{n+1}{r}\right)|x|$ is positive. Putting this < 1, we find that the inequality holds so long as

$$r(|x| - 1) < (n+1)|x|.$$

If $|x| \leqq 1$, this is true for all values of r (of course $> n+1$).

If $|x| > 1$, there will be a *least* term T_{r+1} if r is the greatest integral value of r consistent with the inequality $r < (n+1)\frac{|x|}{|x|-1}$.

Hence in all cases there is a greatest term (or terms) before the sth, and (1) if $|x| \leqq 1$ the terms thereafter continually diminish, (2) if $|x| > 1$ there is a least term (or terms) on or beyond the sth, and thereafter the terms continually increase.

Various interesting possibilities may arise according as the maximum and minimum are single or double or coalesce.

The maximum will be double if $\frac{|x|}{|x|+1}(n+1)$ is an integer, and the minimum will be double if $\frac{|x|}{|x|-1}(n+1)$ is an integer. These contingencies may happen simultaneously, *e.g.*, if $n = \frac{13}{2}$, $x = 4$.

In general the minimum is less than the maximum, but if they are consecutive they may be equal, *e. g.*

$$(1+6)^{\frac{4}{3}} = 1 + 8 + 8 - \tfrac{32}{3} + \tfrac{80}{3} - \ldots$$

Here there is no true maximum or minimum.

Further, each may be doubled, and they may be consecutive and equal. This gives *three consecutive terms numerically equal* (not four, for the ratio T_{r+1}/T_r can $= 1$ at most twice). In order that this may happen, we must have $x = 2k+1$ and $n = \frac{2k^2-1}{2k+1}$, where k is a positive integer; then T_k, T_{k+1} and T_{k+2} are numerically equal. For if $T_k = T_{k+1} = -T_{k+2}$, then $\dfrac{|x|}{|x|+1}(n+1) = k$ and $\dfrac{|x|}{|x|-1}(n+1) = k+1$; solving these equations simultaneously we obtain the above values of $|x|$ and n. Thus

$$(1+5)^{\frac{7}{5}} = 1 + 7 + 7 - 7 + 14 - \dots$$

Finally, for all values of n the first term is the greatest if $|nx| < 1$.

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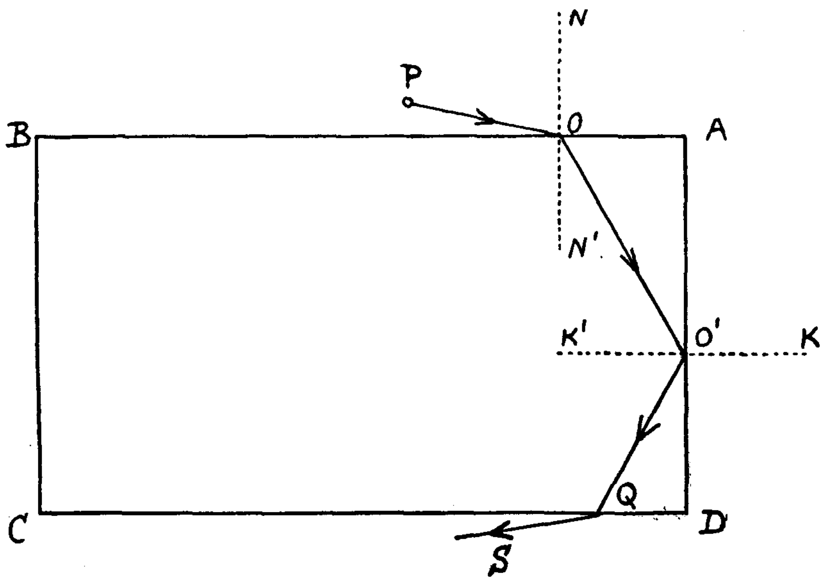
An Experiment in Light.—Let ABCD be a horizontal section of a rectangular slab of glass. A pin is set up vertically at P, close to the face AB, or at a short distance from it. It is possible to see the pin P through the glass if we look through the face CD; but the pin is invisible if we look through the face AD. If, however, we look through the face CD, *into the face* AD, we shall see an image of the pin in AD, which acts like a mirror. The experiment illustrates total reflection of light; the explanation is easy. Let PO be a ray of light which after passing into the glass will be incident on AD at O′. Let NON′ be the normal to AB at O,



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