THE SPECTRAL MATRIX AND GREEN'S FUNCTION FOR SINGULAR SELF-ADJOINT BOUNDARY VALUE PROBLEMS

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1. Introduction. Let L denote the formal ordinary differential operator

$$L = p_0(d/dx)^n + p_1(d/dx)^{n-1} + \ldots + p_n,$$

where we assume the p_k are complex-valued functions with $n-k$ continuous derivatives on an open real interval $a < x < b$ ($a = -\infty$, $b = +\infty$, or both may occur), $p_0(x) \neq 0$ on $a < x < b$, and L coincides with its Lagrange adjoint L^+ given by

$$L^+ = (-1)^n(d/dx)^n(\bar{p}_0 \cdot) + (-1)^{n-1}(d/dx)^{n-1}(\bar{p} \ \cdot) + \ldots + \bar{p}_n.$$

The purpose of this paper is to prove the existence and uniqueness of Green's function for certain self-adjoint boundary value problems associated with L on (a, b) (Theorems 2 and 3 below), and to show how the existence and uniqueness of the spectral matrix occurring in the Parseval equality may be obtained from Green's function (Theorem 5 below). Further we prove the formula which gives the spectral matrix in terms of Green's function (Theorem 5).

The case $n = 2$, p_k real, was treated initially by Weyl **(11)**. Later Titchmarsh **(9)** gave new proofs of the Weyl results using the residue calculus, and obtained a formula relating Green's function and the spectral matrix. Other proofs of the main results in this case have been given by Stone **(8)**, Kodaira **(2)**, Levinson **(4)**, **(5)**, and Yosida **(12)**. Kodaira **(3)** extended Weyl's results to the case where n is even and the p_k are real, using Hilbert space methods. Levitan **(7)** has considered this case also, and has shown the existence of at least one spectral matrix. It is not difficult to see that his proof carries over to the case of the L considered here. In **(1)** we proved, among other things, the uniqueness of the spectral matrix in two important cases. Another proof along the lines of **(5)** has recently been given by Levinson **(6)**. Here we prove the existence and uniqueness of Green's function and the spectral matrix for these two cases by a simple limiting process on self-adjoint boundary value problems on finite sub-intervals of (a, b). The formula relating Green's function to the spectral matrix reduces to that given by Kodaira **(3)** for the cases he considered.

In §2 we state results we require concerning self-adjoint problems in the non-singular case. Section 3 is first devoted to the proof of the compactness of the set of Green's functions $\{G_\delta\}$ associated with self-adjoint boundary value problems on closed bounded sub-intervals δ of (a, b). The essential idea of the proof (Lemma 3) is an outgrowth of our reading the paper by Titchmarsh **(10)** on

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the Laplace operator in the plane. We then consider two cases: the first where no boundary conditions at a or b are required to obtain a self-adjoint boundary value problem; the second where the end-point a is finite and (a, b) can be replaced by $[a, b)$, and a self-adjoint boundary value problem results by imposing boundary conditions at a alone. For $n = 2$, p_k real, the first case corresponds to the situation where L is of the limit point type at a and b, the second where L is of the limit circle type at a and limit point type at b. For our two cases we prove the existence and uniqueness of Green's function, and in §4 we do the same for the spectral matrix, and give the formula relating the two.

2. The spectral matrix and Green's function in the non-singular case. In this section we collect together several well-known facts concerning self-adjoint boundary value problems on a closed bounded sub-interval $\delta = [\tilde{a}, \tilde{b}]$ of (a, b) and show how the spectral matrix is related to Green's function in this case.

Let $\mathfrak{D}$ denote the set of all complex-valued functions u of class $\mathfrak{L}^2(a, b)$ which have continuous derivatives up to order $n - 1$ on (a, b), $u^{(n-1)}$ is absolutely continuous on every closed sub-interval of (a, b), and Lu is of class $\mathfrak{L}^2(a, b)$. For functions u, v in $\mathfrak{D}$ we have Green's formula

$$\int_y^x (\bar{v}\, Lu - u\overline{Lv}) = [uv](x) - [uv](y), \tag{2.1}$$

where $a < y < x < b$, and $[uv](x)$ is the form in $(u, u', \ldots, u^{(n-1)})$ and $(v, v', \ldots, v^{(n-1)})$ given by

$$[uv](x) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j\, u^{(k)}(x)\, (p_{n-m}\, \bar{v})^{(j)}(x). \tag{2.2}$$

We can write (2.2) as

$$[uv](x) = \sum_{j,k=0}^{n-1} B_{jk}(x)\, u^{(k)}(x)\, \bar{v}^{(j)}(x),$$

and it is readily seen that the matrix $B(x) = (B_{jk}(x))$ is non-singular; in fact its determinant is $(p_0(x))^n$. From Green's formula one finds that

$$[uv](x) = -\overline{[vu](x)}.$$

Suppose

$$U_{\delta j}(u) = \sum_{k=1}^{n} M_{\delta jk}\, u^{(k-1)}(\tilde{a}) + N_{\delta jk}\, u^{(k-1)}(\tilde{b}) = 0 \qquad (j = 1, \ldots, n), \tag{2.3}$$

represent n homogeneous self-adjoint (relative to L) boundary conditions for functions $u \in \mathfrak{D}$ on the interval $\delta = [\tilde{a}, \tilde{b}]$. Here $M_{\delta jk}$ and $N_{\delta jk}$ are complex constants, and the linearly independent conditions (2.3) are self-adjoint if and only if

$$M_\delta\, B^{-1}(\tilde{a})\, M_\delta^* = N_\delta\, B^{-1}(\tilde{b})\, N_\delta^*,$$

where M_δ, N_δ are matrices with the elements $M_{\delta jk}$, $N_{\delta jk}$ respectively, B^{-1} is the reciprocal of the matrix B of the form (2.2), and the asterisk indicates the conjugate transposed matrix (cf. **1**). As is known the self-adjoint boundary value problem

$$Lu = lu, \qquad U_{\delta j}(u) = 0 \qquad (j = 1, \ldots, n), \tag{2.4}$$

(l a complex parameter) possesses a real discrete spectrum. Let $\{\chi_{\delta k}\}$ be a complete orthonormal set of eigenfunctions for this problem, and $\{\lambda_{\delta k}\}$ the corresponding eigenvalues.

The inner product and norm in the space $\mathfrak{L}^2(\delta)$ will be denoted by $(\ ,\ )_\delta$ and $\|\ \|_\delta$ respectively, and thus if $u, v \in \mathfrak{D}$,

$$(u, v)_\delta = \int_\delta u\bar{v}, \quad \|u\|_\delta = \left(\int_\delta |u|^2\right)^{\frac{1}{2}}.$$

In case of the interval (a, b), the inner product and norm will be denoted by $(\ ,\ )$ and $\|\ \|$. For $u, v \in \mathfrak{L}^2(\delta)$ one has the Parseval equality

$$(u, v)_\delta = \sum_{k=1}^{\infty} (u, \chi_{\delta k})_\delta \overline{(v, \chi_{\delta k})_\delta}, \tag{2.5}$$

or in the case $u = v$,

$$\|u\|_\delta^2 = \sum_{k=1}^{\infty} |(u, \chi_{\delta k})_\delta|^2. \tag{2.6}$$

Let $s_j = s_j(x, l)$ $(j = 1, \ldots, n)$ be a set of n linearly independent solutions of the equation $Lu = lu$, where l is a complex number, and to be definite, suppose that for some c, $\tilde{a} < c < \tilde{b}$, we have

$$s_j^{(k-1)}(c, l) = \delta_{jk} \qquad (j, k = 1, \ldots, n), \tag{2.7}$$

δ_{jk} being the Kronecker delta. Then the s_j are such that the functions $s_j^{(k-1)}$ $(j, k = 1, \ldots, n)$ are entire in l. In terms of such a linearly independent set we have

$$\chi_{\delta k}(x) = \sum_{j=1}^{n} r_{\delta kj} s_j(x, \lambda_{\delta k}), \tag{2.8}$$

where the $r_{\delta kj}$ are complex constants. Placing (2.8) into (2.5) and (2.6) we can rewrite these in the forms

$$(u, v)_\delta = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \bar{\psi}_{\delta j}(\lambda)\ \phi_{\delta k}(\lambda)\ d\rho_{\delta jk}(\lambda), \tag{2.9}$$

$$\|u\|_\delta^2 = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \bar{\phi}_{\delta j}(\lambda)\ \phi_{\delta k}(\lambda)\ d\rho_{\delta jk}(\lambda). \tag{2.10}$$

Here

$$\phi_{\delta k}(\lambda) = (u, s_k(\lambda))_\delta, \quad \psi_{\delta j}(\lambda) = (v, s_j(\lambda))_\delta,$$

where $s_k(\lambda)$ is s_k considered as a function of x for the fixed real $l = \lambda$. The matrix $\rho_\delta = (\rho_{\delta jk})$, called the *spectral matrix* associated with the self-adjoint problem (2.4), consists of step functions with jumps at the eigenvalues $\lambda_{\delta k}$ and possesses the properties:

(i) ρ_δ is hermitian,

(ii) $\rho_\delta(\Delta) = \rho_\delta(\lambda) - \rho_\delta(\mu)$ is positive semi-definite if $\lambda > \mu$ ($\Delta = (\mu, \lambda]$) (we say ρ_δ is non-decreasing),

(iii) the total variation of $\rho_{\delta jk}$ is finite on every finite λ-interval.

In fact we have

$$\rho_{\delta jk}(\lambda_{\delta p} + 0) - \rho_{\delta jk}(\lambda_{\delta p} - 0) = \sum_{\lambda_{\delta m} = \lambda_{\delta p}} r_{\delta mj}\, \bar{r}_{\delta mk}.$$

It is further assumed that $\rho_\delta(0) = 0$ and $\rho_\delta(\lambda + 0) = \rho_\delta(\lambda)$; this justifies the term "the" spectral matrix.

We turn now to the relation between the spectral matrix ρ_δ and Green's function $G_\delta = G_\delta(x, y, l)$ for the problem (2.4). The latter function is a continuous function defined on the square $\tilde{a} \leqslant x, y \leqslant \tilde{b}$ with the properties:

(i) $\partial^k G_\delta / \partial x^k$ $(k = 0, 1, \ldots, n-2)$ exist, are continuous on the square $\tilde{a} \leqslant x, y \leqslant \tilde{b}$, and $\partial^{n-1} G_\delta / \partial x^{n-1}$, $\partial^n G_\delta / \partial x^n$ are continuous on each of the triangles $\tilde{a} \leqslant x \leqslant y \leqslant \tilde{b}$ and $\tilde{a} \leqslant y \leqslant x \leqslant \tilde{b}$,

(ii) $$\frac{\partial^{n-1} G_\delta}{\partial x^{n-1}}(y + 0, y, l) - \frac{\partial^{n-1} G_\delta}{\partial x^{n-1}}(y - 0, y, l) = \frac{1}{p_0(y)}, \qquad \tilde{a} < y < \tilde{b},$$

(iii) as a function of x, G_δ satisfies the equation $Lu = lu$, if $x \neq y$,

(iv) as a function of x, G_δ satisfies the boundary conditions

$$U_{\delta j}(u) = 0 \quad (j = 1, \ldots, n) \text{ for } \tilde{a} < y < \tilde{b}.$$

Further, since the problem (2.4) is self-adjoint, if $\mathfrak{J} l \neq 0$ ($\mathfrak{J}$ = imaginary part of) no function in $\mathfrak{L}^2(\delta)$, except the zero function, satisfies $Lu = lu$ and $U_{\delta j}(u) = 0$ $(j = 1, \ldots, n)$. Hence the solution of the non-homogeneous problem

$$Lu = lu + f, \quad U_{\delta j}(u) = 0 \qquad (j = 1, \ldots, n),$$

where $f \in \mathfrak{L}^2(\delta)$ $(f \neq 0)$, is given by

$$u(x) = \int_\delta G_\delta(x, y, l)\, f(y)\, dy.$$

Let $\mathfrak{D}_\delta$ be the set of all $u \in \mathfrak{L}^2(\delta)$ for which $u^{(n-1)}$ is absolutely continuous on δ, $Lu \in \mathfrak{L}^2(\delta)$, and $U_{\delta j}(u) = 0$, $(j = 1, \ldots, n)$. Then the operator L_δ defined by $L_\delta u = Lu$ for $u \in \mathfrak{D}_\delta$ is a self-adjoint operator in the Hilbert space $\mathfrak{L}^2(\delta)$, and the integral operator $G_\delta(l)$ defined for all $f \in \mathfrak{L}^2(\delta)$ by

$$G_\delta(l)\, f(x) = \int_\delta G_\delta(x, y, l)\, f(y)\, dy$$

is the resolvent of L_δ, that is, $G_\delta(l) = (L_\delta - l)^{-1}$.

In the following it will be convenient to denote G_δ, considered as a function of x (for fixed y and l) as $G_\delta(\ \ , y, l)$, and similarly when considered as a function of y alone (for fixed x and l) we denote it by $G_\delta(x, \ \ , l)$. Also we define H_δ by

$$H_\delta(x, y, l) = G_\delta(x, y, l) - G_\delta(x, y, \bar{l}).$$

We remark that the $(n-1)$ *st* derivative of H_δ with respect to x is continuous at $x = y$.

LEMMA 1. *If $\Im l \neq 0$ then*

$$2i\Im l \int_\delta \frac{\partial^j G_\delta}{\partial x^j}(x, t, l) \frac{\partial^k G_\delta}{\partial y^k}(t, y, \bar{l})\, dt = \frac{\partial^{j+k} H_\delta}{\partial x^j \partial y^k}(x, y, l), \tag{2.11}$$

for $j, k = 0, 1, \ldots, n-1$.

Proof. If $\Im l \neq 0$, $\Im m \neq 0$, then, as is known,

$$(m - l) \int_\delta G_\delta(t, y, m)\, \bar{G}_\delta(t, x, \bar{l})\, dt = G_\delta(x, y, m) - \bar{G}_\delta(y, x, \bar{l}). \tag{2.12}$$

This is the equation satisfied by the kernels of the resolvent operators $G_\delta(l)$, $G_\delta(m)$. A proof results by applying Green's formula (2.1) to the functions $u = G_\delta(\ \ , y, m)$, $v = G_\delta(\ \ , x, \bar{l})$ on the intervals $[\tilde{a}, y - 0]$, $[y + 0, x - 0]$, $[x + 0, \tilde{b}]$ and adding. Choosing $m = l$ in (2.12) we have

$$\bar{G}_\delta(y, x, \bar{l}) = G_\delta(x, y, l),$$

and as a consequence we have from (2.12) with $m = \bar{l}$,

$$2i\Im l \int_\delta G_\delta(x, t, l)\, G_\delta(t, y, \bar{l})\, dt = G_\delta(x, y, l) - G_\delta(x, y, \bar{l}), \tag{2.13}$$

which is (2.11) for $j = k = 0$. Using the differentiability properties of G_δ it is now obvious that (2.11) follows from (2.13), thus proving the lemma.

LEMMA 2. *The spectral matrix $\rho_\delta = (\rho_{\delta jk})$ satisfies the identity*

$$2i\Im l \int_{-\infty}^{\infty} \frac{d\rho_{\delta jk}(\lambda)}{|\lambda - l|^2} = \frac{\partial^{j+k-2} H_\delta}{\partial x^{j-1} \partial y^{k-1}}(c, c, l), \tag{2.14}$$

where $\Im l \neq 0$, and $j, k = 1, \ldots, n$.

Proof. Since

$$L\chi_{\delta m} = \lambda_{\delta m} \chi_{\delta m} = l\chi_{\delta m} + (\lambda_{\delta m} - l)\, \chi_{\delta m},$$

we have

$$\chi_{\delta m}(x) = (\lambda_{\delta m} - l) \int_\delta G_\delta(x, t, l)\, \chi_{\delta m}(t)\, dt,$$

or

$$\bar{\chi}_{\delta m}(y) = (\lambda_{\delta m} - \bar{l}) \int_\delta G_\delta(t, y, \bar{l})\, \bar{\chi}_{\delta m}(t)\, dt.$$

It follows from this that

$$\bar{\chi}^{(k)}_{\delta m}(y) = (\lambda_{\delta m} - \bar{l}) \int_\delta \frac{\partial^k G_\delta}{\partial y^k}(t, y, \bar{l})\, \bar{\chi}_{\delta m}(t)\, dt,$$

for $k = 0, 1, \ldots, n - 1$. Therefore the mth Fourier coefficient of $\partial^k G_\delta(\ \ , y, \bar{l})/\partial y^k$ with respect to the set $\{\chi_{\delta m}\}$ is

$$\bar{\chi}^{(k)}_{\delta m}(y)/(\lambda_{\delta m} - \bar{l}).$$

From the Parseval relation (2.5) applied to the functions $u = \partial^k G_\delta(\ \ , y, \bar{l})/\partial y^k$ and $v = \partial^j G_\delta(\ \ , x, \bar{l})/\partial y^j$ we obtain

$$\int_\delta \frac{\partial^k G_\delta}{\partial y^k}(t, y, \bar{l}) \frac{\partial^j \bar{G}_\delta}{\partial y^j}(t, x, \bar{l})\, dt = \int_\delta \frac{\partial^j G_\delta}{\partial x^j}(x, t, l) \frac{\partial^k G_\delta}{\partial y^k}(t, y, \bar{l})\, dt$$

$$= \sum_{m=1}^{\infty} \frac{\chi_{\delta m}^{(j)}(x)\, \bar{\chi}_{\delta m}^{(k)}(y)}{|\lambda_{\delta m} - l|^2},$$

and in view of Lemma 1 there results

$$2i\Im l \sum_{m=1}^{\infty} \frac{\chi_{\delta m}^{(j)}(x)\, \bar{\chi}_{\delta m}^{(k)}(y)}{|\lambda_{\delta m} - l|^2} = \frac{\partial^{j+k} H_\delta}{\partial x^j \partial y^k}(x, y, l). \tag{2.15}$$

Using (2.8) we may write, for $\Delta = (\mu, \lambda]$,

$$\sum_{\lambda_{\delta m} \in \Delta} \chi_{\delta m}^{(j)}(x)\, \bar{\chi}_{\delta m}^{(k)}(y) = \int_\Delta \sum_{p,q=1}^{n} s_p^{(j)}(x, \lambda)\, \bar{s}_q^{(k)}(y, \lambda)\, d\rho_{\delta pq}(\lambda),$$

and therefore (2.15) yields

$$2i\Im l \int_{-\infty}^{\infty} \sum_{p,q=1}^{n} s_p^{(j)}(x, \lambda)\, \bar{s}_q^{(k)}(y, \lambda)\, |\lambda - l|^{-2}\, d\rho_{\delta pq}(\lambda) = \frac{\partial^{j+k} H_\delta}{\partial x^j\, \partial y^k}(x, y, l).$$

Setting $x = y = c$ in this formula and recalling (2.7) we obtain (2.14), as desired. We are indebted to the referee for pointing out that $H_\delta/2i\Im l$ is the reproducing kernel of the class of all functions satisfying the boundary conditions $U_{\delta j}(u) = 0$ with a norm given by

$$\|u\|^2 = \int_\delta |Lu - lu|^2\, dx.$$

Formula (2.15) follows directly from this. The equation $u = G_\delta(l)(L - l)u$ expresses the reproducing property of $H_\delta/2i\Im l$ in this class.

3. Green's function for singular cases. Here we establish the existence and uniqueness of Green's functions for two self-adjoint boundary value problems on (a, b). Indeed we show that these functions are the limit (uniform in any finite (x, y, l)-region, $\Im l \neq 0$) of Green's functions constructed for self-adjoint problems (2.4) on closed bounded sub-intervals δ of (a, b).

Let $\delta_0 = [a_0, b_0]$ be any closed bounded interval interior to (a, b). Then, as is known, there exists a *fundamental solution* $K_0 = K_0(x, y)$ of the equation $Lu = 0$ on δ_0. This function exists on the square $a_0 \leqslant x, y \leqslant b_0$, and enjoys all the properties of a Green's function for $Lu = 0$ on δ_0 except that it need not satisfy a set of self-adjoint boundary conditions. (Green's function for $Lu = 0$ on δ_0 need not exist, but a fundamental solution always exists.) In order to exhibit the special properties of a fundamental solution we give a construction for one such. Let s_{0j} $(j = 1, \ldots, n)$ denote the set of n linearly independent solutions of $Lu = 0$ given by

$$s_{0j}(x) = s_j(x, 0),$$

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satisfying (2.7) with $l = 0$. Using Green's formula one finds that $[s_{0j}\, s_{0k}](x)$ is a constant $[s_{0j}\, s_{0k}]$, independent of x. Let S denote the matrix with element $[s_{0j}\, s_{0k}]$ in the jth row and kth column. From the non-degeneracy of the form $[uv](x)$ and the linear independence of the s_{0j} it follows that the matrix S is non-singular. Moreover we have S is skew-hermitian, since

$$[s_{0j}\, s_{0k}] = -\overline{[s_{0k}\, s_{0j}]}.$$

Let $S^{-1} = (S^{-1}_{jk})$ denote the inverse matrix to S; it is also skew-hermitian. Define K_0 by

$$K_0(x, y) = \bar{K}_0(y, x) = (\tfrac{1}{2}) \sum_{j,k=1}^{n} S^{-1}_{jk}\, s_{0k}(x)\, \bar{s}_{0j}(y) \qquad (x \geqslant y). \tag{3.1}$$

It is an easy task to check that this function is a fundamental solution for $Lu = 0$ on any $\delta_0 \subset (a, b)$, and we shall always mean by K_0 this function which is defined on the whole square $a < x, y < b$.

Suppose $\delta_1 = [a_1, b_1]$ is any other closed bounded sub-interval of (a, b) containing δ_0 properly. Let μ be any real-valued function of class C^∞ on $a < x < b$ such that $\mu(x) = 1$ for $x \in \delta_0$ and $\mu(x) = 0$ when x is outside

$$\tfrac{1}{2}(a_0 + a_1) \leqslant x \leqslant \tfrac{1}{2}(b_0 + b_1).$$

Then define J_1 by

$$J_1(x, y) = \mu(x)\, \mu(y)\, K_0(x, y).$$

Clearly $J_1(x, y) = \bar{J}_1(y, x)$. The basis for our results on Green's function is the following representation of G_δ in terms of J_1.

LEMMA 3. *If $\delta \supset \delta_1 \supset \delta_0$ then for $x \in \delta_0$, $y \in \delta$, and $\mathfrak{I}l \neq 0$,*

$$G_\delta(x, y, l) = J_1(x, y) + \int_{\delta_1} G_\delta(t, y, l)\{\overline{lJ_1(t, x)} - \overline{L_t J_1(t, x)}\}\, dt, \tag{3.2}$$

where L_t denotes L applied to $J_1(\ \ , x)$ for fixed x.

Proof. The function $u = G_\delta(\ \ , y, l) - J_1(\ \ , y)$ for $y \in \delta_0$ is in the set $\mathfrak{D}_\delta$, and hence $u = G_\delta(l)(L_t - l)u$. Using $(L_t - l)\, G_\delta\, (t, y, l) = 0$ for $t \neq y$, the symmetry $G_\delta(x, y, l) = \bar{G}_\delta(y, x, \bar{l})$, and $J_1(\ \ , y) = 0$ for y outside δ_1 we see (3.2) results.

Since G_δ is unique it may seem a paradox that it can be expressed in terms of the function J_1 which was defined by an almost arbitrary function μ. This "dilemma" disappears when one observes that (3.2) may be written as

$$\begin{aligned} G_\delta(x, y, l) &= K_0(x, y) + \int_{\delta_0} K_0(x, t)\, l\, G_\delta(t, y, l)\, dt \\ &\quad + [G_\delta(\ \ , y, l)\, K_0(\ \ , x)](a_0) - [G_\delta(\ \ , y, l)\, K_0(\ \ , x)](b_0), \end{aligned} \tag{3.3}$$

when $x, y \in \delta_0$, and here the values of J_1, for x, y not on the square $a_0 \leqslant x, y \leqslant b_0$ do not enter at all. Now (3.3) results by splitting the integration range in (3.2) into the two ranges δ_0 and $\delta_1 - \delta_0$ and using Green's formula to evaluate the integral over $\delta_1 - \delta_0$. On δ_0, of course $L_t J_1(t, x) = 0$, $x \neq t$.

LEMMA 4. *The set of functions* $\{G_\delta\}$ *is uniformly bounded and equicontinuous on every compact* (x, y, l)*-region where* $\Im l \neq 0$.

Proof.[1] From Lemma 3 we have, using the Schwarz inequality, if $\delta \supset \delta_1 \supset \delta_0$ and $x, y \in \delta_0$,

$$(3.4)\quad |G_\delta(x, y, l)| \leqslant M_0 + \|G_\delta(\ , y, l)\|_{\delta_1} \| \bar{l}\, J_1(\ , x) - L_t\, J_1(\ , x)\|_{\delta_1},$$

where

$$M_0 = \max |K_0(x, y)| \qquad (x, y \in \delta_0).$$

The uniform boundedness of the G_δ for $\delta \supset \delta_1$ on the square $a_0 \leqslant x, y \leqslant b_0$, and l ranging over some compact set Λ with $\Im l \neq 0$ will follow from (3.4) once we have shown that $\|G_\delta(\ , y, l)\|_{\delta_1}$ is bounded uniformly for $y \in \delta_0$, $l \in \Lambda$. However this follows from the fact that the resolvent $G_\delta(l)$ is a bounded operator with bound not exceeding $|\Im l|^{-1}$, that is, if $u = G_\delta(l)f$, where $f \in \mathfrak{L}^2(\delta)$, then

$$(3.5)\qquad \|u\|_\delta \leqslant |\Im l|^{-1} \|f\|_\delta.$$

Applying this to $u = G_\delta(\ , y, l) - J_1(\ , y)$ for $y \in \delta_0$ we see that

$$(3.6)\quad \|G_\delta(\ , y, l)\|_\delta \leqslant |\Im l|^{-1} \|L_t\, J_1(\ , y) - lJ_1(\ , y)\|_{\delta_1} + \|J_1(\ , y)\|_{\delta_1},$$

and this inequality implies the uniform boundedness of $\|G_\delta(\ , y, l)\|_{\delta_1}$ for $y \in \delta_0$, $l \in \Lambda$.

From (3.2) it follows that[2]

$$\frac{\partial G_\delta}{\partial x}(x, y, l) = \frac{\partial K_0}{\partial x}(x, y) + \int_{\delta_1} G_\delta(t, y, l) \frac{\partial}{\partial x}\{l\, \bar{J}_1\,(t, x) - \overline{L_t\, J_1}\,(t, x)\}\, dt,$$

and using the same argument as above we see $\partial G_\delta / \partial x$ is uniformly bounded for $a_0 \leqslant x, y \leqslant b_0$, $l \in \Lambda$. The symmetry $G_\delta(x, y, l) = \bar{G}_\delta(y, x, \bar{l})$ implies the same for $\partial G_\delta / \partial y$. From (2.12) we have

$$\frac{\partial G_\delta}{\partial l}(x, y, l) = \int_\delta G_\delta(t, y, l)\, \bar{G}_\delta(t, x, \bar{l})\, dt,$$

and using the Schwarz inequality and (3.6) we find that $\partial G_\delta / \partial l$ is also uniformly bounded for $a_0 \leqslant x, y \leqslant b_0$, $l \in \Lambda$. The uniform boundedness of all first partial derivatives of G_δ implies the equicontinuity of the set $\{G_\delta\}$ thus completing the proof of the lemma.

Certain important conclusions may now be drawn from Lemmas 3 and 4. Firstly, an application of the Ascoli lemma together with Lemma 4 proves that there exists a sequence of intervals $\delta_m \subset (a, b)$ $(m = 1, 2, \ldots)$, $\delta_m \to (a, b)$, such that the corresponding Green's functions $G_m = G_{\delta_m}$ tend uniformly, on any compact subset of $a < x, y < b$, $\Im l > 0$ (or $\Im l < 0$), to a limit function G. This G is defined for $a < x, y < b$, $\Im l \neq 0$, and, being the uniform limit of continuous functions, is continuous. Since the G_m are analytic in l for $\Im l > 0$ (or $\Im l < 0$),

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[1]Strictly speaking our proof is given for the cases $n \geqq 2$, but a slight modification shows the validity for $n = 1$; cf. (3.7) and (3.8) below. If $n = 1$, equicontinuity holds for $x \neq y$.

[2]This relation must be modified in case $n = 1$, cf. footnote 1.

the same holds for G. The relation $G_m(x, y, l) = \bar{G}_m(y, x, \bar{l})$ implies that $G(x, y, l) = \bar{G}(y, x, \bar{l})$.

From (3.2) we have, if $a_0 \leqslant x, y \leqslant b_0$, $\Im l \neq 0$, $\delta_0 \subset \delta_1 \subset \delta$,

$$\frac{\partial^j G_\delta}{\partial x^j}(x, y, l) = \frac{\partial^j K_0}{\partial x^j}(x, y) + \int_{\delta_1} G_\delta(t, y, l)\frac{\partial^j}{\partial x^j}\{l\bar{J}_1(t, x) - \overline{L_t J_1(t, x)}\}dt, \tag{3.7}$$

$$(j = 0, 1, \ldots, n - 1).$$

Recall that since $\partial^{n-1}G_\delta/\partial x^{n-1}$ and $\partial^{n-1}K_0/\partial x^{n-1}$ have the same discontinuity at $x = y$, their difference is continuous there. Moreover, from (3.7), if $x \neq y$,

$$\begin{aligned}\frac{\partial^n G_\delta}{\partial x^n}(x, y, l) = \frac{\partial^n K_0}{\partial x^n}(x, y) + \frac{l\,G_\delta(x, y, l)}{p_0(x)} \\ + \int_{\delta_1} G_\delta(t, y, l)\frac{\partial^n}{\partial x^n}\{l\,\bar{J}_1(t, x) - \overline{L_t\,J_1(t, x)}\}\,dt.\end{aligned} \tag{3.8}$$

Observing (3.2) with $\delta = \delta_m$, and letting $m \to \infty$, we obtain

$$G(x, y, l) = K_0(x, y) + \int_{\delta_1} G(t, y, l)\{l\bar{J}_1(t, x) - \overline{L_t\,J_1(t, x)}\}\,dt,$$

and therefore the partial derivatives $\partial^j G/\partial x^j$ exist and

$$\frac{\partial^j G}{\partial x^j}(x, y, l) = \frac{\partial^j K_0}{\partial x^j}(x, y) + \int_{\delta_1} G(t, y, l)\frac{\partial^j}{\partial x^j}\{l\bar{J}_1(t, x) - \overline{L_t J_1(t, x)}\}\,dt,$$

$$(j = 0, 1, \ldots, n - 1;\ a_0 \leqslant x, y \leqslant b_0;\ \Im l \neq 0),$$

$$\begin{aligned}\frac{\partial^n G}{\partial x^n}(x, y, l) = \frac{\partial^n K_0}{\partial x^n}(x, y) + \frac{l\,G(x, y, l)}{p_0(x)} \\ + \int_{\delta_1} G(t, y, l)\frac{\partial^n}{\partial x^n}\{l\bar{J}_1(t, x) - \overline{L_t J_1(t, x)}\}\,dt \qquad (x \neq y).\end{aligned} \tag{3.9}$$

From (3.9) we infer that as a function of x, G satisfies $Lu = lu$, provided $x \neq y$. (This may perhaps be seen better by observing the relation (3.3) with G_δ replaced by G.) Further, $\partial^{n-1}G/\partial x^{n-1}$ has the same jump at $x = y$ as $\partial^{n-1}K_0/\partial x^{n-1}$, namely,

$$\frac{\partial^{n-1}G}{\partial x^{n-1}}(y + 0, y, l) - \frac{\partial^{n-1}G}{\partial x^{n-1}}(y - 0, y, l) = \frac{1}{p_0(y)}.$$

Since the right sides of (3.7) and (3.8) with $\delta = \delta_m$ tend (as $m \to \infty$) to the right sides of the corresponding formulas in (3.9) we see that

$$\frac{\partial^j G_m}{\partial x^j} \to \frac{\partial^j G}{\partial x^j} \qquad (j = 0, 1, \ldots, n), \tag{3.10}$$

uniformly on any compact (x, y, l) region where $\Im l \neq 0$, and provided $x \neq y$ when $j = n - 1, n$. The relations $G_\delta(x, y, l) = \bar{G}_\delta(y, x, \bar{l})$ and $G(x, y, l) = \bar{G}(y, x, \bar{l})$ imply that

$$\frac{\partial^j G_m}{\partial y^j} \to \frac{\partial^j G}{\partial y^j} \qquad (j = 0, 1, \ldots, n),$$

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under the same conditions that (3.10) is valid. Returning to (3.7), (3.8), and (3.9) it is easy to see that the mixed derivatives $\partial^{j+k}G/\partial x^j\,\partial y^k$ $(j, k = 0, 1, \ldots, n-1)$ all exist, and

$$\frac{\partial^{j+k}G_m}{\partial x^j\,\partial y^k} \to \frac{\partial^{j+k}G}{\partial x^j\,\partial y^k} \qquad (j, k = 0, 1, \ldots, n-1),$$

uniformly on any compact (x, y, l) region where $\Im l \neq 0$ and $x \neq y$ if j or k is $n-1$.

It follows from (3.6) that there exists a constant c_1 (depending on δ_0 and δ_1 only) such that

$$\|G_\delta(\ \ , y, l)\|_\delta \leqslant c_1|\Im l|^{-1}\,(2|l| + 1) \qquad (y \in \delta_0).$$

But $\|G_\delta(\ \ , y, l)\|_{\bar\delta} \leqslant \|G_\delta(\ \ , y, l)\|_\delta$ for $\bar\delta \subset \delta$, and letting first $\delta \to (a, b)$ through the sequence δ_m and then $\bar\delta \to (a, b)$ we infer, for any fixed (y, l), $\Im l \neq 0$, that $G(\ \ , y, l) \in \mathfrak{L}^2(a, b)$. This implies that for fixed (x, l), $\Im l \neq 0$, $G(x, \ \ , l) \in \mathfrak{L}^2(a, b)$. If $f \in \mathfrak{L}^2(a, b)$ then the integral

$$\int_a^b G(x, t, l)\, f(t)\, dt \qquad (a < x < b, \Im l \neq 0),$$

converges absolutely (and uniformly for x in any finite sub-interval of (a, b)), defines a function v, and using the properties developed above for G it is not difficult to see that v has continuous derivatives up to order $n-1$, $v^{(n-1)}$ is absolutely continuous on every closed sub-interval of (a, b), and

$$Lv = lv + f.$$

For example, to prove the existence and continuity of v', one first shows by means of (3.5) applied to $u = (\partial/\partial x)(G_\delta(x, \ \ , l) - J_1(x, \ \ ))$ that $\|\partial G(x, \ \ , l)/\partial x\|$ is bounded for fixed l, $\Im l \neq 0$, uniformly for x on any finite sub-interval of (a, b). Thus the integral

$$\int_a^b \frac{\partial G}{\partial x}(x, t, l)\, f(t)\, dt$$

converges uniformly for x on any finite sub-interval of (a, b), and hence represents a continuous function on (a, b) which is easily verified to be v'.

From (3.5) for $\delta = \delta_m$ we obtain, letting $m \to \infty$,

$$\|v\| \leqslant |\Im l|^{-1}\, \|f\|,$$

and since $Lv = lv + f$, we have also

$$\|Lv\| \leqslant (1 + |l|/|\Im l|)\|f\|.$$

In short v belongs to the class of functions $\mathfrak{D}$ defined at the beginning of §2, and $Lv = lv + f$. We summarize our information concerning G in the following statement.

THEOREM 1. *Let G be the limit of any convergent sequence $\{G_m\}$ of the set $\{G_\delta\}$ of Green's functions associated with given self-adjoint boundary value problems*

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$$Lu = lu, \quad U_{\delta j}(u) = 0 \qquad (j = 1, \ldots, n),$$

on closed bounded sub-intervals δ of (a, b). Then G is continuous for $a < x, y < b$, analytic in l for $\Im l > 0$ (and $\Im l < 0$), and possesses the properties:

(i) *$\partial^k G/\partial x^k$ $(k = 0, 1, \ldots, n-2)$ exist, are continuous on $a < x, y < b$, and $\partial^{n-1}G/\partial x^{n-1}$, $\partial^n G/\partial x^n$ are continuous on each of the regions $x \leqslant y$ and $y \leqslant x$,*

(ii) $$\frac{\partial^{n-1}G}{\partial x^{n-1}}(y+0, y, l) - \frac{\partial^{n-1}G}{\partial x^{n-1}}(y-0, y, l) = \frac{1}{p_0(y)}, \qquad a < y < b,$$

(iii) *as a function of x, G satisfies $Lu = lu$ if $x \neq y$,*

(iv) $$\frac{\partial^{j+k}G_m}{\partial x^j \partial y^k} \to \frac{\partial^{j+k}G}{\partial x^j \partial y^k} \qquad (j, k = 0, 1, \ldots, n-1),$$

uniformly on any compact (x, y, l) region where $\Im l \neq 0$, and $x \neq y$ if j or k is $n-1$,

(v) $$G(x, y, l) = \bar{G}(y, x, \bar{l}),$$

(vi) $$G(x, \quad, l) \in \mathfrak{L}^2(a, b) \qquad (a < x < b).$$

If $f \in \mathfrak{L}^2(a, b)$ the function v defined by

$$v(x) = \int_a^b G(x, t, l) f(t)\, dt \qquad (\Im l \neq 0),$$

is an element of $\mathfrak{D}$ and

$$Lv = lv + f.$$

We do not call G a Green's function yet for we have not considered the question of boundary conditions on (a, b). Here we shall consider two cases, which reduce to the most important ones when $n = 2$, and show how G may then be considered as a Green's function.

For Case I we assume that there are no functions $u \in \mathfrak{D}$ other than $u = 0$ satisfying $Lu = iu$ or $Lu = -iu$. Then the problem

$$Lu = lu \qquad (u \in \mathfrak{D}), \tag{3.11}$$

is a self-adjoint boundary value problem on (a, b)—with no boundary conditions required other than $u \in \mathfrak{D}$. Another way of putting this is the following: the operator T defined in the Hilbert space $\mathfrak{L}^2(a, b)$ with domain $\mathfrak{D}$ by

$$Tu = Lu \qquad (u \in \mathfrak{D}),$$

is self-adjoint. For the cases treated by Weyl, Case I corresponds to the situation where L is of the limit point type at a and b. It is known that if $\Im l > 0$ ($\Im l < 0$) the number of solutions $u \in \mathfrak{D}$ of $Lu = lu$ is equal to the number of solutions of $Lu = iu$ ($Lu = -iu$), and therefore in Case I the equation $Lu = lu$ has no solutions $u \in \mathfrak{D}$ other than the zero function for any l for which $\Im l \neq 0$.

THEOREM 2. *In Case* I, *G is unique, and indeed*

$$G_\delta \to G \qquad (\delta \to (a, b)), \tag{3.12}$$

uniformly on any compact (x, y, l) *region where* $\Im l \neq 0$, *independent of the choice of the self-adjoint boundary conditions on* δ. (G *is called Green's function for* (3.11) *in Case* I.)

Proof. Let G° be any other function having the same properties as the G of Theorem 1. Then from Theorem 1 the function v° given by

$$v*(x) = \int_a^b G^{\circ}(x, t, l) f(t)\, dt \qquad (\Im l \neq 0),$$

exists for any $f \in \mathfrak{L}^2(a, b)$, and

$$Lv^{\circ} = lv^{\circ} + f.$$

Thus

$$L(v^{\circ} - v) = l(v^{\circ} - v),$$

and this implies, in Case I, that $v^{\circ} = v$. One readily infers from this, and the symmetry of G and G°, that $G = G^{\circ}$. This proves the uniqueness of G, and that every convergent sequence of $\{G_{\delta}\}$ tends to G, thus giving (3.12).

Before going to Case II we interpret our results in terms of the operator T. Let $G(l)$ denote the operator with domain $\mathfrak{L}^2(a, b)$ defined by

$$G(l) f(x) = \int_a^b G(x, t, l) f(t)\, dt.$$

It is an integral operator of Carleman type, and

$$G(l) = (T - l)^{-1}. \tag{3.13}$$

Indeed Theorem 1 shows that $G(l)f \in \mathfrak{D}$ for every $f \in \mathfrak{L}^2(a,b)$, and $(L-l)G(l)f=f$. Conversely, let $u \in \mathfrak{D}$, and put $(L - l)u = f$. Then clearly $w = u - G(l)f$ is in $\mathfrak{D}$, and $(L - l)w = (L - l)u - (L - l)G(l)f = f - f = 0$. Thus $w = 0$, or $u = G(l)f$, and thereby $G(l)(L - l)u = u$, proving (3.13).

For *Case* II we consider a half-closed interval $[a, b)$, where a is finite. We assume that the coefficients p_k in L are of class C^{n-k} on $[a, b)$, and $p_0(x) \neq 0$ on $[a, b)$. Let $\mathfrak{D}$ now represent those $u \in \mathfrak{L}^2([a, b))$ with continuous derivatives up to order $n - 1$ on $[a, b)$, $u^{(n-1)}$ absolutely continuous on every closed sub-interval of $[a, b)$ of the form $[a, \tilde{b}]$, $a < \tilde{b} < b$, and $Lu \in \mathfrak{L}^2([a, b))$. Suppose

$$U_j(u) = \sum_{k=1}^{n} M_{jk}\, u^{(k-1)}(a) = 0 \qquad (j = 1, \ldots, \omega), \tag{3.14}$$

is a set of ω linearly independent boundary conditions for functions $u \in \mathfrak{D}$. In Case II we assume that the problem for $u \in \mathfrak{D}$,

$$Lu = lu, \qquad U_j(u) = 0 \qquad (j = 1, \ldots, \omega), \tag{3.15}$$

is a self-adjoint boundary value problem on $[a, b)$. In other words

$$(Lu, v) = (u, Lv)$$

holds for every $u, v \in \mathfrak{D}$ which satisfy the boundary conditions (3.14), and there are no functions $u \in \mathfrak{D}$ (other than $u = 0$) satisfying (3.14) and $Lu = iu$ or

$Lu = -iu$. The latter statement is true if and only if there are no $u \in \mathfrak{D}$ satisfying (3.14) (other than the zero function) and $Lu = lu$ for any l for which $\Im l \neq 0$. In a previous work **(1)** we have shown that Case II can only arise when $n = 2\omega$. In the situation treated by Weyl ($n = 2$) Case II occurs when L is of the limit circle type at a and limit point type at b.

Let $\mathfrak{D}^\circ$ denote the set of all $u \in \mathfrak{D}$ satisfying the boundary conditions (3.14). Then to say that the problem (3.15) is self-adjoint is equivalent to saying that the operator T° defined in $\mathfrak{L}^2([a, b))$ by

$$T^\circ u = Lu \qquad (u \in \mathfrak{D}^\circ),$$

is a self-adjoint operator.

In Case II we consider closed bounded sub-intervals δ of $[a, b)$ of the form $\delta = [a, \tilde{b}]$, $a < \tilde{b} < b$, and add to the set (3.14) any $n - \omega = \omega$ linearly independent boundary conditions $U_{\delta k}(u) = 0$ such that the combined set

$$U_j(u) = 0, \quad U_{\delta k}(u) = 0 \qquad (j, k = 1, \ldots, \omega), \tag{3.16}$$

is a self-adjoint set on δ. Let G_δ represent Green's function for $\Im l \neq 0$ of the self-adjoint problem

$$Lu = lu, \quad U_j(u) = 0, \quad U_{\delta k}(u) = 0 \qquad (j, k = 1, \ldots, \omega).$$

Then it is clear that Lemma 4 and Theorem 1 remain valid for the set $\{G_\delta\}$. If G is the limit of any convergent sequence $\{G_m\}$ of $\{G_\delta\}$, then since every G_δ (considered as a function of x) satisfies the conditions $U_j(u) = 0$ ($j = 1, \ldots, \omega$) for fixed y, $a < y < \tilde{b}$, we have that G, as a function of x, satisfies $U_j(u) = 0$ ($j = 1, \ldots, \omega$) each fixed y, $a < y < b$. We infer from this that if $f \in \mathfrak{L}^2([a, b))$ the function

$$v(x) = \int_a^b G(x, t, l) f(t)\, dt \qquad (\Im l \neq 0),$$

satisfies the boundary conditions $U_j(u) = 0$ ($j = 1, \ldots, \omega$). From these considerations it is easy to show, just as in Theorem 2, that every convergent sequence of $\{G_\delta\}$ must tend to the same limit, and hence $G_\delta \to G$ as $\delta = [a, \tilde{b}] \to [a, b)$, i.e., as $\tilde{b} \to b$. Moreover G is unique; it is the only G with the properties listed in Theorem 1 which satisfies the conditions $U_j(u) = 0$ ($j = 1, \ldots, \omega$).

THEOREM 3. *In Case* II *G is unique, and $G_\delta \to G$, as $\tilde{b} \to b$, uniformly on any compact (x, y, l) region where $\Im l \neq 0$, independent of the choice of the boundary conditions $U_{\delta k}(u) = 0$ $(k = 1, \ldots, \omega)$ on δ. (In Case* II *G is called Green's function for the problem* (3.15)).

In terms of the operator T° defined above, we have shown that the operator $G(l)$ defined by

$$G(l) f(x) = \int_a^b G(x, t, l) f(t)\, dt \qquad (f \in \mathfrak{L}^2([a, b))),$$

is an integral operator of Carleman type, and

$$G(l) = (T^\circ - l)^{-1}.$$

4. The spectral matrix for singular cases. We are now in a position to exploit the relation (2.14) in Lemma 2, which connects the spectral matrix ρ_δ with Green's function G_δ. For brevity we put

$$P_{\delta jk}(l) = \frac{\partial^{j+k-2} H_\delta}{\partial x^{j-1} \partial y^{k-1}} (c, c, l) \qquad (j, k = 1, \ldots, n),$$

where we recall

$$H_\delta(x, y, l) = G_\delta(x, y, l) - G_\delta(x, y, \bar{l}).$$

Also we write

$$P_{jk}(l) = \frac{\partial^{j+k-2} H}{\partial x^{j-1} \partial y^{k-1}} (c, c, l) \qquad (j, k = 1, \ldots, n),$$

where

$$H(x, y, l) = G(x, y, l) - G(x, y, \bar{l}),$$

and G is the limit of any convergent sequence $\{G_m\}$ of the set $\{G_\delta\}$.

THEOREM 4. *Let $\{G_m\}$ be any convergent sequence of the set $\{G_\delta\}$, and let the corresponding spectral matrices be $\rho_m = (\rho_{mjk})$. Then there exists an hermitian, non-decreasing matrix $\rho = (\rho_{jk})$ whose elements are of bounded variation on every finite λ-interval, such that if $\Delta = (\mu, \lambda]$ is a finite interval whose end-points are continuity points for ρ_{jk}, then*

$$\rho_{mjk}(\Delta) \to \rho_{jk}(\Delta) \qquad (m \to \infty).$$

Further, if $G_m \to G$, then in terms of P_{jk} constructed above,

$$\rho_{jk}(\Delta) = \frac{1}{2\pi i} \lim_{\epsilon \to +0} \int_\mu^\lambda P_{jk}(\sigma + i\epsilon)\, d\sigma. \tag{4.1}$$

Proof. Using the results developed in §3 the proof reduces to the argument given by Levinson **(4)** in the case $n = 2$, p_k real. From (2.14) we have

$$2i\Im l \int_{-\infty}^{\infty} \frac{d\rho_{mjk}(\lambda)}{|\lambda - l|^2} = P_{mjk}(l) \qquad (\Im l \neq 0), \tag{4.2}$$

and for $l = i$,

$$\int_{-\mu}^{\mu} \frac{d\rho_{mjj}(\lambda)}{1 + \lambda^2} \leqslant \int_{-\infty}^{\infty} \frac{d\rho_{mjj}(\lambda)}{1 + \lambda^2} = \frac{P_{mjj}(i)}{2i}, \tag{4.3}$$

since ρ_{mjj} is non-decreasing. The right side of (4.3) is bounded uniformly in m since by Theorem 1 (iv) $P_{mjj} \to P_{jj}$ uniformly on any compact subset Λ of $\Im l > 0$ (or $\Im l < 0$). Therefore, from (4.3), there exists a constant C not depending on m such that

$$\int_{-\mu}^{\mu} d\rho_{mjj}(\lambda) < C(1 + \mu^2) \qquad |\rho_{mjj}(\lambda)| < C(1 + \lambda^2).$$

But $|\rho_{mjk}(\Delta)|^2 \leqslant \rho_{mjj}(\Delta)\, \rho_{mkk}(\Delta)$, and hence the total variation of the ρ_{mjk} on any finite λ-interval is bounded independent of m. By the Helly selection theorem there exists a subsequence of the matrices $\rho_m = (\rho_{mjk})$ tending to a limit matrix

ρ, which is hermitian, non-decreasing, and of bounded variation on every finite λ-interval.

From (4.2) we readily infer that

$$\int_{-\infty}^{\infty} \frac{d\rho_{jk}(\lambda)}{|\lambda - l|^2}$$

converges absolutely. If $\Im l \neq 0$, $\Im l^\circ \neq 0$, then the integral

$$\int_{-\infty}^{\infty} \left(\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l^\circ|^2} \right) d\rho_{m\,jk}(\lambda) \tag{4.4}$$

tends, as $m \to \infty$ through the subsequence, to

$$\int_{-\infty}^{\infty} \left(\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l^\circ|^2} \right) d\rho_{jk}(\lambda).$$

But (4.4) is just

$$\frac{P_{m\,jk}(l)}{2i\Im l} - \frac{P_{m\,jk}(l^\circ)}{2i\Im l^\circ},$$

which, by Theorem 1 (iv), tends to

$$\frac{P_{jk}(l)}{2i\Im l} - \frac{P_{jk}(l^\circ)}{2i\Im l^\circ}.$$

Therefore we see that

$$\frac{P_{jk}(l)}{2i\Im l} - \int_{-\infty}^{\infty} \frac{d\rho_{jk}(\lambda)}{|\lambda - l|^2}$$

is a constant independent of l for $\Im l \neq 0$. The relation (4.1) now follows readily from this. Indeed

$$\begin{aligned}
\lim_{\epsilon \to +0} \int_{\mu}^{\lambda} P_{jk}(\sigma + i\epsilon)\, d\sigma &= 2i \lim_{\epsilon \to +0} \int_{\mu}^{\lambda} \left(\int_{-\infty}^{\infty} \frac{\epsilon\, d\rho_{jk}(\nu)}{(\nu - \sigma)^2 + \epsilon^2} \right) d\sigma \\
&= 2i \int_{-\infty}^{\infty} \lim_{\epsilon \to +0} \left[\tan^{-1}\left(\frac{\lambda - \nu}{\epsilon}\right) - \tan^{-1}\left(\frac{\mu - \nu}{\epsilon}\right) \right] d\rho_{jk}(\nu) \\
&= 2\pi i\, (\rho_{jk}(\lambda) - \rho_{jk}(\mu)),
\end{aligned}$$

provided λ, μ are points of continuity for ρ_{jk}. This proves (4.1).

It is now clear from the relation (4.1) that every convergent subsequence of $\{\rho_{m\,jk}(\Delta)\}$ must tend to the same limit, and therefore $\rho_{m\,jk}(\Delta) \to \rho_{jk}(\Delta)$ if the end-points of Δ are continuity points of ρ_{jk}. This completes the proof of Theorem 4.

For Cases I and II we showed that $G_\delta \to G$ as $\delta \to (a, b)$ ($\delta = [a, \tilde{b}] \to [a, b)$ in Case II), and from Theorem 4 it is obvious that the corresponding spectral matrices $\rho_\delta = (\rho_{\delta\,jk})$ satisfy $\rho_\delta(\Delta) \to \rho(\Delta)$, if the end-points of Δ are continuity points of ρ. Further relation (4.1) holds.

THEOREM 5. *In Case* I *or* II *there exists an hermitian, non-decreasing matrix* $\rho = (\rho_{jk})$ *whose elements are of bounded variation on every finite* λ*-interval, and*

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which is essentially unique, in the sense that if $\Delta = (\mu, \lambda]$ *is a finite interval whose end-points are continuity points for* ρ_{jk} *then*

$$\rho_{\delta jk}(\Delta) \to \rho_{jk}(\Delta) \qquad \begin{array}{l}(\delta \to (a, b) \text{ in Case I},\\ \delta \to [a, b) \text{ in Case II}).\end{array}$$

Further

$$\rho_{jk}(\Delta) = \frac{1}{2\pi i} \lim_{\epsilon \to +0} \int_{\mu}^{\lambda} P_{jk}(\sigma + i\epsilon)\, d\sigma,$$

where

$$P_{jk}(l) = \frac{\partial^{j+k-2} H}{\partial x^{j-1} \partial y^{k-1}}(c, c, l), \quad H(x, y, l) = G(x, y, l) - G(x, y, \bar{l}),$$

and G is Green's function for Case I *or Case* II.

In Case I or II we call ρ the *spectral matrix*. If ρ is any limit matrix given by Theorem 4 there always exists a Parseval equality and expansion theorem valid for functions $u \in \mathfrak{L}^2(a, b)$. The proof given by Levinson **(4)** can be carried over to this case; cf. also Levitan **(7)**. Let $\phi = (\phi_1, \ldots, \phi_n)$, $\psi = (\psi_1, \ldots, \psi_n)$ be vector functions of λ, and introduce the inner product

$$(\phi, \psi) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \bar{\psi}_j(\lambda)\, \phi_k(\lambda)\, d\rho_{jk}(\lambda).$$

Since ρ is non-decreasing, $(\phi, \phi) \geqq 0$ and we can define a norm $\|\ \ \|^*$ by $\|\phi\|^* = (\phi, \phi)^{\frac{1}{2}}$. Let $\mathfrak{H}^*$ be the Hilbert space of all ϕ such that $\|\phi\|^* < \infty$.

THEOREM 6. *Let* ρ *be any limit matrix given by Theorem* 4. *If* $u \in \mathfrak{L}^2(a, b)$ *the vector* $\phi = (\phi_j)$ *where*

$$\phi_j(\lambda) = \int_a^b \bar{s}_j(x, \lambda)\, u(x)\, dx,$$

converges in norm in $\mathfrak{H}^*$, *and*

$$\|\phi\|^* = \|u\| \qquad \text{(Parseval equality)}.$$

In terms of this ϕ,

$$u(x) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} s_j(x, \lambda)\, \phi_k(\lambda)\, d\rho_{jk}(\lambda) \quad \text{(Expansion theorem)}$$

where the integral converges in norm in $\mathfrak{L}^2(a, b)$.

The convergence of ϕ in norm in $\mathfrak{H}^*$ is meant in the sense that the vector $\phi_\delta = (\phi_{\delta j})$, where

$$\phi_{\delta j}(\lambda) = \int_\delta \bar{s}_j(x, \lambda)\, u(x)\, dx,$$

converges strongly in $\mathfrak{H}^*$ to ϕ when $\delta \to (a, b)$. Similarly the expansion theorem is meant in the sense that

$$\int_\Delta \sum_{j,k=1}^{n} s_j(x, \lambda)\, \phi_k(\lambda)\, d\rho_{jk}(\lambda)$$

converges strongly in $\mathfrak{L}^2(a, b)$ to u as $\Delta \to (-\infty, \infty)$.

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