A Graphic Method of Solving n Simultaneous Linear Equations involving n Unknowns

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§ 1.

The method here explained affords a complete solution of the problem to determine by geometric construction the values of any number of unknowns connected by an equal number of equations of the first degree. The construction consists entirely of straight lines, and can be carried out by the aid of a straight-edge and a scale; or, in its modified form, in which parallel lines are required, by these instruments along with a set-square.

§ 2.

The Case of two Simultaneous Equations

$$a_1 x + b_1 y + c_1 = 0 \qquad (1)$$
$$a_2 x + b_2 y + c_2 = 0 \qquad (2)$$

Draw a triangle OXY, and in OX take X_1, X_2, so that

$$a_1 OX_1 + c_1 X_1 X = 0$$
$$a_2 OX_2 + c_2 X_2 X = 0$$

Similarly, in OY take the points Y_1, Y_2 so that

$$b_1 OY_1 + c_1 Y_1 Y = 0$$
$$b_2 OY_2 + c_2 Y_2 Y = 0$$

Let $X_1 Y_1$ meet $X_2 Y_2$ in P
" YP " OX " X_{12}
" XP " OY " Y_{12}

Then $x = OX_{12}/X_{12}X$; $y = OY_{12}/Y_{12}Y$
are the values which satisfy (1) and (2)

To verify that they satisfy (1) we may proceed thus:—

$$a_1 \frac{OX_{12}}{X_{12}X} + b_1 \frac{OY_{12}}{Y_{12}Y} + c_1 = c_1 \left(- \frac{X_1 X}{OX_1} \cdot \frac{OX_{12}}{X_{12}X} - \frac{Y_1 Y}{OY_1} \cdot \frac{OY_{12}}{Y_{12}Y} + 1 \right)$$
$$= c_1 \left(- \frac{XP}{PY_{12}} \cdot \frac{Y_{12}Y_1}{Y_1 O} \cdot \frac{PY_{12}}{XP} \cdot \frac{YO}{Y_{12}Y} - \frac{Y_1 Y}{OY_1} \cdot \frac{OY_{12}}{Y_{12}Y} + 1 \right)$$

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(applying Menelaus' Theorem to the transversals X_1PY_1 and $X_{12}PY$ of the triangle OXY_{12})

$$=c_1\left(-\frac{Y_{12}Y_1}{Y_1O}\cdot\frac{YO}{Y_{12}Y}-\frac{Y_1Y}{OY_1}\cdot\frac{OY_{12}}{Y_{12}Y}+1\right)$$

$$=c_1\frac{Y_{12}Y_1\,.\,YO+YY\,.\,OY_{12}+OY_1\,.\,Y_{12}Y}{OY_1\,.\,Y_{12}Y}=0.$$

Similarly we can show that the same values of x and y satisfy (2).

The process of constructing the points X_{12}, Y_{12} when the points X_1, Y_1, X_2, Y_2 are given on the triangle OXY may be called for brevity "solving X_1Y_1 and X_2Y_2 graphically, as to x and y", the points X_{12}, Y_{12} being called "the solution".

§ 3.

The Case of three Equations

$$a_1x+b_1y+c_1z+d_1=0 \qquad (3)$$
$$a_2x+b_2y+c_2z+d_2=0 \qquad (4)$$
$$a_3x+b_3y+c_3z+d_3=0 \qquad (5)$$

Draw any tetrastigm $OXYZ$, and let X_1, Y_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3 be determined in the same way as X_1, Y_1, X_2, Y_2, were before; *e.g.* X_1 in OX makes $a_1\,OX_1/X_1X+d_1=0$ and Z_2 in OZ makes $c_2\,OZ_2/Z_2Z+d_2=0$

Then solve

X_1Z_1, X_2Z_2	graphically	as to x, z,	and let	X_{12}, Z_{12}	be the solution
X_1Z_1, X_3Z_3	„	„ „	„	X_{13}, Z_{13}	„ „
Y_1Z_1, Y_2Z_2	„	„ y, z	„	Y_{12}, Z_{12}	„ „
Y_1Z_1, Y_3Z_3	„	„ „	„	Y_{13}, Z_{13}	„ „

Next solve $X_{12}Y_{12}$, $X_{13}Y_{13}$ graphically as to x, y and let $X_{123}Y_{123}$ be the solution. In a similar manner, by interchanging, say X and Z, we can determine Y_{123}, Z_{123}.

Then $x=OX_{123}/X_{123}X$, $y=OY_{123}/Y_{123}Y$, $z=OZ_{123}/Z_{123}Z$ are the values of x, y, z satisfying the equations (3), (4), (5).

§ 4.

The Case of n Equations

The extension of the method to any number of equations will now be obvious.

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If, for example, we have worked the method up to the solution of $n-1$ simultaneous equations, then for the case of n equations we first solve graphically as to x and y two sets of $n-1$ equations each, viz.

(i) Equations (1), (3), (4)......(n) giving solutions $X_{134\ldots n}$, $Y_{134\ldots n}$
(ii) „ (1), (2), (4)......(n) „ „ $X_{124\ldots n}$, $Y_{124\ldots n}$

Then solve graphically as to x and y the four points thus determined on OX and OY, giving the solution $X_{123\ldots n}$, $Y_{123\ldots n}$. Then the values $x = OX_{123\ldots n}/X_{123\ldots n}X$, $y = OY_{123\ldots n}/Y_{123\ldots n}Y$, form part of the solution of the n given equations; and the values of the other unknowns can be worked out in a similar manner.

It is clear that a great variety of constructions exist which can be got by interchanging pairs of letters or of suffixes, and that a limited number of these is sufficient to determine all the unknowns.

§ 5.

Let us now consider the *rationale* of the method. It is obviously suggested by the well-known simple method of solving two simultaneous equations of the first degree by the intersection of two linear graphs. The rule for three simultaneous equations was arrived at by considering the projection upon a plane of a figure consisting of a tetrahedral coordinate system and its intersections with three planes representing the three given equations. But it is exactly parallel to the algebraic method of eliminating one unknown in two ways so as to get two equations involving only two unknowns which are then solved by eliminating successively the two remaining unknowns. And this parallel holds in the general case. It will be sufficient to point out the parallel in detail in the case of four equations with four unknowns.

We have $a_r\dfrac{OX_r}{X_rX}+e_r=0$, &c. (where $r=1, 2, 3, 4$) so that the rth equation can be put in the form

$$\frac{x}{OX_r/X_rX}+\frac{y}{OY_r/Y_rY}+\frac{z}{OZ_r/Z_rZ}+\frac{u}{OU_r/U_rU}=\rho$$

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Where ρ is the letter introduced to make the equations homogeneous, to which the value 1 is given, then the points X_{12}, Y_{12}, Z_{12}, &c., are such that

$$\frac{x}{OX_{12}/X_{12}X} + \frac{y}{OY_{12}/Y_{12}Y} + \frac{z}{OZ_{12}/Z_{12}Z} = \rho$$

$$\frac{x}{OX_{13}/X_{13}X} + \frac{y}{OY_{13}/Y_{13}Y} + \frac{z}{OZ_{13}/Z_{13}Z} = \rho$$

$$\frac{x}{OX_{14}/X_{14}X} + \frac{y}{OY_{14}/Y_{14}Y} + \frac{z}{OZ_{14}/Z_{14}Z} = \rho$$

and further

$$\frac{x}{OX_{123}/X_{123}X} + \frac{y}{OY_{123}/Y_{123}Y} = \rho, \quad \frac{x}{OX_{124}/X_{124}X} + \frac{y}{OY_{124}/Y_{124}Y} = \rho$$

and lastly $$\frac{x}{OX_{1234}/X_{1234}X} = \rho, \quad \frac{y}{OY_{1234}/Y_{1234}Y} = \rho$$

§ 6.

Modified Construction.

By sending the points XYZ ... off to infinity, while O remains near, we get an extreme case of the general method. The only modifications entailed in the construction are first, that OX_1 is cut off from OX to be equal to $-\frac{c_1}{a_1}$, instead of OX being divided in that ratio, and second, that when such a pair of lines as X_1Y_1, X_2Y_2 intersect in P, X_{12} is got by drawing through P a line parallel to OY, instead of passing through Y.

We might further modify the construction by using only two axes of reference, say OA, OB, and letting each step of the graphic solution be worked on these axes. This would involve the transference of some of the points from one axis to another. For instance, in § 2, if OX_1, OX_2, OX_3 were cut off from OA and OZ_1, OZ_2, OZ_3 from OB then the points X_{12}, X_{13} would fall on OA; if then the same process were carried through with Y instead of X, we should have the points Y_{12}, Y_{13} also on OA, and they would have to be transferred to OB before we could graphically solve X_{12}, Y_{12}, X_{13}, Y_{13}.

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§ 7.

Let us now estimate the least number of lines required in addition to OX, OY, OZ, &c., to find *one* unknown, say x, from n simultaneous linear equations.

For $n = 2$ the number is clearly $2 + 1 = 3$

For $n = 3$, if we proceed as in § 3, each of the three points Z_1, Z_2, Z_3 must be joined to 2 points (6 lines) giving 4 intersections through which other 4 lines have to be drawn; and the final stage requires 3 more lines; so that the total for $n = 3$ is 13 lines.

For $n = 4$, the number of lines required in the first stage will be $4 \times 3 + 9$, in the second stage 13, and in the third 3. Total 34.

A little consideration will show that in the general case the number of lines required to find x is $3 + 2 . 5 + 3 . 7 + \ldots\ldots + (n-1)(2n-1)$, which amounts to $n(n-1)(4n+1)/6$, or if we include the n lines of reference OX, OY, &c., the total is $n(4n^2 - 3n + 5)/6$.

§ 8.

VERIFICATION OF THE METHOD BY THE AID OF DETERMINANTS.

Let us take for simplicity the modified method in which X, Y, Z...are at infinity and (for § 1) $OX_1 = -\dfrac{c_1}{a_1}$ &c.

In § 1 the solution of (1) and (2) is given by

$$x = \begin{vmatrix} b_1 c_1 \\ b_2 c_2 \end{vmatrix} \div \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = (b_1 c_2) \div (a_1 b_2)$$

$$y = -\begin{vmatrix} a_1 c_1 \\ a_2 c_2 \end{vmatrix} \div \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = -(a_1 c_2) \div (a_1 b_2)$$

Hence, in § 2, since

$$\begin{aligned} OX_{12} &= (c_1 d_2) \div (a_1 c_2) \\ OX_{13} &= (c_1 d_3) \div (a_1 c_3) \\ OY_{12} &= (c_1 d_2) \div (b_1 c_2) \\ OY_{13} &= (c_1 d_3) \div (b_1 c_3) \end{aligned}$$

the equation to $X_{12} Y_{12}$ is $\dfrac{x(a_1 c_2)}{(c_1 d_2)} + \dfrac{y(b_1 c_2)}{(c_1 d_2)} = 1$

and that to $X_{13} Y_{13}$ is $x(a_1 c_3) + y(b_1 c_3) = (c_1 d_3)$

Hence $OX_{123} = \begin{vmatrix} (c_1 d_2), & (b_1 c_2) \\ (c_1 d_3), & (b_1 c_3) \end{vmatrix} \div \begin{vmatrix} (a_1 c_2), & (b_1 c_2) \\ (a_1 c_3), & (b_1 c_3) \end{vmatrix}$

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Now, if we denote by $A_1\ A_2 \ldots$ the co-factors of $a_1, a_2 \ldots$ in the determinant $(a_1\, b_2\, c_3)$ the determinant after the $\div$ sign is $= -\begin{vmatrix} B_3 & A_3 \\ B_2 & A_2 \end{vmatrix} = -\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix}$ and therefore, by the theory of reciprocal determinants and their minors, is $= -\,c_1\,(a_1\,b_2\,c_3)$. Similarly, the determinant before the sign $\div = +\,c_1\,(a_1\,b_2\,c_3) = +\,c_1\,(b_1\,c_2\,d_3)$

Thus
$$\begin{aligned} OX_{123} &= -\,(b_1\,c_2\,d_3) \div (a_1\,b_2\,c_3) \\ &= -\,(d_1\,b_2\,c_3) \div (a_1\,b_2\,c_3) \end{aligned}$$

which we know to be the value of x got from equations (3), (4), (5).

To extend the proof, by mathematical induction, to the general case of n equations, we assume

$OX_{123\ldots(n-1)} = -\,(h_1\,b_2\,c_3 \ldots g_{n-1}) \div (a_1\,b_2\,c_3 \ldots g_{n-1})$, and then, starting with n equations we have

$OX_{123\ldots(n-1)} = -\,(h_1\,c_2\,d_3 \ldots h_{n-1}) \div (a_1\,c_2\,d_3 \ldots h_{n-1})$

$OX_{123\ldots(n-2)n} = -\,(k_1\,c_2\,d_3 \ldots g_{n-2}h_n) \div (a_1\,c_2\,d_3 \ldots g_{n-2}h_n)$ and two similar results with X and Y, a and b interchanged.

Then
$$OX_{12\ldots\ldots n} = \begin{vmatrix} -\,(k_1\,c_2\,d_3 \ldots h_{n-1}), & (b_1\,c_2\,d_3 \ldots h_{n-1}) \\ -\,(k_1\,c_2\,d_3 \ldots g_{n-2}h_n), & (b_1\,c_2\,d_3 \ldots g_{n-2}h_n) \end{vmatrix}$$
$$\div \begin{vmatrix} (a_1\,c_2\,d_3 \ldots h_{n-1}), & (b_1\,c_2\,d_3 \ldots h_{n-1}) \\ (a_1\,c_2\,d_3 \ldots g_{n-2}h_n), & (b_1\,c_2\,d_3 \ldots g_{n-2}h_n) \end{vmatrix}$$

The numerator
$$= \begin{vmatrix} B_n(-1)^n, & A_n(-1)^{n-1} \\ B_{n-1}(-1)^{n-1}, & A_{n-1}(-1)^n \end{vmatrix} = \begin{vmatrix} A_{n-1} & B_{n-1} \\ A_n & B_n \end{vmatrix}$$

when A_n is the co-factor of a_n in the determinant $(a_1\,b_2 \ldots h_n)$, &c. By the theory of Reciprocal Determinants this gives us $(c_1\,d_2 \ldots h_{n-2})\ (a_1\,b_2\,c_3 \ldots h_n)$ similarly the denominator reduces to $-\,(c_1\,d_2 \ldots h_{n-2})\ (k_1\,b_2\,c_3 \ldots h_n)$

Hence $OX_{12\ldots\ldots n} = -\,(k_1\,b_2\,c_3 \ldots\ldots h_n) \div (a_1\,b_2 \ldots\ldots h_n)$, which completes the proof.

§ 9.

To test the practicability of the graphic method explained in this example, I worked out on an ordinary piece of squared paper, 7 inches by 9, with very ordinary instruments, the graphic solution of the following set of four simultaneous equations.

$$\begin{aligned} x + \tfrac{y}{3} + \tfrac{z}{2} + \tfrac{u}{2} &= 1 \\ \tfrac{x}{2} + y + z - u &= 1 \\ 2x - 2y - 2z - \tfrac{u}{2} &= 1 \\ -\,x + \tfrac{y}{2} - \tfrac{z}{3} + \tfrac{u}{4} &= 1 \end{aligned}$$

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The numerical process of solution gives $x = \frac{584}{529} \doteqdot 1{\cdot}104$, $y = \frac{1437}{529} \doteqdot 2{\cdot}72$, $z = -\frac{1134}{529} \doteqdot -2{\cdot}15$, $u = \frac{66}{529} \doteqdot {\cdot}125$. The graphic process gave $x = 1{\cdot}14$, $y = 2{\cdot}8$, and again $x = 1{\cdot}2$, $z = -1{\cdot}95$. The axes were taken so as to divide space into octants, each of the lines OX, OY, OZ, OU making an angle of 45° with the one preceding it. The second value of x, got by solving for x and z, is less accurate than the first value, and this is, probably, due to the fact that the points X_{124} Z_{124} come very close together, giving an ill-conditioned determination of the line joining them. Some of the intersections lay several inches beyond the paper, and additional pieces of paper were temporarily used. This might have been avoided by using one of the well-known methods of drawing a line through a given point or in a given direction, towards a point of intersection of two lines when it lies out of reach. I did not time myself, either during the construction or during the calculation, but my impression is that the latter took more time—and certainly it would do so in the case of equations with less simple numerical coefficients. Against this possible gain there must be set the very limited degree of accuracy of the graphical method. But I believe this method may have useful applications to a good many technical problems.

[*Added 18th June, 1908.*] After the foregoing paper was in print, my attention was called by the Editor to a Note on the same topic by Mr F. Boulad in Vol. 7 of the 4th Series of the "Nouvelles Annales," and I find by reference to the "Encyclopædie der Math." that several papers on the subject have appeared, the most important of which seems to be that of Van den Berg (Verslagen en Mededeelingen der koninklijke Akademie van Wetenschappen, Amsterdam, 1888.) The only article on the subject I have found published in English is that by Mr F. J. Vaes in "Engineering," Vol. 66, p. 867 (1898) which explains a method differing considerably from that given above.

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