Proof of the sufficiency of the determinant condition for the consistency of a system of n homogeneous linear equations in n variables.—For $n=1$, the theorem is that, if $a=0$, the equation $ax=0$ has a solution for which x is not zero. This is obviously true. And the case of n variables can easily be made to depend on that of $n-1$ variables. For brevity we show the method by taking $n=3$.

Consider the system

$$\left.\begin{aligned} a_1x+b_1y+c_1z=0,\\ a_2x+b_2y+c_2z=0,\\ a_3x+b_3y+c_3z=0.\end{aligned}\right\} \qquad \text{(1)}$$

It is given that

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0, \qquad \text{(2)}$$

and we have to show that the system (1) has a solution in which the variables are not all zero.

If c_1, c_2, c_3 are all zero, (1) has the non-null solution (0, 0, 1). If not, let $c_1 \neq 0$. Then (1) are equivalent to

$$\left.\begin{aligned} a_1x+b_1y+c_1z=0,\\ c_1(a_2x+b_2y+c_2z)-c_2(a_1x+b_1y+c_1z)=0,\\ c_1(a_3x+b_3y+c_3z)-c_3(a_1x+b_1y+c_1z)=0.\end{aligned}\right\} \qquad \text{(3)}$$

Assuming the general theorem known for the case $n=2$, the two last equations of (3) have a non-null solution (X, Y), if

$$\begin{vmatrix} c_1a_2-c_2a_1, & c_1b_2-c_2b_1 \\ c_1a_3-c_3a_1, & c_1b_3-c_3b_1 \end{vmatrix} = 0. \qquad \text{(4)}$$

Since the first equation of (3) gives a definite value Z for z when $x=\text{X}$, $y=\text{Y}$, on account of $c_1 \neq 0$, it follows that (4) is a sufficient condition for the "consistency" of (1). But the determinant of (4) is, since $c_1 \neq 0$, equivalent to

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ c_1a_2-c_2a_1, & c_1b_2-c_2b_1, & 0 \\ c_1a_3-c_3a_1, & c_1b_3-c_3b_1, & 0 \end{vmatrix} = 0 \qquad \text{(5)}$$

To the second row of this determinant add the first multiplied by c_2, to the third row add the first multiplied by c_3; then divide the second and third rows by c_1. Thus we find that (2) is a sufficient condition for the consistency of (1).

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