Appendix F
Solutions to Selected Exercises

Chapter 1

Exercise 1.1.1 *Let* $0 < p < 1$. *We prove that the curve* $\beta\colon [0,1] \to L^p[0,1]$, $\beta(t) := [\mathbf{1}_{[0,t[}]$ *is an injective* C^1*-curve with* $\beta'(t) = 0$*, for all* $t \in [0,1]$.

Let us show that for every $x \in]0,1[$ the derivative of β vanishes. Then the claim follows by continuity also for the boundary points. Consider for h small the differential quotient $[h^{-1}(\beta(x+h) - \beta(x))] = h^{-1}[\mathbf{1}_{[x,x+h[}]$ converges to 0 with respect to the L^p-metric:

$$\begin{aligned} d([h^{-1}(\beta(x+h) - \beta(x))], [0]) &= \int_0^1 |h^{-1}\mathbf{1}_{[x,x+h[}(s)|^p \mathrm{d}s \\ &= |h|^{-p} \int_0^1 \mathbf{1}_{[x,x+h[}(s)\mathrm{d}s = |h|^{1-p}. \end{aligned}$$

Taking the limit $h \to 0$, we see that the derivative must be 0 and, in particular, β is a C^1-function. Now let $x < y$. Then $\beta(y) - \beta(x) = [\mathbf{1}_{[x,y[}] \neq [0]$, so β is injective.

Exercise 1.2.2(c) *We will show that* $D\colon C^\infty([0,1],\mathbb{R}) \to C^\infty([0,1],\mathbb{R})$, $c \mapsto c'$ *is continuous linear.*

Clearly the differential operator is linear with respect to pointwise addition of functions. Now let 0 be the constant 0-function. Then linearity of D together with Lemma A.5 implies that D will be continuous if the preimage of every 0-neighbourhood $U \subseteq C^\infty([0,1],\mathbb{R})$ is a 0-neighbourhood. Thus we pick an open 0-neighbourhood U. Shrinking U, we may assume that U is a ball $B_r^n(0)$ of radius $r > 0$ for the seminorm $\|\cdot\|_n$ where we have chosen suitable $n \in \mathbb{N}_0$ (and have exploited that these seminorms form a fundamental system by Example A.14). Now as the kth derivative of a function coincides with the $(k-1)$th derivative of its derivative, we observe that $D(B_{n+1}^r(0)) \subseteq B_n^r(0)$.

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In other words, $B^r_{n+1}(0) \subseteq D^{-1}(U)$ and the preimage is a 0-neighbourhood. We deduce that D is continuous.

Exercise 1.3.3. Schwarz' theorem *We will show that for a C^k-map $f\colon E \supseteq U \to F$ and $x \in U$ the map $d^r f(x;\cdot)\colon E^r \to F$ is symmetric for every $2 \leq r \leq k$ (and $r < \infty$).*

Remark: There are several possibilities to prove this, for example, it suffices to show that the directional derivatives D_{v_1} and D_{v_2} commute for all $v_1, v_2 \in E$ (then the general case follows from $d^r f(x; v_1, \ldots, v_r) = D_{v_r} \cdots D_{v_1} f(x)$). However, we will reduce the problem to the well-known finite-dimensional case.

Use the Hahn–Banach theorem: It suffices to prove that $d^r(\lambda \circ f)(x;\cdot) = \lambda(d^r f(x;\cdot))$ is symmetric for every continuous linear functional λ. Hence without loss of generality $F = \mathbb{R}$. Now pick $2 \leq r \leq k$ $v_1, \ldots, v_r \in E$. Then there is $\varepsilon > 0$ such that $x + \sum_{i=1}^r t_i v_i$ is contained in U for all $|t_i| < \varepsilon$. Thus we can define the auxiliary function

$$h\colon \mathbb{R}^r \supseteq]-\varepsilon, \varepsilon[^r \to \mathbb{R}, \quad (t_1, \ldots, t_r) \mapsto f(x + t_1 v_1 + \cdots + t_r v_r).$$

By the chain rule h is C^k. Note that by the finite-dimensional version of Schwarz' theorem the partial derivatives of h commute. Now the statement for f follows from the chain rule and the observation (see Remark 1.15) that

$$\left.\frac{\partial^r}{\partial_{t_1} \cdots \partial_{t_r}}\right|_{t_1, \ldots, t_r = 0} h(t_1, \ldots, t_r) = d^r f(x; v_1, \ldots, v_r).$$

Exercise 1.3.5 *For a C^2 map f and C^1 maps g, h we derive a formula for the derivative of $\phi := df \circ (g, h)$.*

This is an exercise in applying the chain rule and the rule on partial differentials (Proposition 1.20):

$$\begin{aligned} d\phi(x;y) &= d(df \circ (g,h))(x;y) \\ &= (d_1 df)(g(x), h(x); dg(x;y)) + (d_2 df)(g(x), h(x); dh(x;y)) \\ &= d^2 f(g(x); h(x), dg(x;y)) + df(g(x); dh(x;y)). \end{aligned}$$

Here we have used the fact that the derivative of df with respect to the first component is just $d^2 f$ and that $df(g(x);\cdot)$ is continuous linear.

Exercise 1.5.1 *We construct charts turning* $\operatorname{graph}(f) := \{(m, f(m)) \mid m \in M\}$ *into a split C^r submanifold of $M \times N$ for a C^r-function f.*

It suffices to construct submanifold charts for every point $(m, f(m))$. To this end, pick (U_φ, φ) a chart of M and (U_ψ, ψ) a chart of N with $m \in U_\varphi$, $f(m) \in U_\psi$. Assume that $\varphi \times \psi$ is a mapping into the locally convex space

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$E \times F$. We will now construct a chart around $(m, f(m))$ mapping all elements in graph$(f) \cap U_\varphi \times U_\psi$ to the (complemented) subspace $E \times \{0\} \subseteq E \times F$. Since the vector space operations are continuous (bi)linear, they are smooth in the Bastiani sense and we obtain a C^r-map for $M \times N$ via

$$\kappa\colon U_\varphi \times U_\psi \to E \times F, \quad (m,n) \mapsto (\varphi(m), \psi(n) - \psi(f(m))).$$

This mapping is a C^r-diffeomorphism as its inverse is given by the formula $\kappa^{-1}(x,y) := (\varphi^{-1}(x), \psi^{-1}(y + \psi(f(\varphi^{-1}(x)))))$ (we leave it as an exercise to show that this mapping is well defined on an open subset of $E \times F$). Thus κ is a chart for $M \times N$. By construction $\kappa(m, f(m)) = (\varphi(m), \psi(f(m)) - \psi(f(m))) = (\varphi(m), 0)$. Thus κ is a submanifold chart for the graph.

Exercise 1.5.4 *We prove that a locally compact manifold M is necessarily finite dimensional* (note that the exercise asks for compact manifolds, but the argument only requires local compactness).

Let $\varphi\colon U_\varphi \to V_\varphi \subseteq E$ be a chart for M. Since M is locally compact, there exists a compact neighbourhood C of $x \in U_\varphi$ such that $C \subseteq U_\varphi$. Then $\varphi(C) \subseteq E$ is a compact neighbourhood of $\varphi(x)$. Since translations are homeomorphisms in E, the translated set $\varphi(C) - \varphi(x) = \{y = m - \varphi(x) \mid m \in \varphi(C)\}$ is a compact 0-neighbourhood in E. Thus E is finite dimensional by Proposition A.3 and M is a finite-dimensional manifold.

Exercise 1.7.1 *We show that the composition $g \circ f$ of the submersions $f\colon M \to N$ and $g\colon N \to L$ is a submersion.*

The submersion property is local, whence we can restrict to chart neighbourhoods of submersion charts around $m \in U \subseteq M, f(m) \in V \subseteq N$ and $g(f(m)) \in W \subseteq L$ such that:

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\varphi"] & V \arrow[r, equal] \arrow[d, "\psi_1"] & V \arrow[r, "g"] \arrow[d, "\psi_2"] & W \arrow[d, "\kappa"] \\
F \times X \arrow[r, "\mathrm{pr}_F"] & F \arrow[r, "\cong"] & Y \times Z \arrow[r, "\mathrm{pr}_Y"] & Y.
\end{tikzcd}$$

Now via the typical insertion of charts:

$$\begin{aligned} \kappa \circ g \circ f \circ \varphi^{-1} &= \kappa \circ g \circ \psi_2^{-1} \circ \psi_2 \circ \psi_1^{-1} \circ \psi_1 \circ f \circ \varphi^{-1} \\ &= \mathrm{pr}_Y \circ \psi_2 \circ \psi_1^{-1} \circ \mathrm{pr}_F. \end{aligned} \tag{F.1}$$

Note that the change of charts $\psi_2 \circ \psi_1^{-1}$ is a diffeomorphism on its domain (which we will now call O). Shrinking U, we may assume that $\varphi(U) = O \times D$. We obtain a modified chart $\tilde{\varphi} := ((\psi_2 \circ \psi_1^{-1}) \times \mathrm{id}_W) \circ \varphi$. If we now insert $\tilde{\varphi}$ into (F.1) we see that $\kappa \circ g \circ f \circ \tilde{\varphi}^{-1} = \mathrm{pr}_Y : F \times X \cong Y \times Z \times X \to Y$. In other

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words, we have constructed submersion charts for the composition which thus turns out to be a submersion.

Chapter 2

Exercise 2.2.1 *We assume that the exponential law holds for all relevant function spaces in this exercise! If $f \in C^\infty(U \times M, O)$ and $p \in C^\infty(O, N)$, we will prove that the pushforward satisfies $p_* \circ (f^\vee) = (p \circ f)^\vee$.*

Pick $(u, m) \in U \times M$. Then

$$\begin{aligned} p_* \circ (f^\vee)(u)(m) &= p_*(f^\vee(u))(m) = (p \circ f(u, \cdot))(m) = p(f(u, m)) \\ &= (p \circ f)^\vee(u)(m) \end{aligned}$$

and this proves the assertion. (Why did we need to assume that the exponential law holds if the calculation does not use it?)

Exercise 2.3.1 *Let $h\colon L \to K$ be a smooth map. Assume that $C^\infty(K, M)$ and $C^\infty(L, M)$ are canonical manifolds. We prove that*

(a) *the pullback $h^*\colon C^\infty(K, M) \to C^\infty(L, M)$, $f \mapsto f \circ h$ is smooth;*
(b) *if K, L are compact and M admits a local addition, then $TC^\infty(K, M) \cong C^\infty(K, TM)$ (see* C.12*). This identifies $T(h^*)$ with $h^*\colon C^\infty(K, TM) \to C^\infty(L, TM)$.*

(a) The pullback is a partial map of the full composition, hence smooth by Proposition 2.23. Alternatively, smoothness follows directly from the exponential law, as h^* is smooth if and only if the adjoint $(h^*)^\wedge\colon C^\infty(K, M) \times L \to M$, $(f, \ell) \mapsto f(h(\ell)) = \mathrm{ev}(f, h(\ell))$ is smooth. Since $C^\infty(K, M)$ is canonical, Lemma 2.16 shows that the evaluation is smooth. Now smoothness of the adjoint follows, since ev and h are smooth.

(b) We only need the assumptions to identify $TC^\infty(K, M) \cong C^\infty(K, TM)$. To compute the tangent, we pick $c\colon]-\varepsilon, \varepsilon[\to C^\infty(K, M)$ smooth with $c(0) = f$ and $\dot{c}(0) = V_f$. Under the identification we can interpret $V_f(x) = \frac{\partial}{\partial t}\big|_{t=0} c^\wedge(t, x)$ as a function $K \to TM$. Now

$$Th^*(V_f)(x) = \frac{\partial}{\partial t}\bigg|_{t=0} h^*(c^\wedge(t, x)) = \frac{\partial}{\partial t}\bigg|_{t=0} c^\wedge(t, h(x)) = V_f(h(x)) = h^*(V_f).$$

Thus we have identified the tangent map as $h^*\colon C^\infty(K, TM) \to C^\infty(L, TM)$.

Exercise 2.3.4 *For K a compact manifold and M a manifold with local addition, we endow $C^\infty(K, M)$ with its canonical manifold structure and compute the tangent map of the evaluation* $\mathrm{ev}\colon C^\infty(K, M) \times K \to M$.

We apply the rule on partial differentials for manifolds (Exercise 1.6.1):

$$T_{(\varphi, k)}\,\mathrm{ev}(v_\varphi, v_k) = T_\varphi\,\mathrm{ev}(\cdot, k)(v_\varphi) + T_k\,\mathrm{ev}(\varphi, \cdot)(v_k).$$

To evaluate the first term, pick a curve $c\colon]-\varepsilon,\varepsilon[\to C^\infty(K,M)$ with $c(0)=\varphi$ and $\dot{c}(0)=v_\varphi$. If we identify $TC^\infty(K,M)\cong C^\infty(K,TM)$ via C.12 we can interpret v_φ as the smooth mapping $\frac{\partial}{\partial t}\big|_{t=0} c^\wedge\colon K\to TM$. Then we compute

$$T_\varphi\,\mathrm{ev}(\cdot,k)(v_\varphi)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathrm{ev}(c(t),k)=\left.\frac{\partial}{\partial t}\right|_{t=0} c^\wedge(t,k)=v_\varphi(k)=\mathrm{ev}_k(v_\varphi).$$

Here $\mathrm{ev}_k\colon C^\infty(K,TM)\to TM$ is the evaluation in k and we have exploited the exponential law in the computation. For the second term in the sum, it is immediately clear that we get $T_k\varphi(v_k)$. Thus we get as a formula for the tangent mapping,

$$T\,\mathrm{ev}(v_\varphi,v_k)=\mathrm{ev}_k(v_\varphi)+T\varphi(v_k).$$

Chapter 3

Exercise 3.1.5 *Let $H=\{(x_n)_{n\in\mathbb{N}}\in\ell^2\mid x_n\in\frac{1}{n}\mathbb{Z},n\in\mathbb{N}\}$. Then we prove that*

(a) *Every 0-neighbourhood in the subspace topology of H contains at least one non-zero element.*
(b) *There is no 0-neighbourhood in H which contains a continuous path connecting 0 with a non-zero element. Thus H is not a (sub)manifold.*

(a) It suffices to consider the intersection of norm balls with H, and in particular, we only need to find such elements in $B_{1/m}(0)\cap H$ for $m\in\mathbb{N}$. However, for such a ball, it is clear that the sequence $x_n^m=0$ if $m\neq n$ and $x_m^m=1/(2m)$ is contained in the intersection.

(b) Assume that there is a 0-neighbourhood in H which is path-connected. Then it contains an element $(x_k)_{k\in\mathbb{N}}\neq 0$. Pick $\ell\in\mathbb{N}$ with $x_\ell\neq 0$. If $c\colon[0,1]\to H$ is a continuous path connecting 0 and $(x_k)_{k\in\mathbb{N}}$, then $\pi_\ell\circ c$ is a continuous path in $\mathbb{R}$ connecting 0 and $x_\ell\neq 0$. Since the path c takes its values in H, $\pi_\ell\circ c$ can take only values in a discrete subset of $\mathbb{R}$. Contradiction! Thus there is no 0-neighbourhood of H in the subspace topology which is path-connected and H is therefore not locally homeomorphic to a locally convex space. We conclude that it cannot be a (sub-)manifold of ℓ^2.

Exercise 3.3.7 *(Mini Lie–Palais) Every Lie algebra morphism $\phi\colon\mathfrak{g}\to\mathcal{V}(M)$ from a finite-dimensional Lie algebra $\mathfrak{g}$ to the Lie algebra of vector fields of a compact manifold M (with the negative of the usual bracket) gives rise to a Lie group action $G\times M\to M$ (with $\mathbf{L}(G)=\mathfrak{g}$).*

Note first that since $\mathfrak{g}$ is finite dimensional and ϕ is linear, ϕ is automatically continuous, hence a morphism of locally convex Lie algebras. By Lie's third theorem (Hilgert and Neeb, 2012, Theorem 9.4.11) there exists a connected,

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simply connected Lie group G such that $\mathbf{L}(G) = \mathfrak{g}$. Now G is finite dimensional and thus regular and $\mathrm{Diff}(M)$ is regular by Example 3.36. Hence we can apply Lie's second theorem for regular Lie groups, Proposition E.14, to integrate ϕ to a Lie group morphism $\Phi\colon G \to \mathrm{Diff}(M)$, that is, $\mathbf{L}(\Phi) = \phi$. Exploiting that $\mathrm{Diff}(M) \Subset C^\infty(M,M)$ and $C^\infty(M,M)$ is a canonical manifold, Lemma C.11, the adjoint map $\Phi^\wedge\colon G \times M \to M$ is a Lie group action (naturally induced by ϕ).

Chapter 5

Exercise 5.1.1(c) *We check the details of Proposition* 5.3 *and show that the second derivative of the vertical part of S_* is given by the pushforward of the second derivative of the vertical part of S.*

We already saw in the proposition that we can instead compute the partial derivative (with respect to the first variable) of

$$(S_*)^\wedge\colon C^\infty(\mathbb{S}^1, TM) \times \mathbb{S}_1 \to T^2M, \quad (h,\theta) \mapsto S(h(\theta)).$$

Pick $\theta \in \mathbb{S}^1$ and $h \in C^\infty(\mathbb{S}^1, TM)$ together with a chart (U,φ) of M such that $h(\theta) \in TU \cong U \times E$ (for E a locally convex space). By continuity, there is a compact set L such that $h \in \lfloor L, TU \rfloor$ and $\theta \in L^\circ$. For a curve $c\colon]-\varepsilon,\varepsilon[\to \lfloor L, TU \rfloor$, apply again the exponential law, Theorem 2.12, to see that $c(t)|_{L^\circ} = T\varphi^{-1} c_E(t)$ for some smooth $c_E\colon]-\varepsilon,\varepsilon[\to C^\infty(L^\circ, U \times E)$. Plugging this into $(S_*)^\wedge$ we take derivatives to obtain

$$\begin{aligned} &T^2\varphi \circ (S_*)^\wedge (T\varphi^{-1})_* \circ c_E(t), \theta) \\ &\quad = T^2\varphi S(T\varphi^{-1}(c_E(t)(\theta)) \\ &\quad = (\pi_M \circ h(\theta), c_E(t)(\theta), c_E(t)(\theta), S_{U,2}(h(\theta), c_E(t)(\theta))). \end{aligned}$$

Here $S_{U,2}$ is the non-trivial vector part of the spray S. We conclude that after projecting onto the fourth component and after fixing the parameter θ, the second derivative of the vertical part of S_* can be identified with the pushforward of the second derivative of $S_{U,2}$. This proves that B_{S_*} is the pushforward of B.

Exercise 5.1.2 *We establish the formula* (5.5)*: $\nabla_{\dot{c}} \alpha(\cdot)(x) = \nabla^g_{\dot{c}^\wedge(\cdot,x)} \alpha^\wedge(\cdot,x)$ for all $x \in \mathbb{S}^1$.*

The trick is to avoid at all costs working in charts of the manifold of mappings $C^\infty(\mathbb{S}^1, M)$. However, the object $\nabla_{\dot{c}}\alpha$ is defined via the local formula (4.13). Taking a look at the local formula for the connector (4.11), we see that $\nabla_{\dot{c}}\alpha = K_*(\dot{\alpha})$, where K_* is the connector associated to the covariant derivative ∇. As already shown in the notation (and proved in Proposition 5.7), the connector of ∇ is the pushforward of the connector K associated to the Riemannian metric g on M (and thus also to the covariant derivative ∇^g). Setting in now these relations, we obtain the desired identity

$$\nabla_{\dot{c}}\alpha(t)(x) = K_*(\dot{\alpha}(t))(x) = (K \circ \alpha^\wedge)(t,x) = \nabla_{c^\wedge(t,x)}\alpha^\wedge(t,x).$$

Exercise 5.2.3 *The elastic metric* (5.9) *is invariant under reparametrisations with elements φ in* $\mathrm{Diff}(\mathbb{S}^1)$ *which satisfy $T_\theta\varphi(1) > 0$ for all $\theta \in \mathbb{S}^1$.*

We have seen in Proposition 5.17 that the elastic metric is the pullback of the L^2-metric via the SRVT. In Exercise 5.2.1(c) we have seen that for a diffeomorphism φ in $\mathrm{Diff}(\mathbb{S}^1)$ which satisfies $T_\theta\varphi(1) > 0$, for all $\theta \in \mathbb{S}^1$ one has $\mathcal{R}(c \circ \varphi) = \dot{\varphi} \cdot \mathcal{R}(c) \circ \varphi$. Plugging this in the L^2-inner product, we see that invariance follows from the usual transformation rule for integrals.

Chapter 6

Exercise 6.2.3 *We show that for a Banach Lie groupoid $G \rightrightarrows M$ the multiplication map* $\mathbf{m}\colon G \times_M G \to G$ *is a submersion.*

Remark: I do not know whether this proof (which was shared with me by D. M. Roberts (Adelaide)) or even the corresponding statement generalises beyond the Banach setting.

We exploit the following characterisation of submersions between Banach manifolds: The map $\mathbf{m}$ is a submersion if and only if it admits local sections, that is, for every $(g_1,g_2) \in G \times_M G$ there exists a smooth map $\varphi\colon U \to G \times_M G$ such that $g := \mathbf{m}(g_1,g_2) \in U$, $\varphi(g) = (g_1,g_2)$ and $\mathbf{m} \circ \varphi = \mathrm{id}_U$ (see Margalef-Roig and Domínguez, 1992, Proposition 4.1.13). Thus we fix g_1,g_2 and g as above and write $\varphi = (\varphi_1,\varphi_2) \in C^\infty(G, G \times G)$ (exploiting that $G \times_M G$ is a submanifold of the cartesian product, whence it suffices to obtain two smooth maps with values in G such that their combination takes values in the fibre-product). Now exploiting the groupoid structure, we observe that $g_2 = g_1^{-1} \cdot g$. Ignoring for a moment that multiplication is not globally defined, we see that for any smooth map φ_1 with $\varphi_1(g) = g_1$, the smoothness of the groupoid operations yields a smooth map φ_2 via

$$\varphi_2(x) = \varphi_1(x)^{-1} \cdot x = \mathbf{m}(\mathbf{i}(\varphi_1(x)), x). \tag{F.2}$$

Setting in $x = g$, we immediately see that $\varphi_2(g) = g_2$. However, to make the formula (F.2) well defined we need to require that $\mathbf{s}(\varphi_1(x)^{-1}) = \mathbf{t}(x)$. Since inversion intertwines source and target this yields $\mathbf{t} \circ \varphi_1 = \mathbf{t}$. We deduce that it suffices to construct a certain smooth map $\varphi_1 : U \to G$ on some neighbourhood U of g with $\mathbf{t} \circ \varphi_1 = \mathbf{t}$ and $\varphi_1(g) = g_1$.

Set $y := \mathbf{t}(g)$ and observe that $\mathbf{t}(g) = \mathbf{t}(g_1g_2) = \mathbf{t}(g_1)$. Since $\mathbf{t}$ is a submersion, there is an open neighbourhood O_y of y together with a smooth section

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$\sigma\colon O_y \to G$ such that $\sigma(y) = g_1$ and $\mathbf{t}\circ\sigma = \mathrm{id}_{O_y}$. We can now choose an open neighbourhood $g \in U$ such that $\mathbf{t}(U) \subseteq O_y$ and define $\varphi_1\colon U \to G$, $x \mapsto \sigma \circ \mathbf{t}(x)$. Then φ_1 is smooth and satisfies $\varphi_1(g) = g_1$ and $\mathbf{t}\circ\varphi_1 = \mathbf{t}$. We conclude that $\mathbf{m}$ admits local sections and is thus a submersion.

Exercise 6.2.4 *We check that the gauge groupoid associated to a principal G-bundle (E,p,M,F) is a Lie groupoid if E, M and G are Banach manifolds.*

(a) We begin with the construction of charts for $(E\times E)/G$. Let $(U_i)_{i\in I}$ be an open cover of M such that there exist smooth local sections $\sigma_i\colon U_i \to E$ of p. This yields an atlas $(U_i,\kappa_i)_{i\in I}$ of local trivialisations of the bundle $p\colon E \to M$ which are given by

$$\kappa_i\colon p^{-1}(U_i) \to U_i \times G, \quad x \mapsto (p(x), \mathrm{d}(\sigma_i(p(x)),x)),$$

with $\mathrm{d}\colon E\times_M E \to G$, $(x,y) \mapsto x^{-1}\cdot y$. Here we use $x^{-1}\cdot y$ as the suggestive notation for the unique element $g \in G$ that satisfies $x\cdot g = y$.
The local trivialisations commute with the right G-action on E since

$$\kappa_i(x\cdot g) = (p(x\cdot g), \mathrm{d}(\sigma_i(p(x\cdot g)), x\cdot g) = (p(x), \mathrm{d}(\sigma_i(p(x)),x))\cdot g.$$

In particular, the trivialisations descend to manifold charts for the arrow manifold of the gauge groupoid:

$$\begin{aligned}&K_{ij}\colon (p^{-1}(U_i)\times p^{-1}(U_j))/G \to U_i \times U_j \times G,\\ &[x_1,x_2] \mapsto (p(x_1), p(x_2), \mathrm{d}(\sigma_i(p(x_1)),x_1)\mathrm{d}(\sigma_j(p(x_2)),x_2)^{-1}).\end{aligned}$$

To see that the projection is a submersion, it suffices to prove this locally in charts. In the trivialisations and the charts, the quotient becomes the map

$$(U_i\times G)\times(U_j\times G) \to U_i\times U_j\times G, \quad \big((u_i,g_i),(u_j,g_j)\big) \mapsto (u_i,u_j,g_ig_j^{-1}).$$

While we have the identity in the u components, the G component is the composition of inversion in the second component with the Lie group multiplication. Inversion in a Lie group is a diffeomorphism, while the multiplication in the Banach Lie group G is a submersion by Exercise 6.2.3. Now the composition of submersions is a submersion by Exercise 1.7.1 and by Exercise 1.7.2, the quotient map is a submersion.

(b) Smoothness of the mappings follows from Exercise 1.7.6 by composing them with the submersion $q\colon E\times E \to (E\times E)/G$ and observing that the resulting mappings are smooth on $E\times E$. In particular, $\mathbf{s}\circ q = p$ is a surjective submersion, whence by Margalef-Roig and Domínguez (1992, Proposition 4.1.5), $\mathbf{s}$ is a surjective submersion.

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Chapter 7

Exercise 7.3.1 *Let (M,g) be a Riemannian metric with volume form μ. We show that the L^2-metric is a right invariant Riemannian metric on* $\mathrm{Diff}_\mu(M)$.

We have already seen (for $M = \mathbb{S}^1$ but the general case is similar) that the L^2-inner product $\langle X,Y\rangle_{L^2} := \int_M g_m(X(m),Y(m))\mathrm{d}\mu(m)$ is a continuous inner product on $\mathcal{V}(M)$. From the formula for the tangent of the right multiplication in $\mathrm{Diff}(M)$ (see 2.22), we see that the right-invariant metric induced by the L^2-inner product is given by

$$\langle X \circ \varphi, Y \circ \varphi\rangle_\varphi = \langle X \circ \varphi \circ \varphi^{-1}, Y \circ \varphi \circ \varphi^{-1}\rangle_{L^2} = \langle X,Y\rangle_{L^2}.$$

Let us assume now that φ is a volume-preserving diffeomorphism. Then by diffeomorphism invariance of the integral (see e.g. Lee, 2013, Proposition 16.6), we derive now that

$$\begin{aligned}\langle X \circ \varphi, Y \circ \varphi\rangle_\varphi = \langle X,Y\rangle_{L^2} &= \int_M \varphi^* g(X,Y)\mathrm{d}\mu \\ &= \int_M g_{\varphi(m)}(X \circ \varphi(m), Y \circ \varphi(m))\mathrm{d}\mu(m).\end{aligned}$$

Thus for every volume-preserving diffeomorphism, the L^2-metric coincides with the right-invariant Riemannian metric induced by the L^2-inner product on the vector fields. We conclude that the L^2-metric is right invariant on the subgroup $\mathrm{Diff}_\mu(M)$.

Exercise 7.3.1 *For (M,g) a compact Riemannian manifold consider the L^2-metric g^{L^2}* (7.8) *on* $\mathrm{Diff}(M)$. *Let S be the metric spray of g and K the associated connector. (a) Then S_* and K_* define a spray and a connector on* $\mathrm{Diff}(M)$. *Moreover, by (b)–(c) $\nabla^{L^2}_X Y := K_* \circ TY \circ X$ defines the metric derivative on g^{L^2}.*

(a) As $\mathrm{Diff}(M) \Subset C^\infty(M,M)$ and $C^\infty(M,M)$ is a canonical manifold, the pushforwards are smooth. That they form a spray and a connector follows directly from the identification of $T^k\,\mathrm{Diff}(M) \Subset T^k C^\infty(M,M) \cong C^\infty(M,T^k M)$.

(b)–(c) Details for this proof are recorded in Ebin and Marsden (1970, Proof of Theorem 9.1).

Chapter 8

Exercise 8.2.4 *Let $N \in \mathbb{N} \cup \{\infty\}, d \in \mathbb{N}$ and $X\colon [0,1] \to \mathbb{R}^d$ be a smooth path. Define $DX\colon [0,1] \to \mathcal{T}^N(\mathbb{R}^d)$, $t \mapsto (0, X'_t, 0, \ldots)$. We will then:*

(a) *Show that* $\frac{\mathrm{d}}{\mathrm{d}t} S_N(X)_{s,t} = S_N(X)_{s,t} \otimes (DX)_t, s < t \leq 1, S_N(X)_{s,s} = \mathbf{1}$.
(b) *Establish Chen's relation*

$$S_N(X)_{s,t} = S_N(X)_{s,u} \otimes S_N(X)_{u,t}, \qquad 0 \leq s \leq u \leq t \leq 1.$$

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(c) *Thus* $S_N(X)_{s,t} = S_N(X)_{0,s}^{-1} \otimes S_N(X)_{0,t}$ *holds.*

First note that the claims will follow for $N \in \mathbb{N} \cup \{\infty\}$ if we can establish the identity for the projection to every finite degree $k \in \mathbb{N}_0$.

(a) For $s = 0$ the claim is just (8.7). For arbitrary s the claim follows from inspecting the Volterra series. Namely, projecting to homogeneous elements of degree $k \geq 1$, the component of $S_N(X)_{s,t}$ is

$$\begin{aligned}
&\int_s^t \int_s^{r_{k-1}} \cdots \int_s^{r_1} \mathrm{d}X_{r_1} \otimes \cdots \otimes \mathrm{d}X_{r_k} \\
&\quad = \int_s^t \left(\int_s^{r_{k-1}} \cdots \int_s^{r_1} \mathrm{d}X_{r_1} \otimes \cdots \otimes \mathrm{d}X_{r_{k-1}} \right) \otimes \mathrm{d}X_{r_k} \\
&\quad = \int_s^t \pi_{k-1}^N \left(S_N(X)_{s,r_k} \right) \otimes \mathrm{d}X_{r_k}.
\end{aligned}$$

In other words, the signature satisfies the integral equation $S_N(X)_{s,t} = 1 + \int_s^t S_N(X)_{s,r} \otimes \mathrm{d}X_r$ in $\mathcal{T}^N(\mathbb{R}^d)$, whence it solves the desired ODE.

(b) We proceed by induction on k. Note that the identity is trivially true for $k = 0$ since it reads $1 = 1 \cdot 1$. Assume now that we have established the claim now for every $s < u < t \in [0,1]$ and $\ell \leq k$, and so $S_k(X)_{s,t} = S_k(X)_{s,u} \otimes S_k(X)_{u,t}$. We work now in the truncated tensor algebra $\mathcal{T}^{k+1}(\mathbb{R}^d)$ (and note that the following identities hold precisely by truncating after degree $k+1$):

$$S_{k+1}(X)_{s,u} = 1 + \int_s^u S_{k+1}(X)_{s,r} \otimes \mathrm{d}X_r = \int_s^u S_k(X)_{s,r} \otimes \mathrm{d}X_r,$$

$$S_{k+1}(X)_{s,u} \otimes \int_u^t S_N(X)_{u,r} \otimes \mathrm{d}X_r = S_k(X)_{s,u} \otimes \int_u^t S_k(X)_{u,r} \otimes \mathrm{d}X_r.$$

Applying the induction hypothesis to split the $S_k(X)$ for $s < u < r < t$, we obtain

$$\begin{aligned}
S_{k+1}(X)_{s,t} &= 1 + \int_s^u S_k(X)_{s,r} \mathrm{d}X_r + \int_u^t S_k(X)_{s,u} \otimes S_k(X)_{u,r} \otimes \mathrm{d}X_r \\
&= S_{k+1}(X)_{s,u} + S_{k+1}(X)_{s,u} \otimes \left(\int_u^t S_k(X)_{t,r} \otimes \mathrm{d}X_r \right) \\
&= S_{k+1}(X)_{s,u} \otimes \left(1 + (S_{k+1}(X)_{u,t} - 1)\right) \\
&= S_{k+1}(X)_{s,u} \otimes S_{k+1}(X)_{u,t}.
\end{aligned}$$

(c) Multiplying Chen's relation for $S_N(X)_{0,t}$ from the left with the inverse of $S_N(X)_{0,s}$ immediately yields the desired identity.

Appendix A

Exercise A.6.2(a) *We show that the box topology turns* $E := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid \text{almost all } x_n = 0\}$ *into a locally convex space and makes the right shift continuous*.

Note that we have $\lambda(\prod_n U_n) + (\prod_n V_n) = \prod_{n\in\mathbb{N}}(\lambda U_n + V_n)$ for $U_n, V_n \Subset \mathbb{R}$ and $\lambda \in \mathbb{R}$. If $(x_n) + (y_n) \in \prod_n U_n$ we exploit that $\mathbb{R}$ is a topological vector space to find for every $n \in \mathbb{N}$ $x_n \in A_n \Subset \mathbb{R}$ and $y_n \in B_n \Subset \mathbb{R}$ with $A_n + B_n \subseteq U_n$. We conclude that $(\prod_n A_n) + (\prod_n B_n) \subseteq (\prod_n U_n)$ and vector addition is continuous. For scalar multiplication we note that if $\lambda \cdot (x_n) \in \prod_n U_n$ we can exploit that $x_n \neq 0$ for only finitely many n, to construct an open λ-neighbourhood $\lambda \in O \Subset \mathbb{R}$ together with $(x_n) \in \prod_n V_n$ such that $O \cdot \prod_n V_n \subseteq \prod_n U_n$. Hence scalar multiplication is continuous and E is a TVS.

Now let $\prod_n U_n$ be a 0-neighbourhood. This implies that every U_n is a 0-neighbourhood in $\mathbb{R}$. Since $\mathbb{R}$ is locally convex, for every n there is a convex 0-neighbourhood $C_n \subseteq U_n$. Then $\prod_n C_n \subseteq \prod_n U_n$ is a convex 0-neighbourhood and thus E is locally convex. The right shift is continuous as $R^{-1}(\prod_n U_n) = \prod_n U_{n+1}$ if $0 \in U_1$ and $\emptyset$ otherwise.

Appendix B

Exercise B.2.4 *We show that the set* $\Omega' := \{f \in C(K,Y) \mid \operatorname{graph}(f) \subseteq \Omega\}$ *is open in the compact-open topology if K is compact and* $\Omega \Subset K \times Y$.

Let us show that Ω' is a neighbourhood for each $f \in \Omega'$. Since Ω is open in $K \times Y$ with the product topology, we find for each $x \in K$ open subsets $U_x \Subset K, V_x \Subset Y$ with $(x, f(x)) \in U_x \times V_x \Subset \Omega$. Shrinking the U_x, we may also assume that $\overline{U}_x \times V_x \subseteq \Omega$ and $f(\overline{U}_x) \subseteq V_x$.

By compactness $\overline{U}_x$ is compact and we can cover K with finitely many of the U_x, say, $K = \bigcup_{1\leq k\leq n} U_{x_k}$. Then by construction $N_f := \bigcap_{1\leq k\leq n} \lfloor \overline{U}_{x_k}, V_{x_k} \rfloor$ is open in the compact-open topology and $f \in N_f$. Moreover, if $h \in N_f$, then we have for $x \in U_{x_k}$ that $(x, h(x)) \in U_x \times V_x \subseteq \Omega$. Since the U_x cover K, we deduce that $h \in \Omega'$ and thus $N_f \subseteq \Omega'$.

Appendix C

Exercise C.2.2 We show that the pullback bundle $f^*(E)$ is a split submanifold of $K \times E$. The idea is similar to the proof that the graph of a smooth function is a split submanifold. The proof is essentially Lemma 1.60: By definition of a fibre bundle $p\colon E \to M$ is a submersion. In the notation of that lemma we have $f^*(E) = K \times_M E$ is a split submanifold of $K \times E$ for each $f \in C^\infty(K, M)$.

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Appendix E

Exercise E.3.5 *For a Riemannian manifold (M, g) with metric derivative ∇, we establish the formula*

$$g(Z, \operatorname{grad} g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \ \textit{for} \ X, Y, Z \in \mathcal{V}(M).$$

From the definition of the gradient and the compatibility of the metric derivative with g we deduce that

$$g(Z, \operatorname{grad} g(X,Y)) = dg(X,Y)(\cdot; Z) = Z \,.\, g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

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