A system of nonagons nonuply in perspective.

By Dr William P. Milne.

(*Received 15th January 1913. Read 9th May 1913*).

In investigating the properties of the cubic curve C, we are led to consider its Hessian C′, its Cayleyan Γ and the Hessian of its Cayleyan Γ′. In the present paper we propose to deal with the intersections of the systems $C + \lambda C'$ and $\Gamma + \mu\Gamma'$.

We note that all the members of the system $C + \lambda C'$ have the same set of points and lines as flexes and harmonic lines respectively. Similarly the system $\Gamma + \mu\Gamma'$ has the same set of points and lines as harmonic points and cuspidal tangents respectively.

If we project any one of the flexes to infinity and take its harmonic line as the axis of x, every member of $C + \lambda C'$ has the form

$$uy^2 + v = 0$$

where u and v are polynomials in x of degrees 1 and 3 respectively. Furthermore, referred to the same axes $\Gamma + \mu\Gamma'$ has the tangential form

$$pm^2 + q = 0$$

where p and q are polynomials in l of degrees 1 and 3 respectively. Hence $C + \lambda C'$ and $\Gamma + \mu\Gamma'$ are each symmetrical about the axis of x. But $\Gamma + \mu\Gamma'$ is of degree 6; therefore, $C + \lambda C'$ intersects $\Gamma + \mu\Gamma'$ in 18 points, which divide themselves into two sets of 9, the images of one another in the axis of x. We thus obtain two nonagons in perspective, the flex at infinity on the axis of y being the centre of perspective. Dealing thus in succession with each flex and harmonic line, we see that $C + \lambda C'$ *intersects* $\Gamma + \mu\Gamma'$ *in two nonagons nonuply in perspective, the centres of perspective being the nine flexes.*

If we take $2\pi_1$, $2\pi_2$, $2\pi_3$ to be the periods of the Weierstrassian Elliptic Functions, the parameters of the flexes of the cubic $x = \wp(u)$, $y = \wp'(u)$ are as usual:

$$\begin{array}{ccc} 0 & \dfrac{2\pi_1}{3} & \dfrac{4\pi_1}{3} \\[2ex] \dfrac{2\pi_2}{3} & \dfrac{2\pi_1}{3} + \dfrac{2\pi_2}{3} & \dfrac{4\pi_1}{3} + \dfrac{2\pi_2}{3} \\[2ex] \dfrac{4\pi_2}{3} & \dfrac{2\pi_1}{3} + \dfrac{4\pi_2}{3} & \dfrac{4\pi_1}{3} + \dfrac{4\pi_2}{3} \end{array} \qquad (1)$$

Now if we take u as the parameter of any point on $C+\lambda C'$ and join u to each of the flexes in turn, we get as one nonagon of the system the third points of intersection of these chords respectively

$$-u,\ -u-\frac{2\pi_1}{3},\ -u-\frac{4\pi_1}{3},\ -u-\frac{2\pi_2}{3},\ -u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right),$$
$$-u-\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right),\ -u-\frac{4\pi_2}{3},\ -u-\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right),\ -u-\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right)$$

The other nonagon with which the above is nonuply in perspective may easily be found by joining the above points to any one flex (say 0) and taking the points in which these chords cut the cubic again.

We obtain:

$$u,\ u+\frac{2\pi_1}{3},\ u+\frac{4\pi_1}{3},\ u+\frac{2\pi_2}{3},\ u+\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right),\ u+\left(\frac{4\pi_1}{3}+\frac{2\pi_2}{3}\right),$$
$$u+\frac{4\pi_2}{3},\ u+\left(\frac{2\pi_1}{3}+\frac{4\pi_2}{3}\right),\ u+\left(\frac{4\pi_1}{3}+\frac{4\pi_2}{3}\right).$$

We may represent the collineations diagrammatically as follows: where a point in the upper line joined to a point in the same column passes through the flex in the same horizontal row. Thus $u+\frac{4\pi_2}{3}$ from the upper line joined to $-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$ from the same column passes through the flex $\frac{2\pi_1}{3}+\frac{4\pi_2}{3}$ in the same horizontal line as $-u-\left(\frac{2\pi_1}{3}+\frac{2\pi_2}{3}\right)$. (See subjoined diagram (2)).

The way in which the points of the above scheme replace one another may be more easily seen if we use the points themselves instead of their parameters. Thus let the flexes be

$$\begin{matrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{matrix} \qquad (3)$$

instead of (1), where three flexes are collinear if the letters are the same (*e.g.* $G_1G_2G_3$), or if the suffixes are the same (*e.g.* $F_2G_2H_2$), or if the letters and suffixes are all different (*e.g.* $F_3G_2H_1$). The scheme (2) then becomes

https://doi.org/10.1017/S0013091500034180 Published online by Cambridge University Press

P_1	P_2	P_3	Q_1	Q_2	Q_3	R_1	R_2	R_3	Flexes
X_1	X_2	X_3	Y_1	Y_2	Y_3	Z_1	Z_2	Z_3	F_1
X_2	X_3	X_1	Y_2	Y_3	Y_1	Z_2	Z_3	Z_1	F_2
X_3	X_1	X_2	Y_3	Y_1	Y_2	Z_3	Z_1	Z_2	F_3
Y_1	Y_2	Y_3	Z_1	Z_2	Z_3	X_1	X_2	X_3	G_1
Y_2	Y_3	Y_1	Z_2	Z_3	Z_1	X_2	X_3	X_1	G_2
Y_3	Y_1	Y_2	Z_3	Z_1	Z_2	X_3	X_1	X_2	G_3
Z_1	Z_2	Z_3	X_1	X_2	X_3	Y_1	Y_2	Y_3	H_1
Z_2	Z_3	Z_1	X_2	X_3	X_1	Y_2	Y_3	Y_1	H_2
Z_3	Z_1	Z_2	X_3	X_1	X_2	Y_3	Y_1	Y_2	H_3

(4)

If two points of the above system coincide, it is plain that all the others will coincide in pairs. For let the coincident pair not lie on the axis of x. Then it is plain that their images with respect to Ox will also coincide. If we use the property of symmetry with respect to all the harmonic lines as explained in the beginning of the paper, we see that we obtain nine pairs of coincident points. With respect to the particular harmonic line Ox, four of these coincident pairs must lie above Ox and four must be their images below. The ninth point must lie on Ox, and the tangent thereat to the cubic curve must be parallel to Oy, that is, must pass through the flex corresponding to that particular harmonic line. Thus, *when the eighteen points coincide in pairs, the tangent at each of the resulting nine points passes through one or other of the flexes. If the tangent at one of these points pass through the flex* F, *the other eight points lie two by two on four chords passing through* F.

We shall show that *the cross-ratio of the above four chords passing through* F *is equi-anharmonic.*

Let the point of contact of the tangent through F be T. Let p be a chord drawn from F, and let the two points in which p cuts the cubic be joined to T and cut the curve again. It can easily be shown that the chord p' joining these latter two points passes through F. Hence a one-to-one algebraic correspondence exists between p and p', and hence the cross-ratio of any four positions of p is equal to that of the corresponding four positions of p'. Take the four positions of p on which lie the flexes other than F. We thus get a pencil of four lines whose cross-ratio is known to be equi-anharmonic. (5)

Let the flex F have 0 as its parameter.

Let T have as its parameter π_1.

Then if p_1 cut the cubic in the flexes $\frac{2\pi_1}{3}$ and $-\frac{2\pi_1}{3}$, plainly p_1' cuts the cubic in the points $-\pi_1 - \frac{2\pi_1}{3}$ and $-\pi_1 + \frac{2\pi_1}{3}$, and so on.

If we examine the scheme (2) and substitute throughout π_1 instead of u, we see that p_1', p_2', p_3', p_4' cut the cubic in the four pairs of coincident points required.

Hence $\quad [p_1', p_2', p_3', p_4'] \equiv [p_1 p_2 p_3 p_4]$

$\equiv$ equi-anharmonic ratio (by (5)).

Examples of cases of two coincident nonagons are :—

"The Hessian touches the Cayleyan at the points where the inflexional tangents meet their respective harmonic lines."

If we isolate any one of the points in which the inflexional tangents meet their respective harmonic lines, the other eight lie two by two on four lines passing through that isolated flex, and the cross-ratio of this pencil is equi-anharmonic.

"The cusps of a class-cubic lie on a cubic curve having the same system of flexes and harmonic lines."

If we isolate any one of the cusps of a class-cubic, the other eight cusps lie two by two on four lines passing through the harmonic point of that isolated cuspidal tangent, and the cross-ratio of this pencil is equi-anharmonic.

If C be a nodal cubic, C′ is also a nodal cubic having the same nodal tangents and flexes. Again Γ and Γ' may be considered as a system of conics touching the nodal tangents at the points H and K, in which they meet the common line of flexes. Project H and K into the Circular Points at Infinity. We thus see that $\Gamma + \mu\Gamma'$ is a system of concentric circles, having the node as their common centre, cutting $C + \lambda C'$ in sets of six points forming the vertices of a hexagon consisting of two equilateral triangles. The Hessian touches the Cayleyan at three points forming the vertices of an equilateral triangle.

https://doi.org/10.1017/S0013091500034180 Published online by Cambridge University Press