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RESEARCH ARTICLE

Pressure of a dilute spin-polarized Fermi gas: Lower bound

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Abstract

We consider a dilute fully spin-polarized Fermi gas at positive temperature in dimensions $d \in \{1, 2, 3\}$. We show that the pressure of the interacting gas is bounded from below by that of the free gas plus, to leading order, an explicit term of order $a^d \rho^{2+2/d}$, where a is the p-wave scattering length of the repulsive interaction and ρ is the particle density. The results are valid for a wide range of repulsive interactions, including that of a hard core, and uniform in temperatures at most of the order of the Fermi temperature. A central ingredient in the proof is a rigorous implementation of the fermionic cluster expansion of Gaudin, Gillespie and Ripka (Nucl. Phys. A, 176.2 (1971), pp. 237–260).

1. Introduction

The study of dilute quantum gases [GPS08] has received much interest from the mathematical physics community in the recent decades. In particular, much work has been done pertaining to the ground state energies of both Fermi and Bose gases in the thermodynamic limit.

For Bose gases in 3 dimensions, the leading term of the ground state energy was first shown by Dyson [Dys57] as an upper bound and by Lieb–Yngvason [LY98] as a lower bound. The leading term depends only on the density and the s-wave scattering length of the interaction. More recently, the second order correction, known as the Lee–Huang–Yang correction, was shown [FS20; FS23; YY09]. Also, the 2-dimensional [FGJMO24; LY01] and 1-dimensional [Age23; ARS22] settings have been studied.

The fermionic setting has been similarly studied in the 3-dimensional [FGHP21; Gia23; Lau23; LS24a; LS24b; LSS05], 2-dimensional [LS24a; LS24b; LSS05] and 1-dimensional [Age23; ARS22; LS24b] case. For fermions, the spin is important. For nonzero spin, the leading correction to the energy of the free gas is similar to the leading term for bosons and depends only on the density and the s-wave scattering length of the interaction. For fully spin-polarized (i.e., effectively spin-0) fermions, the behavior is different. By the Pauli exclusion principle, the probability of two fermions of the same spin being close enough to interact is suppressed. As such, the leading correction to the energy of the free gas depends on the p-wave scattering length of the interaction instead and is much smaller for dilute gases, which makes its analysis significantly harder.

A natural question to consider is the extension of these results on the ground state energy to positive temperature. This has been done both for bosons [DMS20; HHNST23; MS20; Sei08; Yin10] and nonzero spin fermions [Sei06]. In this paper, we consider the extension for fully spin-polarized fermions. More precisely, we consider the problem of finding the pressure $\psi(\beta, \mu)$ at positive temperature $T = 1/\beta$ and chemical potential μ in the setting of a spin-polarized Fermi gas. We are interested in the dilute limit $a^d\overline{\rho} \ll 1$, where a denotes the *p-wave scattering length* of the interaction and $\overline{\rho}$ denotes the particle density. In this dilute limit, we show the lower bound in dimensions $d \in \{1, 2, 3\}$

$$\psi(\beta,\mu) \geq \psi_0(\beta,\mu) - c_d(\beta\mu) a^d \overline{\rho}^{2+2/d}(1+o(1)) \qquad \text{as } a^d\overline{\rho} \to 0,$$

for an explicit (temperature dependent) coefficient $c_d(\beta\mu)$. Here, ψ, respectively ψ_0, denotes the pressure of the interacting respectively noninteracting system at inverse temperature β and chemical potential μ.

As discussed in more details in Remark 1.6 below, the term $c_d(\beta\mu)a^d\overline{\rho}^{2+2/d}$ arises naturally from the two-body interaction and the fact that the two-body density vanishes quadratically for incident particles. In the low-temperature limit $\beta\mu \to \infty$, the coefficients $c_d(\beta\mu)$ converge to the corresponding zero-temperature constants [ARS22; LS24a; LS24b]. The temperature dependence of this term can then be understood via the temperature dependence of the two-particle density of the free state.

The result is valid for temperatures T at most of the order of the Fermi temperature $T_F \sim \overline{\rho}^{2/d}$ of the free gas. For larger temperatures, one should expect thermal effects to become larger than quantum effects, and thus the gas should behave more like a (high temperature) classical gas. The natural parameter capturing the temperature is the *fugacity* $z = e^{\beta\mu}$. In terms of the fugacity, the constraint that the temperature satisfies $T \lesssim T_F$ reads $z \gtrsim 1$.

In contrast, for nonzero spin fermions, the pressure in the dilute limit is in 3 dimensions [Sei06]

$$\psi(\beta,\mu) = \psi_0(\beta,\mu) - 4\pi\left(1 - q^{-1}\right) a_s \overline{\rho}^2(1+o(1)) \qquad \text{as } a_s^3\overline{\rho} \to 0,$$

with ψ and ψ_0 the pressures of the interacting, respectively noninteracting, system, $q \geq 2$ the number of spin sectors and a_s the s-wave scattering length of the interaction. Notably, here the coefficient $4\pi(1 - q^{-1})$ does not depend on the temperature.

Our method of proof is split in two cases depending on the temperature. For sufficiently small temperatures, the result follows by a simple comparison to the zero-temperature setting and using the result of [LS24b]. In the more interesting case of higher temperatures, our method of proof consists of computing the pressure of a Jastrow-type trial state using a rigorous implementation [Lau23; LS24b] (given in Lemma 4.4) of the fermionic cluster expansion of Gaudin–Gillespie–Ripka [GGR71]. (More precisely in [Lau23; LS24b], we found conditions under which the formulas of [GGR71] are convergent.) A similar method was employed in the zero-temperature setting [LS24b], with the important difference that, because of the smoothness of the momentum distribution, the condition for convergence we obtain at positive (not too small) temperature is uniform in the volume (see Theorem 4.3). Thus, we can compute the thermodynamic limit directly, without appealing to a box method of localizing a trial state into large but finite boxes as done in [LS24b].

1.1. Precise statement

To state our main theorem precisely, define the (spin-polarized) fermionic Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} L_a^2([0,L]^{dn};\mathbb{C}) = \bigoplus_{n=0}^{\infty} \bigwedge^n L^2([0,L]^d;\mathbb{C})$. On this space, we define the free Hamiltonian $\mathcal{H}$, the number operator $\mathcal{N}$ and interaction operator $\mathcal{V}$ as follows (in natural units where $\frac{\hbar}{2m} = 1$):

$$\begin{aligned}
\mathcal{H} &= (0, H_1, \ldots, H_n, \ldots), \qquad & H_n &= \sum_{j=1}^{n} -\Delta_{x_j}, \\
\mathcal{N} &= (0, 1, \ldots, n, \ldots), \\
\mathcal{V} &= (0, 0, V_2, \ldots, V_n, \ldots), \qquad & V_n &= \sum_{1 \leq i < j \leq n} v(x_i - x_j).
\end{aligned}$$

The interacting Hamiltonian is then $\mathcal{H} + \mathcal{V}$. In the calculations below, we will use periodic boundary conditions for convenience. The pressure does not depend on the choice of boundary conditions [Rob71], and hence, we are free to choose the most convenient ones. We are interested in determining the pressure of the system described by this Hamiltonian at inverse temperature β and chemical potential μ. We denote this by

$$\psi(\beta,\mu) = \lim_{L\to\infty} \sup_{\Gamma} P[\Gamma], \qquad -L^d P[\Gamma] = \mathrm{Tr}_{\mathcal{F}}[(\mathcal{H} - \mu\mathcal{N} + \mathcal{V})\Gamma] - \frac{1}{\beta} S(\Gamma),$$

where $S(\Gamma) = -\mathrm{Tr}\,\Gamma \log \Gamma$ is the entropy of the state Γ and $P[\Gamma]$ is the pressure functional. By *state* we mean a density matrix (i.e., a positive trace-class operator on $\mathcal{F}$ of unit trace). (We suppress from the notation the dependence on the dimension d and the length L.) We denote moreover by

$$\psi_0(\beta,\mu) = \lim_{L\to\infty} \sup_{\Gamma} P_0[\Gamma], \qquad -L^d P_0[\Gamma] = \mathrm{Tr}_{\mathcal{F}}[(\mathcal{H} - \mu\mathcal{N})\Gamma] - \frac{1}{\beta} S(\Gamma)$$

the pressure and pressure functional of the free gas. The supremum is a maximum and is achieved for the Gibbs state

$$\Gamma = Z^{-1} \exp(-\beta(\mathcal{H} - \mu\mathcal{N})) = Z^{-1}(\Gamma_0, \Gamma_1, \ldots, \Gamma_n, \ldots), \qquad \Gamma_n = e^{\beta\mu n} e^{-\beta H_n}. \tag{1.1}$$

Then [Hua87, Equation (8.63)]

$$\begin{aligned}\psi_0(\beta,\mu) &= \lim_{L\to\infty} \frac{1}{L^d}\left[-\mathrm{Tr}_{\mathcal{F}}[(\mathcal{H} - \mu\mathcal{N})\Gamma] + \frac{1}{\beta} S(\Gamma)\right] = \lim_{L\to\infty} \frac{1}{L^d\beta} \log Z \\ &= \frac{1}{\beta(2\pi)^d} \int_{\mathbb{R}^d} \log\left(1 + e^{\beta\mu - \beta|k|^2}\right) \mathrm{d}k.\end{aligned} \tag{1.2}$$

To state our main theorem, we moreover define the *p-wave scattering length a*. (See also [LY01, Appendix A] and [SY20, Equations (2.9), (4.3)].)

Definition 1.1 [LS24b, Definitions 1.1, 1.9 and 1.11]. The *p-wave scattering length a* of the interaction v in dimension d is defined by

$$c_d a^d = \inf\left\{\int_{\mathbb{R}^d} \left(|\nabla f_0(x)|^2 + \frac{1}{2} v(x) f_0(x)^2\right)|x|^2 \,\mathrm{d}x : f_0(x) \to 1 \text{ for } |x| \to \infty\right\},$$

where

$$c_d = \begin{cases} 12\pi & d = 3, \\ 4\pi & d = 2, \\ 2 & d = 1. \end{cases} \tag{1.3}$$

The minimizer f_0 is the *p-wave scattering function*. (If $v(x) = +\infty$ for some x [for instance if v has a hard core, $v(x) = +\infty$ for $|x| < R_0$], we interpret $v(x)\,\mathrm{d}x$ as a measure. We suppress from the notation the dependence of a and f_0 on the dimension d.)

The dimensionless parameter measuring the diluteness is then $a^d\bar{\rho}$, with $\bar{\rho}$ the particle density[1] (in infinite volume) given by $\bar{\rho} = \partial_\mu \psi(\beta,\mu)$. We are interested in a dilute limit, meaning that $a^d\bar{\rho} \ll 1$.

[1]For the sake of simplicity of notation, we assume that the derivative $\partial_\mu\psi(\beta,\mu)$ exists. The function $\psi(\beta,\mu)$ being convex in μ always has left and right derivatives. Should these not coincide, we can just replace instances of $\partial_\mu\psi(\beta,\mu)$ with either the left or right derivative.

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Moreover, we are considering temperatures $T \lesssim T_F \sim \overline{\rho}^{2/d}$, meaning that $z \gtrsim 1$. As mentioned in the introduction, small z corresponds to a (high-temperature) classical gas.

We shall prove the following theorem.

Theorem 1.2. *Let $v \geq 0$ be radial and of compact support. If $d = 1$, assume moreover that $\int \left(|\partial f_0|^2 + \frac{1}{2} v f_0^2 \right) dx < \infty$. For any $z_0 > 0$, there exists $c > 0$ such that if $a^d \overline{\rho}_0 < c$, then, uniformly in $z = e^{\beta\mu} \geq z_0$, we have the lower bound*

$$\psi(\beta,\mu) \geq \psi_0(\beta,\mu) - 2\pi c_d \frac{-\operatorname{Li}_{d/2+1}(-z)}{(-\operatorname{Li}_{d/2}(-z))^{1+2/d}} a^d \overline{\rho}_0^{2+2/d} [1 + \delta_d],$$

where $\overline{\rho}_0 = \partial_\mu \psi_0(\beta,\mu)$ is the particle density of the free gas (in infinite volume), the constants c_d are defined in Equation (1.3) and

$$|\delta_d| \leq \begin{cases} C(a^3\overline{\rho}_0)^{1/39} \left|\log a^3\overline{\rho}_0\right|^{12/13} & d = 3, \\ C(a^2\overline{\rho}_0)^{1/5} \left|\log a^2\overline{\rho}_0\right|^{6/5} & d = 2, \\ C(a\overline{\rho}_0)^{1/7} \left|\log a\overline{\rho}_0\right|^{12/7} & d = 1. \end{cases} \tag{1.4}$$

Here, Li_s denotes the *polylogarithm*. It satisfies [NIS, Equation 25.12.16]

$$-\operatorname{Li}_s(-e^x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t-x}+1} \, \mathrm{d}t \tag{1.5}$$

with Γ the Gamma function.

We expect that the lower bound of Theorem 1.2 is, in fact, an equality (with a potentially different bound on the error term). It remains an open problem to prove this.

Remark 1.3. For better comparison with the zero-temperature result in [LS24b], we find it convenient to write the correction to the pressure of the free gas in terms of the particle density (of the free gas) $\overline{\rho}_0$. The latter is given explicitly as

$$\overline{\rho}_0 = -\frac{1}{(4\pi\beta)^{d/2}} \operatorname{Li}_{d/2}(-z). \tag{1.6}$$

This follows from an elementary computation, which we give in Lemma 3.6 below.

To leading order $\overline{\rho} \simeq \overline{\rho}_0$, more precisely,

Corollary 1.4. *Under the same assumptions as in Theorem 1.2, we have for the particle density*[2] *$\overline{\rho} = \partial_\mu \psi(\beta,\mu)$*

$$\overline{\rho} = \overline{\rho}_0 \left[1 + O((a^d\overline{\rho}_0)^{1/2}) \right].$$

We shall give the proof at the end of this section. In particular, the conditions of small $a^d\overline{\rho}$ and of small $a^d\overline{\rho}_0$ are equivalent. Moreover, the error terms of Theorem 1.2 can equally well be written with $\overline{\rho}_0$ replaced by $\overline{\rho}$.

Remark 1.5. The additional assumption on v in dimension $d = 1$ is discussed in [LS24b, Remark 1.13]. If v is either smooth or has a hard core (meaning that $v(x) = +\infty$ for $|x| \leq a_0$ for some $a_0 > 0$), this assumption is satisfied.

Remark 1.6. The term of order $a^d \overline{\rho}_0^{2+2/d}$ depends on the temperature. This is different from the setting of spin-$\frac{1}{2}$ fermions, where the analogous term (in 3 dimensions) is $2\pi a \overline{\rho}_0^2$ [Sei06] uniformly in the

[2] Should the left and right derivatives of $\psi(\beta,\mu)$ not coincide, the statement holds for either derivative.

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temperature. That the term of order $a^d\overline{\rho}_0^{2+2/d}$ should depend on the temperature may be heuristically understood as follows: This term arises from the fact that the two-body density vanishes quadratically for incident particles. The rate at which it vanishes depends on the exact state, and thus the temperature. Concretely, the two-particle density of the free gas (in infinite volume) satisfies

$$\overline{\rho}^{(2)}(x_1,x_2) = 2\pi \frac{-\operatorname{Li}_{d/2+1}(-z)}{(-\operatorname{Li}_{d/2}(-z))^{1+2/d}}\overline{\rho}_0^{2+2/d}|x_1-x_2|^2\left[1+O\left(\overline{\rho}_0^{2/d}|x_1-x_2|^2\right)\right], \tag{1.7}$$

where $O\left(\overline{\rho}_0^{2/d}|x_1-x_2|^2\right)$ is understood as being bounded by $C\overline{\rho}_0^{2/d}|x_1-x_2|^2$ uniformly. This follows from an elementary computation, which we give in Lemma 3.6 below.

In the low-temperature limit $z\to\infty$, we recover the zero-temperature constants in the terms of order $a^d\overline{\rho}_0^{2+2/d}$. The zero-temperature results read [LS24b, Theorems 1.3, 1.10, 1.12]

$$e(\overline{\rho}_0) \le e_0(\overline{\rho}_0) + c_{0,d}a^d\overline{\rho}_0^{2+2/d}[1+\delta_d],$$

with $e(\overline{\rho}_0), e_0(\overline{\rho}_0)$ denoting the ground state energy density of the interacting, respectively the free, gas and

$$c_{0,d} = \begin{cases} \frac{12\pi}{5}(6\pi^2)^{2/3} & d=3,\\ 4\pi^2 & d=2,\\ \frac{2\pi^2}{3} & d=1,\end{cases} \qquad |\delta_d| \lesssim \begin{cases} a^2\overline{\rho}^{2/3} & d=3,\\ a^2\overline{\rho}_0|\log a^2\overline{\rho}_0|^2 & d=2,\\ (a\overline{\rho}_0)^{13/17} & d=1.\end{cases} \tag{1.8}$$

Indeed, we claim that

$$2\pi c_d \frac{-\operatorname{Li}_{d/2+1}(-z)}{(-\operatorname{Li}_{d/2}(-z))^{1+2/d}} = c_{0,d} + O((\log z)^{-2}) \quad \text{as } z\to\infty. \tag{1.9}$$

To see this, write (following [Woo92])

$$\begin{aligned}-\operatorname{Li}_s(-e^x) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^{t-x}+1}\,\mathrm{d}t = \frac{1}{\Gamma(s)}\left[\int_0^x t^{s-1}\,\mathrm{d}t - \int_0^x \frac{t^{s-1}}{e^{x-t}+1}\,\mathrm{d}t + \int_x^\infty \frac{t^{s-1}}{e^{t-x}+1}\,\mathrm{d}t\right] \\ &= \frac{x^s}{\Gamma(s+1)} - \frac{1}{\Gamma(s)}\int_0^x \frac{(x-u)^{s-1}-(x+u)^{s-1}}{e^u+1}\,\mathrm{d}u - \frac{1}{\Gamma(s)}\int_x^\infty \frac{(x+u)^{s-1}}{e^u+1}\,\mathrm{d}u,\end{aligned}$$

where we changed variables $t = x\pm u$. The middle and last integrals can easily be bounded as $O(x^{s-2})$ and $O(x^s e^{-x})$, respectively. Thus,

$$-\operatorname{Li}_s(-e^x) = \frac{x^s}{\Gamma(s+1)} + O(x^{s-2}), \tag{1.10}$$

and Equation (1.9) follows.

Remark 1.7. The error bounds in Theorem 1.2 are uniform in z. They arise as the worst cases of two types of bounds, one good for $z\sim 1$ and one good for $z\gg 1$. In particular, for concrete values of z, the error bounds can be improved. See Propositions 1.8 and 1.9 below.

Finally, we give the following:

Proof of Corollary 1.4. Note that $\psi(\beta,\mu)$ is a convex function of μ. Thus, we may bound its derivative by any difference quotient. More precisely, for any $\varepsilon>0$, we have

$$\overline{\rho} = \partial_\mu\psi(\beta,\mu) \le \frac{\psi(\beta,\mu+\varepsilon)-\psi(\beta,\mu)}{\varepsilon}.$$

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Using the trivial upper bound $\psi(\beta,\mu+\varepsilon) \le \psi_0(\beta,\mu+\varepsilon)$ (which is a consequence of the assumed non-negativity of the interaction potential v) and the lower bound of Theorem 1.2, we conclude that

$$\overline{\rho} \le \frac{\psi_0(\beta,\mu+\varepsilon)-\psi_0(\beta,\mu)}{\varepsilon} + Ca^d\overline{\rho}_0^{2+2/d}\varepsilon^{-1} = \overline{\rho}_0 + O\left(|\partial_\mu^2\psi_0|\varepsilon\right) + O\left(a^d\overline{\rho}_0^{2+2/d}\varepsilon^{-1}\right).$$

Using the explicit formula for $\overline{\rho}_0 = \partial_\mu\psi_0$ and optimizing in ε, we get that $\overline{\rho} \le \overline{\rho}_0(1+O((a^d\overline{\rho}_0)^{1/2}))$. For $\varepsilon < 0$, the argument is analogous only the direction of the inequalities is reversed. □

1.2. *Strategy of the proof*

To prove Theorem 1.2, we distinguish two cases: that of a 'low-temperature' setting and that of a 'high-temperature' setting. For sufficiently small temperatures, we compare to the ground state energy studied in [LS24b]. For larger temperatures, we consider a specific trial state Γ_J of Jastrow-type (defined in Equation (3.1) below) and compute the pressure functional evaluated on this trial state. For these computations, we use a rigorous implementation [Lau23; LS24b] of the formal cluster expansion of Gaudin–Gillespie–Ripka [GGR71].

Temperature-dependent errors naturally arise as powers of $\zeta := 1 + |\log z|$. We shall prove the following propositions.

Proposition 1.8. *Let $v \ge 0$ be radial and of compact support. If $d = 1$, assume moreover that $\int\left(|\partial f_0|^2 + \frac{1}{2}vf_0^2\right)dx < \infty$. Then for sufficiently small $a^d\overline{\rho}_0$ and large $z = e^{\beta\mu}$, we have*

$$\psi(\beta,\mu) \ge \psi_0(\beta,\mu) - 2\pi c_d \frac{-\mathrm{Li}_{d/2+1}(-z)}{(-\mathrm{Li}_{d/2}(-z))^{1+2/d}} a^d\overline{\rho}_0^{2+2/d}[1+\delta_d], \tag{1.11}$$

where $\overline{\rho}_0$ is the particle density of the free gas, c_d is defined in Equation (1.3) and

$$|\delta_d| \lesssim \begin{cases} a^2\overline{\rho}_0^{2/3} & + (a^3\overline{\rho}_0)^{-1}\zeta^{-2} \quad d=3,\\ a^2\overline{\rho}_0|\log a^2\overline{\rho}_0|^2 & + (a^2\overline{\rho}_0)^{-1}\zeta^{-2} \quad d=2,\\ (a\overline{\rho}_0)^{13/17} & + (a\overline{\rho}_0)^{-1}\zeta^{-2} \quad d=1.\end{cases} \tag{1.12}$$

Proposition 1.9. *Let $v \ge 0$ be radial and of compact support. If $d = 1$, assume moreover that $\int\left(|\partial f_0|^2 + \frac{1}{2}vf_0^2\right)dx < \infty$. Then for $z = e^{\beta\mu}$ satisfying $z \gtrsim 1$, there exists a constant $c > 0$ such that if $a^d\overline{\rho}_0 < c$ and $a^d\overline{\rho}_0\zeta^{d/2}|\log a^d\overline{\rho}_0| < c$, then*

$$\psi(\beta,\mu) \ge \psi_0(\beta,\mu) - 2\pi c_d \frac{-\mathrm{Li}_{d/2+1}(-z)}{(-\mathrm{Li}_{d/2}(-z))^{1+2/d}} a^d\overline{\rho}_0^{2+2/d}[1+\delta_d],$$

where $\overline{\rho}_0$ is the particle density of the free gas, c_d is defined in Equation (1.3) and

$$|\delta_d| \lesssim \begin{cases} (a^3\overline{\rho}_0)^{6/15}\zeta^{-3/5} + (a^3\overline{\rho}_0)\zeta^{1/2}|\log a^3\overline{\rho}_0|^2 + (a^3\overline{\rho}_0)^{7/3}\zeta^{9/2}|\log a^3\overline{\rho}_0|^3 & d=3,\\ (a^2\overline{\rho}_0)^{1/2}\zeta^{-1/2} + (a^2\overline{\rho}_0)\zeta|\log a^2\overline{\rho}_0| + (a^2\overline{\rho}_0)^2\zeta^3|\log a^2\overline{\rho}_0|^3, & d=2,\\ (a\overline{\rho}_0)^{1/2}|\log a\overline{\rho}_0|^{1/2} + a\overline{\rho}_0\zeta^{3/2}|\log a\overline{\rho}_0|^3 & d=1.\end{cases} \tag{1.13}$$

Proposition 1.8 is a simple corollary of [LS24b, Theorems 1.3, 1.10, 1.13], extending the result to small positive temperatures. Proposition 1.9 is the main new result of this paper. Most of the rest of the paper is concerned with the proof of Proposition 1.9. Theorem 1.2 is an immediate consequence:

Proof of Theorem 1.2. We use the lower bound in Proposition 1.8 for

$$\zeta \geq \zeta_0 := \begin{cases} (a^3\overline{\rho}_0)^{-20/39}\big|\log a^3\overline{\rho}_0\big|^{-6/13} & d=3, \\ (a^2\overline{\rho}_0)^{-3/5}\big|\log a^2\overline{\rho}_0\big|^{-3/5} & d=2, \\ (a\overline{\rho}_0)^{-4/7}\big|\log a\overline{\rho}_0\big|^{-6/7} & d=1 \end{cases}$$

and the lower bound in Proposition 1.9 otherwise. Theorem 1.2 follows. □

We note that for $\zeta \sim \zeta_0$, the last of the summands in Equation (1.13) (in all dimensions) dominate the error term in Proposition 1.9.

Remark 1.10. The proof of Proposition 1.9 uses the Gaudin–Gillespie–Ripka expansion. This expansion consists of formulas for the normalization constant Z_J (defined in Equation (3.1) below) and the reduced densities of the state Γ_J; see Theorem 4.3. Both Z_J and the reduced densities are given as infinite series of diagrams (defined in Definition 4.1). Using these formulas, the 'smallest' diagrams give the corrections of Proposition 1.9 and the remaining diagrams are error terms. To bound the error terms, we calculate the values of (finitely many) 'small' diagrams and give crude bounds for all (infinitely many) 'larger' diagrams.

Remark 1.11. We expect that with the method presented here, one could improve the error bounds in Proposition 1.9 (and consequently Theorem 1.2) slightly by treating more diagrams in the Gaudin–Gillespie–Ripka expansion as small (i.e., calculating their values more precisely). See also [LS24b, Remark 1.8]. This is similar to what is done in [BCGOPS23; Lau23]. (In [BCGOPS23], the hard core Bose gas is treated with a method similar to a cluster expansion. Using such an expansion to sufficiently high order proves the bounds of [BCGOPS23].)

More precisely, we expect that by treating more diagrams as small, one could improve the bounds in Proposition 1.9 to

$$|\delta_d| \lesssim O\left(\begin{cases} (a^3\overline{\rho}_0)^{6/15}\zeta^{-3/5} & d=3 \\ (a^2\overline{\rho}_0)^{1/2}\zeta^{-1/2} & d=2 \\ (a\overline{\rho}_0)^{1/2}\big|\log a\overline{\rho}_0\big|^{1/2} & d=1 \end{cases}\right) + O\left((a^d\overline{\rho}_0)^{-2/d}\left(a^d\overline{\rho}_0\zeta^{d/2}\big|\log a^d\overline{\rho}_0\big|\right)^n\right) \tag{1.14}$$

for any n. This would then propagate to better error terms in Theorem 1.2. More precisely, by using the bound in Proposition 1.8 for $\zeta \geq \zeta_0$ and the bound in Proposition 1.9 with error improved as in Equation (1.14) otherwise and optimising in ζ_0, one would improve the error bound in Theorem 1.2 to

$$|\delta_d| \lesssim \begin{cases} C_\varepsilon(a^3\overline{\rho}_0)^{1/3-\varepsilon} & d=3, \\ (a^2\overline{\rho}_0)^{1/2} & d=2, \\ (a\overline{\rho}_0)^{1/2}\big|\log a\overline{\rho}_0\big|^{1/2} & d=1 \end{cases}$$

for any $\varepsilon > 0$, where C_ε depends on ε, by taking n sufficiently large in Equation (1.14).

The first terms in Equation (1.14) come from the precise evaluation of certain small diagrams. In dimension $d = 2, 3$, one should not expect to get better bounds than this using the method presented here. In dimension $d = 1$, one might be able to do a more precise analysis (see Remark 5.6) and thus improve the bound.

The proof of Proposition 1.8 will be given in Section 2. It is mostly independent of the rest of the paper (Sections 3, 4 and 5), which is devoted to the proof of Proposition 1.9.

Structure of the paper:
First, in Section 2, we give the proof of Proposition 1.8. Then, in Section 3, we define the trial state Γ_J and give some preliminary computations. Next, in Section 4, we compute reduced densities of the trial state Γ_J using the (rigorous implementation of the) Gaudin–Gillespie–Ripka expansion. Finally,

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in Section 5, we calculate the individual terms in the pressure functional and prove Proposition 1.9. In Appendix A, we show that Γ_J has particle density $\approx \overline{\rho}_0$.

2. Low temperature

In this section, we prove Proposition 1.8 by comparing to the zero-temperature problem.

Proof of Proposition 1.8. The pressures ψ, ψ_0 (of the interacting and noninteracting gas, respectively) are the Legendre transforms of the corresponding free energy densities ϕ, ϕ_0. That is,

$$\begin{aligned} \psi(\beta,\mu) &= \sup_{\tilde\rho}[\tilde\rho\mu - \phi(\beta,\tilde\rho)] \geq \overline{\rho}_0\mu - \phi(\beta,\overline{\rho}_0) \\ \psi_0(\beta,\mu) &= \sup_{\tilde\rho}[\tilde\rho\mu - \phi_0(\beta,\tilde\rho)] = \overline{\rho}_0\mu - \phi_0(\beta,\overline{\rho}_0) \end{aligned} \tag{2.1}$$

with $\overline{\rho}_0$ the density of the free gas at chemical potential μ and inverse temperature β, given in Equation (1.6). We may trivially bound the free energy density by the ground state energy density e. The latter is bounded from above in [LS24b, Theorems 1.3, 1.10 and 1.13]. That is,

$$\phi(\beta,\overline{\rho}_0) \leq e(\overline{\rho}_0) \leq e_0(\overline{\rho}_0) + c_{0,d}a^d\overline{\rho}_0^{2+2/d}[1+\delta_d], \tag{2.2}$$

with $e_0(\overline{\rho}_0)$ denoting the ground state energy density of the free gas and $c_{0,d}$ and δ_d as in Equation (1.8). By a straightforward calculation, the ground state energy density of the free gas is

$$e_0(\overline{\rho}_0) = 4\pi\frac{d^{2/d}}{d+2}\left(\frac{d}{2}\right)^{2/d}\Gamma(d/2)^{2/d}\overline{\rho}_0^{1+2/d}.$$

By Equations (1.2), (1.6) and (1.10), we have for large $z = e^{\beta\mu}$ (see also [Hua87, Equation (11.31)]),

$$\begin{aligned} \psi_0(\beta,\mu) &= \beta^{-1-d/2}\frac{\left|\mathbb{S}^{d-1}\right|\Gamma(d/2)}{2(2\pi)^d}(-\operatorname{Li}_{d/2+1}(-e^{\beta\mu})) \\ &= 4\pi\overline{\rho}_0^{1+2/d}\frac{-\operatorname{Li}_{d/2+1}(-e^{\beta\mu})}{(-\operatorname{Li}_{d/2}(-e^{\beta\mu}))^{1+2/d}} = \frac{2}{d}e_0(\overline{\rho}_0) + O\left(\overline{\rho}_0^{1+2/d}(\beta\mu)^{-2}\right), \end{aligned}$$

where $\left|\mathbb{S}^{d-1}\right| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the area of the $(d-1)$-sphere. Thus,

$$\phi_0(\beta,\overline{\rho}_0) = \overline{\rho}_0\mu - \psi_0(\beta,\mu) = e_0 + O\left(\overline{\rho}_0^{1+2/d}(\beta\mu)^{-2}\right).$$

Combining this with Equations (2.1) and (2.2), we conclude the proof of Proposition 1.8. □

The rest of the paper concerns the proof of Proposition 1.9. We start with some preliminary computations.

3. Preliminaries

To prove Proposition 1.9, we will consider a finite system on a cubic box of side length L with periodic boundary conditions and bound $\psi(\beta,\mu)$ from below by the pressure functional evaluated on the trial state

$$\Gamma_J = \frac{Z}{Z_J}F\Gamma F, \qquad F = \bigoplus_{n=0}^{\infty}F_n, \qquad F_n = \prod_{1\leq i<j\leq n}f(x_i - x_j), \tag{3.1}$$

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where f is some cut-off and rescaled scattering function defined in Equation (3.2) below, where Γ is defined in Equation (1.1), and where Z_J is such that this is normalized with $\operatorname{Tr}\Gamma_J = 1$. Concretely, on the n-particle space, Γ_J acts via the kernel

$$Z_J^{-1} F_n(X_n)\Gamma_n(X_n, Y_n)F_n(Y_n).$$

(Recall that Γ acts via the kernel $Z^{-1}\Gamma_n(X_n, Y_n)$.) The function f is more precisely

$$f(x) = \begin{cases} \frac{1}{1-a^d/b^d} f_0(x) & |x| \le b \\ 1 & |x| \ge b, \end{cases} \tag{3.2}$$

where $f_0(x)$ is the p-wave scattering function defined in Definition 1.1 and b is a length to be chosen later. We will choose $a \ll b \le C\rho_0^{-1/d}$. Here and in the following, ρ_0 denotes the particle density of the free gas in finite volume. In particular, for $a^d\rho_0$ small enough, b is larger than the range of v and so f is continuous (since $f_0(x) = 1 - \frac{a^3}{|x|^3}$ for x outside the support of v).

Notation 3.1.

- We will denote expectation values of operators in the free state Γ by $\langle\cdot\rangle_0$ and in the trial state Γ_J by $\langle\cdot\rangle_J$. That is, $\langle\mathcal{A}\rangle_0 = \operatorname{Tr}_{\mathcal{F}}[\mathcal{A}\Gamma]$ and $\langle\mathcal{A}\rangle_J = \operatorname{Tr}_{\mathcal{F}}[\mathcal{A}\Gamma_J]$ for any operator $\mathcal{A}$ on $\mathcal{F}$.
- We denote $g(x) = f(x)^2 - 1$.
- For any function h, we write $h_e = h_{ij} = h(x_i - x_j)$ for an edge $e = (i, j)$.
- Moreover, we write $\gamma_e^{(1)} = \gamma_{ij}^{(1)} = \gamma^{(1)}(x_i; x_j)$ for an edge $e = (i, j)$, where $\gamma^{(1)}$ is the 1-particle density matrix of Γ defined in Equation (3.4) below (see also Notation 3.3).
- We write $X_n = (x_1, \ldots, x_n)$ and $X_{[n,m]} = (x_n, \ldots, x_m)$ if $n \le m$. If $n > m$, then $X_{[n,m]} = \varnothing$.

Remark 3.2. The trial state Γ_J does not have (average) particle density ρ_0. However, we have that

$$\frac{1}{L^d}\langle\mathcal{N}\rangle_J = \rho_0\Big(1 + O(a^d b^2 \rho_0^{1+2/d}) + O\Big((a^d\rho_0)^2\zeta^d(\log b/a)^2\Big)\Big). \tag{3.3}$$

This is not needed for the proof of Proposition 1.9, however. We give the proof of (3.3) in Appendix A.

We normalize q-particle density matrices of a general state $\tilde\Gamma = (\tilde\Gamma_0, \tilde\Gamma_1, \ldots)$ as

$$\gamma_{\tilde\Gamma}^{(q)}(X_q; Y_q) = \sum_{n=q}^{\infty} \frac{n!}{(n-q)!} \int \cdots \int \tilde\Gamma_n(X_q, X_{[q+1,n]}; Y_q, X_{[q+1,n]})\, \mathrm{d}X_{[q+1,n]}. \tag{3.4}$$

Notation 3.3. For the Gibbs state $\Gamma = (Z^{-1}\Gamma_0, Z^{-1}\Gamma_1, \ldots)$ and the trial state Γ_J, we denote their q-particle density matrices by $\gamma^{(q)}(X_q; Y_q) = \gamma_\Gamma^{(q)}(X_q; Y_q)$ and $\gamma_J^{(q)}(X_q; Y_q) = \gamma_{\Gamma_J}^{(q)}(X_q; Y_q)$, respectively. The same applies to the q-particle densities, being then denoted $\rho^{(q)}$ and $\rho_J^{(q)}$.

The Gibbs state Γ is quasi-free and particle preserving. Thus, by Wick's rule (see [BR97, Section 5.2.4], [Sol14, Theorem 10.2]), we have for the q-particle density

$$\rho^{(q)}(X_q) = \gamma^{(q)}(X_q; X_q) = \det\Big[\gamma_{ij}^{(1)}\Big]_{1\le i,j\le q}.$$

Moreover, by translation invariance, we have that $\gamma^{(1)}(x; y)$ is a function of $x - y$ only. With a slight abuse of notation, we then write

$$\gamma^{(1)}(x; y) = \gamma^{(1)}(x - y) = \frac{1}{L^d}\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d} \hat\gamma^{(1)}(k)e^{-ik(x-y)}.$$

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A simple calculation shows that (see [Hua87, Equation (8.65)])

$$\hat{\gamma}^{(1)}(k) = \frac{ze^{-\beta|k|^2}}{1+ze^{-\beta|k|^2}} = \frac{e^{\beta\mu-\beta|k|^2}}{1+e^{\beta\mu-\beta|k|^2}}.$$

For the proof of Proposition 1.9, we compute the pressure of the trial state Γ_J. We have

$$\begin{aligned}\psi(\beta,\mu) &\geq \limsup_{L\to\infty} \frac{1}{L^d}\left[-\langle \mathcal{H}-\mu\mathcal{N}+\mathcal{V}\rangle_J + \frac{1}{\beta}S(\Gamma_J)\right] \\ &= \limsup_{L\to\infty} \frac{1}{L^d}\left[-\langle \mathcal{H}\rangle_J - \mu\langle\mathcal{N}\rangle_J - \frac{1}{2}\iint v_{12}\rho_J^{(2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 + \frac{1}{\beta}S(\Gamma_J)\right],\end{aligned} \tag{3.5}$$

where $\rho_J^{(2)}$ is the two-body reduced density of the trial state Γ_J. We calculate $\rho_J^{(2)}$ in Section 4 using the Gaudin–Gillespie–Ripka expansion, and we compute the individual terms of Equation (3.5) in Section 5 below. First, however, we need some preliminary bounds.

3.1. Useful bounds

We recall some useful bounds on the scattering function (defined in Equation (3.2)) from [LS24b].

Lemma 3.4. *The scattering function f satisfies*

$$\int \left|1-f(x)^2\right||x|^n\,dx \leq \begin{cases} Ca^d\log b/a & n=0 \\ Ca^d b^n & n>0 \end{cases} \tag{3.6}$$

$$\int \left(|\nabla f(x)|^2 + \frac{1}{2}v(x)f(x)^2\right)|x|^2\,dx = c_d a^d\left(1+O(a^d/b^d)\right) \tag{3.7}$$

$$\int \left(|\nabla f(x)|^2 + \frac{1}{2}v(x)f(x)^2\right)|x|^n\,dx \leq \begin{cases} Ca^{n+d-2} & n+d\leq 2d+1 \\ Ca^{n+d-2}\log b/a & n+d=2d+2 \\ Ca^{2d}b^{n-d-2} & n+d\geq 2d+3 \end{cases} \tag{3.8}$$

$$\left|\int f(x)|\nabla f(x)||x|^n\,dx\right| \leq \begin{cases} Ca^{d-1} & n=0 \\ Ca^d\log b/a & n=1 \\ Ca^d b^{n-1} & n\geq 2, \end{cases} \tag{3.9}$$

where c_d is defined in Equation (1.3).

Proof. Equations (3.6), (3.7), (3.8) and (3.9) all follow from the definition of the scattering length, Definition 1.1, and the bounds [LY01, Lemma A.1; LS24b, Lemma 2.2]

$$\left[1-\frac{a^d}{|x|^d}\right]_+ \leq f_0(x) \leq 1, \qquad |\nabla f_0(x)| \leq \frac{da^d}{|x|^{d+1}} \quad \text{for } |x|>a,$$

where the left inequality in the first inequality is an equality for x outside the support of v. We refer to [LS24b, Equations (4.1) to (4.6)] for a detailed proof. ◻

We will need the following technical lemma.

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Lemma 3.5. *Let $\hat{\gamma}(k) = ze^{-\beta|k|^2}$. Let p, n, m be non-negative integers with $1 \leq n \leq m$. Then*

$$\frac{1}{L^d} \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} \frac{|k|^p \hat{\gamma}(k)^n}{(1+\hat{\gamma}(k))^m} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|k|^p \hat{\gamma}(k)^n}{(1+\hat{\gamma}(k))^m} \, dk + O\left(L^{-1}\beta \max\{\beta^{-1}, \mu\}^{\frac{p+d+1}{2}}\right)$$
$$\leq C \max\{\beta^{-1}, \mu\}^{\frac{p+d}{2}}$$

for $z = e^{\beta\mu} \gtrsim 1$ and L sufficiently large.

Note that $\hat{\gamma}(k) \neq \hat{\gamma}^{(1)}(k)$. In fact, $\hat{\gamma}^{(1)}(k) = \frac{\hat{\gamma}(k)}{1+\hat{\gamma}(k)}$.

Proof. We interpret the sum as a Riemann sum and compare it with its corresponding integral

$$I_{p,n,m} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|k|^p \hat{\gamma}(k)^n}{(1+\hat{\gamma}(k))^m} \, \mathrm{d}k.$$

Writing $F_{p,n,m}(k) = \frac{|k|^p \hat{\gamma}(k)^n}{(1+\hat{\gamma}(k))^m}$, then

$$\frac{1}{L^d} \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} \frac{|k|^p \hat{\gamma}(k)^n}{(1+\hat{\gamma}(k))^m} = \frac{1}{(2\pi)^d} \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} \int_{[-\frac{\pi}{L}, \frac{\pi}{L}]^d} \left(F_{p,n,m}(k+\xi) - \int_0^1 \partial_t F_{p,n,m}(k+t\xi) \, \mathrm{d}t\right) \mathrm{d}\xi.$$

The first term is the integral $I_{p,n,m}$. For the second term, we may bound by direct computation (defining $F_{p,n,m} = 0$ for $p < 0$)

$$\begin{aligned} \left|\partial_t F_{p,n,m}(k+t\xi)\right| &\leq C|\xi| \left[F_{p-1,n,m}(k+t\xi) + \beta F_{p+1,n,m}(k+t\xi)\right] \\ &\leq C|\xi| e^{C\beta|\xi||k+\xi| + C\beta|\xi|^2} \left[F_{p-1,n,m}(k+\xi) + \beta F_{p+1,n,m}(k+\xi)\right]. \end{aligned}$$

That is, the second term is bounded by the integral

$$CL^{-1} e^{CL^{-2}\beta} \int_{\mathbb{R}^d} e^{CL^{-1}\beta|k|} (F_{p-1,n,m}(k) + \beta F_{p+1,n,m}(k)) \, \mathrm{d}k.$$

Next, to bound the integral, we note that $F_{p,n,m} \leq F_{p,1,1}$. First, consider $z \geq e$ (i.e., $\beta\mu \geq 1$). Then we bound

$$\begin{aligned} \int_{\mathbb{R}^d} e^{CL^{-1}\beta|k|} F_{p,1,1}(k) \, \mathrm{d}k &\leq C \int_0^{\sqrt{2\mu}} e^{CL^{-1}\beta k} k^{p+d-1} \, \mathrm{d}k + C \int_{\sqrt{2\mu}}^\infty e^{CL^{-1}\beta k} k^{p+d-1} e^{-\beta(k^2-\mu)} \, \mathrm{d}k \\ &\leq C\mu^{\frac{p+d}{2}} e^{CL^{-1}\beta\mu^{1/2}} + C\beta^{-\frac{p+d}{2}} \int_{\sqrt{\beta\mu}}^\infty t^{p+d-1} e^{-t^2 + CL^{-1}\beta^{1/2} t} \, \mathrm{d}t \leq C\mu^{\frac{p+d}{2}} \end{aligned}$$

for L sufficiently large.

Next, for $z < e$, we bound

$$\begin{aligned} \int_{\mathbb{R}^d} e^{CL^{-1}\beta|k|} F_{p,1,1}(k) \, \mathrm{d}k &\leq C \int_0^\infty e^{CL^{-1}\beta k} k^{p+d-1} z e^{-\beta k^2} \, \mathrm{d}k \\ &\leq C\beta^{-\frac{p+d}{2}} \int_0^\infty t^{p+d-1} e^{-t^2 + CL^{-1}\beta^{1/2} t} \, \mathrm{d}t \leq C\beta^{-\frac{p+d}{2}} \end{aligned}$$

for L sufficiently large. The equality in the lemma follows. We may bound $I_{p,n,m}$ in a similar manner and conclude the proof of the lemma. □

Finally, we have the following lemma for the reduced densities of the free state.

Lemma 3.6. *The reduced densities of the free Fermi gas satisfy*

$$\rho^{(1)}(x_1) = \rho_0 = \frac{1}{(4\pi)^{d/2}}\beta^{-d/2}(-\mathrm{Li}_{d/2}(-z))\Big[1 + O(L^{-1}\zeta\rho_0^{-1/d})\Big], \tag{3.10}$$

$$\rho^{(2)}(x_1,x_2) = 2\pi\frac{-\mathrm{Li}_{d/2+1}(-z)}{(-\mathrm{Li}_{d/2}(-z))^{1+2/d}}\rho_0^{2+2/d}|x_1-x_2|^2\Big[1 + O(\rho_0^{2/d}|x_1-x_2|^2) + O(L^{-1}\zeta\rho_0^{-1/d})\Big]. \tag{3.11}$$

Equations (3.10) and (3.11) are the finite volume analogues of Equations (1.6) and (1.7).

Remark 3.7. Note that $\beta \sim \zeta\rho_0^{-2/d}$. (Recall that $\zeta = 1 + |\log z|$.) Indeed, for $z \le C$, this is clear from Equation (3.10). For $z \gg 1$, this follows from the asymptotics of the polylogarithm, Equation (1.10). Moreover, if $\beta\mu \ge 1$, then $\mu \sim \rho_0^{2/d}$. In particular then, Lemma 3.5 may be reformulated as

$$\frac{1}{L^d}\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d}\frac{|k|^p\hat\gamma(k)^n}{(1+\hat\gamma(k))^m} = \frac{1}{L^d}\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d}\frac{|k|^p z^n e^{-n\beta|k|^2}}{\left(1+ze^{-\beta|k|^2}\right)^m} \le C\rho_0^{1+p/d} \tag{3.12}$$

for $z \gtrsim 1$ and L sufficiently large. This is the form we will later use.

Proof. By translation invariance,

$$\rho_0 = \frac{\langle\mathcal{N}\rangle_0}{L^d} = \frac{1}{L^d}\int\rho^{(1)}(x)\,\mathrm{d}x = \rho^{(1)}(0).$$

Moreover, by Lemma 3.5,

$$\begin{aligned}\rho^{(1)}(0) &= \frac{1}{L^d}\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d}\frac{e^{\beta\mu-\beta|k|^2}}{1+e^{\beta\mu-\beta|k|^2}}\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{ze^{-\beta|k|^2}}{1+ze^{-\beta|k|^2}}\,\mathrm{d}k + O\Big(L^{-1}\beta\max\{\beta^{-1},\mu\}^{\frac{d+1}{2}}\Big)\\ &= \frac{\Gamma(d/2)\left|\mathbb{S}^{d-1}\right|}{2(2\pi)^d}\beta^{-d/2}(-\mathrm{Li}_{d/2}(-z))\Big(1+O\Big(L^{-1}\beta\max\{\beta^{-1},\mu\}^{1/2}\Big)\Big),\end{aligned}$$

where $\left|\mathbb{S}^{d-1}\right| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the $(d-1)$-sphere. Using that $\max\{\beta^{-1},\mu\} \sim \rho_0^{2/d}$ (which follow from this equation for L sufficiently large; see Remark 3.7), we conclude the proof of Equation (3.10).

Next, we consider the 2-particle density. By Wick's rule, we have

$$\rho^{(2)}(x_1,x_2) = \rho^{(1)}(x_1)\rho^{(1)}(x_2) - \gamma^{(1)}(x_1;x_2)\gamma^{(1)}(x_2;x_1).$$

By translation invariance, $\gamma^{(1)}(x_1;x_2)$ is a function of $x_1 - x_2$ only. We expand it as a Taylor series in $x_1 - x_2$. By symmetry of reflection in any of the axes, all odd orders and all off-diagonal second order terms vanish. Additionally, all second order terms are equal by the symmetry of permutation of the axes. That is,

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$$\begin{aligned}\gamma^{(1)}(x_1;x_2) &= \frac{1}{L^d}\sum_k \hat{\gamma}^{(1)}(k)e^{ik(x_1-x_2)}\\ &= \frac{1}{L^d}\sum_k \hat{\gamma}^{(1)}(k)\left[1-\frac{1}{2d}|k|^2|x_1-x_2|^2+O(|k|^4|x_1-x_2|^4)\right]\\ &= \rho_0 - \frac{1}{2d}\left[\frac{1}{L^d}\sum_k |k|^2\hat{\gamma}^{(1)}(k)\right]|x_1-x_2|^2+O\left(\left[\frac{1}{L^d}\sum_k |k|^4\hat{\gamma}^{(1)}(k)\right]|x_1-x_2|^4\right).\end{aligned}$$

(Here, $O(|k|^4|x_1-x_2|^4)$ means a term that is bounded by $|k|^4|x_1-x_2|^4$ uniformly in $|k|^4|x_1-x_2|^4$, even if it is large.) For the first sum, we have by and Equation (3.10) (and writing the error term in terms of ρ_0 as above)

$$\begin{aligned}\frac{1}{L^d}\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d}|k|^2\hat{\gamma}^{(1)}(k) &= \frac{1}{(2\pi)^d}\int \frac{ze^{-\beta|k|^2}}{1+ze^{-\beta|k|^2}}|k|^2\,\mathrm{d}k + O\left(L^{-1}\zeta\rho_0^{4/d}\right)\\ &= \frac{\Gamma(d/2+1)\left|\mathbb{S}^{d-1}\right|}{2(2\pi)^d}\beta^{-d/2-1}(-\mathrm{Li}_{d/2+1}(-z))\left(1+O(L^{-1}\zeta\rho_0^{-1/d})\right)\\ &= 2d\pi\frac{-\mathrm{Li}_{d/2+1}(-z)}{(-\mathrm{Li}_{d/2}(-z))^{1+2/d}}\rho_0^{1+2/d}\left(1+O(L^{-1}\zeta\rho_0^{-1/d})\right).\end{aligned}$$

Using again Lemma 3.5 to bound the second sum, we conclude that

$$\begin{aligned}\gamma^{(1)}(x_1;x_2) = \rho_0 - \pi\frac{-\mathrm{Li}_{d/2+1}(-z)}{(-\mathrm{Li}_{d/2}(-z))^{1+2/d}}\rho_0^{1+2/d}|x_1-x_2|^2\\ + O\left(L^{-1}\zeta\rho_0^{1+1/d}|x_1-x_2|^2\right)+O\left(\rho_0^{1+4/d}|x_1-x_2|^4\right).\end{aligned}$$

We conclude the proof of Equation (3.11). □

4. Gaudin–Gillespie–Ripka expansion

We use the Gaudin–Gillespie–Ripka (GGR) expansion [GGR71] to compute Z_J and $\rho_J^{(q)}$, the q-particle reduced densities of the trial state Γ_J. For this, we recall some notation from [LS24b].

Definition 4.1 [LS24b, Definition 3.1]. We define $\mathcal{G}_p^q$ as the set of graphs on q *external* vertices $\{1,\ldots,q\}$ and p *internal* vertices $\{q+1,\ldots,q+p\}$ such that there are no edges between external vertices and such that all internal vertices have degree at least 1 (i.e., there is at least one edge incident to each internal vertex). We replace q and/or p with sets V^* and V, respectively, and write $\mathcal{G}_V^{V^*}$ if we need the external and/or internal vertices to have definite indices V^*, respectively V. Concretely, this means that for a set of edges $E\subset\{\{i,j\}:1\le i<j\le q+p\}$, the corresponding graph is in $\mathcal{G}_p^q$ if and only if

$$\forall(q+1\le i\le q+p)\,\exists(1\le j\le q+p):\{i,j\}\in E,\qquad \forall(1\le i<j\le q):\{i,j\}\notin E.$$

Define $\mathcal{T}_p^q\subset\mathcal{C}_p^q\subset\mathcal{G}_p^q$ as the subset of trees and connected graphs, respectively. (Define similarly $\mathcal{T}_V^{V^*}\subset\mathcal{C}_V^{V^*}\subset\mathcal{G}_V^{V^*}$.) Define the functions

$$W_p^q = W_p^q(x_1,\ldots,x_{p+q}) = \sum_{G\in\mathcal{G}_p^q}\prod_{e\in G} g_e.$$

A *diagram* (π,G) (on q external and p internal vertices) is a pair of a permutation $\pi\in\mathcal{S}_{p+q}$ and a graph $G\in\mathcal{G}_p^q$. We view the permutation π as a directed graph on the $p+q$ vertices. The set of all diagrams on q external and p internal vertices is denoted $\mathcal{D}_p^q$.

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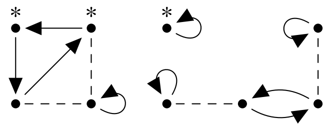

Figure 4.1. *Example of a diagram $(\pi, G) \in \mathcal{D}_6^3$ with three linked components with each linked component containing two (left linked component), one (center top linked component) and two (right linked component) clusters, respectively. Vertices labeled with $*$ denote external vertices, dashed lines denote g-edges and arrows denote γ-edges (i.e., an arrow from i to j denotes that $\pi(i) = j$). Note that all internal vertices have at least one incident g-edge, that external vertices may have none, and that there are no g-edges between external vertices.*

For a diagram (π, G), we will refer to G as the g-graph and π as the γ-graph. The value of a diagram $(\pi, G) \in \mathcal{D}_p^q$ is the function

$$\Gamma_{\pi,G}^q(x_1, \ldots, x_q) = (-1)^\pi \int \cdots \int \prod_{j=1}^{p+q} \gamma^{(1)}(x_j; x_{\pi(j)}) \prod_{e\in G} g_e \, \mathrm{d}X_{[q+1,q+p]}.$$

A diagram $(\pi, G) \in \mathcal{D}_p^q$ is *linked* if the union of π and G is a connected graph. The subset of all linked diagrams is denoted $\mathcal{L}_p^q \subset \mathcal{D}_p^q$.

For $q \geq 1$, define the set $\tilde{\mathcal{L}}_p^q \subset \mathcal{D}_p^q$ as the set of all diagrams such that each linked component contains at least one external vertex. For $q = 0$, we set $\tilde{\mathcal{L}}_p^0 = \mathcal{L}_p^0$.

If $q = 0$, we write $\mathcal{G}_p^q = \mathcal{G}_p$ etc. without a superscript q.

A *cluster* is a connected component of the graph G.

Notation 4.2. By a picture of a diagram, such as Figure 4.1, we will also denote the value of the pictured diagram.

We shall in the remainder of this section prove the following theorem.

Theorem 4.3. *For any q_0, there exists a constant $c_{q_0} > 0$ independently of L such that if $a^d \rho_0 \zeta^{d/2} \log b/a < c_{q_0}$, then*

$$Z_J = Z \exp\left[\sum_{p=2}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\mathcal{L}_p} \Gamma_{\pi,G}\right], \tag{4.1}$$

$$\rho_J^{(q)} = \prod_{1\le i<j\le q} f_{ij}^2 \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\tilde{\mathcal{L}}_p^q} \Gamma_{\pi,G}^q \tag{4.2}$$

for any $q \leq q_0$.

Note that the p-sum in Equation (4.2) starts at $p = 0$ as opposed to that in Equation (4.1). This arises from the fact that diagrams with at least one external vertex may have zero internal vertices, whereas diagrams with no external vertices have at least two internal vertices.

In the proof, we will use the GGR expansion as formulated in [Lau23, Lemma 3.6] and [LS24b, Theorem 3.4]. For convenience, we recall it here. Note that $\frac{1}{L^d} \sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d} \left|\hat{\gamma}^{(1)}(k)\right| = \rho_0$. We continue to abuse notation slightly and treat $\gamma^{(1)}$ as a function and write $\left\|\gamma^{(1)}\right\|_{L^1} = \int_{[0,L]^d} \left|\gamma^{(1)}(x)\right| \mathrm{d}x$.

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Lemma 4.4 [Lau23, Lemma 3.6], [LS24b, Theorem 3.4]. *For any integer q_0, there exists a constant $c_{q_0} > 0$ such that if $\|g\|_{L^1}\|\gamma^{(1)}\|_{L^1}\rho_0 < c_{q_0}$, then*[3]

$$\mathcal{Z} := 1 + \sum_{p=2}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\mathcal{D}_p} \Gamma_{\pi,G} = \exp\left[\sum_{p=2}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\mathcal{L}_p} \Gamma_{\pi,G}\right],$$

$$\frac{1}{\mathcal{Z}} \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\mathcal{D}_p^q} \Gamma_{\pi,G}^q = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\tilde{\mathcal{L}}_p^q} \Gamma_{\pi,G}^q$$

for any $q \le q_0$, the p-sums being absolutely convergent.

We shall bound $\|\gamma^{(1)}\|_{L^1}$ and $\|g\|_{L^1}$ in Lemma 4.6 below.

4.1. Calculation of Z_J

We calculate Z_J. This is analogous to the computation in [LS24b, Section 3.0.1] and [Lau23, Section 3.1]. For simplicity, denote the diagonal of Γ_n by $\Gamma_n = \Gamma_n(X_n) = \Gamma_n(X_n; X_n)$. Then

$$Z_J = \sum_{n=0}^{\infty} \int \cdots \int \prod_{i<j} f_{ij}^2 \Gamma_n(X_n)\,\mathrm{d}X_n = \sum_{n=0}^{\infty} \int \cdots \int \prod_{i<j} (1+g_{ij})\Gamma_n(X_n)\,\mathrm{d}X_n.$$

Expanding the product and grouping all terms where p variables x_i appear in the factors g_{ij}, we find the function W_p (evaluated on the respective p coordinates x_i). Noting further the permutation symmetry of the coordinates, we have

$$= \sum_{n=0}^{\infty} \int \cdots \int \left[1 + \sum_{p=2}^{n} \frac{n!}{(n-p)!p!} W_p(X_p)\right] \Gamma_n(X_n)\,\mathrm{d}X_n$$

since there are $\frac{n!}{(n-p)!p!}$ many ways to choose p coordinates out of n coordinates. Now, if

$$\sum_{n=0}^{\infty} \sum_{p=2}^{n} \frac{n!}{(n-p)!p!} \int \cdots \int |W_p| \Gamma_n\,\mathrm{d}X_n < \infty,$$

then we may interchange the two sums. A criterion for this is given in Lemma 4.5 below. Thus, if the condition of Lemma 4.5 is satisfied – namely, that $\rho_0\|g\|_{L^1}$ is sufficiently small – we have

$$Z_J = Z\left[1 + \sum_{p=2}^{\infty} \frac{1}{p!} \int \cdots \int \mathrm{d}X_p W_p \left[\frac{1}{Z} \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} \int \cdots \int \mathrm{d}X_{[p+1,n]} \Gamma_n\right]\right]$$

$$= Z\left[1 + \sum_{p=2}^{\infty} \frac{1}{p!} \int \cdots \int \mathrm{d}X_p W_p \rho^{(p)}\right].$$

[3]In [Lau23, Lemma 3.6] and [LS24b, Theorem 3.4], the sum $\sum_{(\pi,G)\in\tilde{\mathcal{L}}_p^q} \Gamma_{\pi,G}^q$ is written by decomposing all diagrams $(\pi, G) \in \tilde{\mathcal{L}}_p^q$ into their linked components and noting that $\Gamma_{\pi,G}^q$ factorizes over linked components.

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The free Fermi gas is a quasi-free state, and thus by Wick's rule, we have

$$Z_J = Z\left[1 + \sum_{p=2}^{\infty} \frac{1}{p!} \int \cdots \int \mathrm{d}X_p W_p \det\left[\gamma_{ij}^{(1)}\right]_{1\le i,j\le p}\right].$$

Expanding W_p and the determinant, we get

$$Z_J = Z\left[1 + \sum_{p=2}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\mathcal{D}_p} \Gamma_{\pi,G}\right].$$

Applying then Lemma 4.4, we conclude that if $\rho_0\|g\|_{L^1}\left\|\gamma^{(1)}\right\|_{L^1}$ and $\rho_0\|g\|_{L^1}$ are sufficiently small, then Equation (4.1) holds.

4.2. Calculation of $\rho_J^{(q)}$

Next, we calculate the reduced densities $\rho_J^{(q)}$ of the trial state Γ_J. This is analogous to the computations in [LS24b, Sections 3.0.2–3.0.4] and [Lau23, Section 3.2]. We have

$$\rho_J^{(q)}(X_q) = \frac{1}{Z_J} \sum_{n=q}^{\infty} \frac{n!}{(n-q)!} \int \cdots \int \prod_{1\le i<j\le n} f_{ij}^2 \Gamma_n(X_n)\,\mathrm{d}X_{[q+1,n]}.$$

We write $f_{ij}^2 = 1 + g_{ij}$ if at least one of i, j is an internal vertex and expand the product of the $(1+g_{ij})$'s. Grouping together those terms where p internal vertices are present, we find the function W_p^q. Using additionally the symmetry of permutation of the coordinates, we find

$$= \frac{1}{Z_J} \prod_{1\le i<j\le q} f_{ij}^2 \sum_{n=q}^{\infty} \frac{n!}{(n-q)!} \sum_{p=0}^{n-q} \frac{(n-q)!}{p!(n-q-p)!} \int \cdots \int W_p^q(X_{p+q})\Gamma_n(X_n)\,\mathrm{d}X_{[q+1,n]}.$$

By Lemma 4.5 below, we may interchange the sums if $\rho_0\|g\|_{L^1}$ is sufficiently small. Then

$$\begin{aligned}\rho_J^{(q)} &= \frac{Z}{Z_J} \prod_{1\le i<j\le q} f_{ij}^2 \sum_{p=0}^{\infty} \frac{1}{p!} \int \cdots \int W_p^q \left[\frac{1}{Z} \sum_{n=p+q}^{\infty} \frac{n!}{(n-p-q)!} \int \cdots \int \Gamma_n\,\mathrm{d}X_{[q+p+1,n]}\right] \mathrm{d}X_{[q+1,q+p]} \\ &= \frac{Z}{Z_J} \prod_{1\le i<j\le q} f_{ij}^2 \sum_{p=0}^{\infty} \frac{1}{p!} \int \cdots \int W_p^q \rho^{(p+q)}\,\mathrm{d}X_{[q+1,q+p]}.\end{aligned}$$

Expanding the W_p^q and using the Wick rule for the reduced densities of the free gas as above, we get

$$\rho_J^{(q)} = \frac{Z}{Z_J} \prod_{1\le i<j\le q} f_{ij}^2 \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\mathcal{D}_p^q} \Gamma_{\pi,G}^q.$$

As above, by Lemma 4.4, we get that Equation (4.2) holds for $\rho_0\|g\|_{L^1}\left\|\gamma^{(1)}\right\|_{L^1}$ and $\rho_0\|g\|_{L^1}$ small enough (dependent on q).

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4.3. A convergence criterion

In this section, we show the following:

Lemma 4.5. *There exists a constant $c > 0$ such that if $\rho_0 \|g\|_{L^1} < c$, then*

$$\frac{1}{Z}\sum_{n=0}^{\infty}\sum_{p=2}^{n}\frac{n!}{(n-p)!p!}\int\cdots\int |W_p||\Gamma_n|\,\mathrm{d}X_n \le \exp(CL^d\rho_0\|g\|_{L^1}) < \infty, \tag{4.3}$$

and for any $q \ge 1$,

$$\frac{1}{Z}\sum_{n=q}^{\infty}\sum_{p=0}^{n-q}\frac{n!}{(n-q-p)!p!}\int\cdots\int |W_p^q||\Gamma_n|\,\mathrm{d}X_{[q+1,n]} \le C_q\rho_0^q\exp(CL^d\rho_0\|g\|_{L^1}) < \infty \tag{4.4}$$

uniformly in $x_1, \ldots, x_q$.

Proof. Write

$$\begin{aligned}\frac{1}{Z}\sum_{n=0}^{\infty}\sum_{p=2}^{n}\frac{n!}{(n-p)!p!}\int\cdots\int |W_p||\Gamma_n|\,\mathrm{d}X_n &= \frac{1}{Z}\sum_{p=2}^{\infty}\sum_{n=p}^{\infty}\frac{n!}{(n-p)!p!}\int\cdots\int \mathrm{d}X_n|W_p|\Gamma_n\\ &= \sum_{p=2}^{\infty}\frac{1}{p!}\int\cdots\int \mathrm{d}X_p|W_p|\rho^{(p)}.\end{aligned}$$

By splitting all graphs into their connected components, we have

$$\int\cdots\int \mathrm{d}X_p\,|W_p|\rho^{(p)} = \int\cdots\int \mathrm{d}X_p\left|\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{n_1,\ldots,n_k\ge 2}\binom{p}{n_1,\ldots,n_k}\chi(\textstyle\sum n_\ell = p)\displaystyle\prod_{\ell=1}^{k}\left[\sum_{G_\ell\in\mathcal{C}_{n_\ell}}\prod_{e\in G_\ell} g_e\right]\right|\rho^{(p)}.$$

We abused notation slightly and denote by $\mathcal{C}_{n_\ell}$ the set of connected graphs on n_ℓ specified vertices, say $\{\sum_{\ell'<\ell} n_{\ell'} + 1, \ldots, \sum_{\ell'\le\ell} n_{\ell'}\}$, such that no two G_ℓ's share any vertices. Here, k is the number of connected components having sizes $n_1, \ldots, n_k$. Note that $n_\ell \ge 2$ since each connected component needs at least two vertices since any vertex in a graph $G \in \mathcal{G}_p$ is internal and hence connected to at least one other vertex. The factor $\frac{1}{k!}$ comes from counting the possible labelings of the connected component, and the factor $\binom{p}{n_1\cdots n_k}$ comes from counting the possible labelings of the vertices in the different connected components.

Next, we employ the tree-graph inequality [Uel18]. This reads (since $0 \le g \le 1$)

$$\left|\sum_{G\in\mathcal{C}_n}\prod_{e\in G} g_e\right| \le \sum_{T\in\mathcal{T}_n}\prod_{e\in T}|g_e|.$$

Thus,

$$\int\cdots\int \mathrm{d}X_p\,\frac{1}{p!}|W_p|\rho^{(p)} \le \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{n_1,\ldots,n_k\ge 2}\frac{1}{n_1!\cdots n_k!}\chi(\textstyle\sum n_\ell = p)\displaystyle\int\cdots\int \mathrm{d}X_p\prod_{\ell=1}^{k}\left[\sum_{T_\ell\in\mathcal{T}_{n_\ell}}\prod_{e\in T_\ell}|g_e|\right]\rho^{(p)}.$$

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Next, we bound $\rho^{(p)}$ analogously to [LS24b, Lemma 3.10]. First, $\rho^{(p)} = \det[\gamma^{(1)}_{ij}]_{1\le i,j\le p}$ by the Wick rule. Next, define $\alpha_i(k) = L^{-d/2}e^{ikx_i}\hat{\gamma}^{(1)}(k) \in \ell^2(\frac{2\pi}{L}\mathbb{Z}^d)$. Then $\gamma^{(1)}_{ij} = \langle\alpha_i|\alpha_j\rangle_{\ell^2(\frac{2\pi}{L}\mathbb{Z}^d)}$ and so by the Gram–Hadamard inequality [GMR21, Lemma D.1],

$$\rho^{(p)} = \det\left[\gamma^{(1)}_{ij}\right]_{1\le i,j\le p} \le \prod_{i=1}^{p} \|\alpha_i\|^2_{\ell^2(\frac{2\pi}{L}\mathbb{Z}^d)} = \rho_0^p.$$

Thus,

$$\sum_{p=2}^{\infty}\frac{1}{p!}\int\cdots\int \mathrm{d}X_p|W_p|\rho^{(p)} \le \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{n_1,\ldots,n_k\ge 2}\frac{1}{n_1!\cdots n_k!}\rho_0^{\sum n_\ell}\prod_{\ell=1}^{k}\left[\sum_{T_\ell\in\mathcal{T}_{n_\ell}}\int\cdots\int\prod_{e\in T_\ell}|g_e|\right].$$

For each tree, the integration is over all variables; thus, by the translation invariance, the integration over the variables in the tree T_ℓ gives $L^d(\int|g|)^{n_\ell-1}$. Using moreover Cayley's formula $\#\mathcal{T}_n = n^{n-2} \le C^n n!$, we get

$$\begin{aligned}
&\le \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{n_1,\ldots,n_k\ge 2}\frac{1}{n_1!\cdots n_k!}\rho_0^{\sum n_\ell}C^{\sum n_\ell}n_1!\cdots n_k!\left(\int|g|\right)^{\sum n_\ell - k}L^{dk}\\
&= \sum_{k=1}^{\infty}\frac{1}{k!}\left[C\rho_0 L^d\sum_{n=2}^{\infty}(C\rho_0\|g\|_{L^1})^{n-1}\right]^k\\
&\le \exp\left(CL^d\rho_0\|g\|_{L^1}\right) < \infty
\end{aligned}$$

if $\rho_0\|g\|_{L^1}$ is sufficiently small.

The proof of Equation (4.4) is in spirit the same. Write

$$\frac{1}{Z}\sum_{n=q}^{\infty}\sum_{p=0}^{n-q}\frac{n!}{(n-q-p)!p!}\int\cdots\int|W_p^q||\Gamma_n|\,\mathrm{d}X_{[q+1,n]} = \sum_{p=0}^{\infty}\frac{1}{p!}\int\cdots\int\mathrm{d}X_{[q+1,q+p]}|W_p^q|\rho^{(q+p)}.$$

By decomposing the graphs into their connected components, we have

$$\begin{aligned}
&\int\cdots\int\mathrm{d}X_{[q+1,q+p]}\,|W_p^q|\rho^{(q+p)}\\
&= \int\cdots\int\mathrm{d}X_{[q+1,q+p]}\left|\sum_{\kappa=1}^{q}\frac{1}{\kappa!}\sum_{\substack{(V_1^*,\ldots,V_\kappa^*)\\ \text{partition of }\{1,\ldots,q\}\\ V_\lambda^*\neq\varnothing}}\sum_{n_1^*,\ldots,n_\kappa^*\ge 0}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n_1,\ldots,n_k\ge 2}\chi_{(\sum_\lambda n_\lambda^* + \sum_\ell n_\ell = p)}\right.\\
&\qquad\left.\times\binom{p}{n_1^*,\ldots,n_\kappa^*,n_1,\ldots,n_k}\prod_{\lambda=1}^{\kappa}\left[\sum_{G_\lambda^*\in\mathcal{C}^{V_\lambda^*}_{n_\lambda^*}}\prod_{e\in G_\lambda^*}g_e\right]\prod_{\ell=1}^{k}\left[\sum_{G_\ell\in\mathcal{C}_{n_\ell}}\prod_{e\in G_\ell}g_e\right]\right|\rho^{(q+p)}.
\end{aligned}$$

Here, κ is the number of connected components having external vertices, and k is the number of connected components only with internal vertices. The partition $(V_1^*,\ldots,V_\kappa^*)$ partitions the external vertices into the κ different connected components with external vertices, and the numbers $n_1^*,\ldots,n_\kappa^*$

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are the number of internal vertices in the connected components with external vertices. The numbers $n_1, \ldots, n_k$ and the combinatorial factors are as above.

Using the tree-graph inequality as above, we will obtain a sum of trees. (Technically, we need to use a trivial modification of the tree-graph bound adapted to the setting with external vertices as in [LS24b, Section 3.1.3; Lau23, Section 4.2]. One simply defines $g_e = 0$ for a disallowed edge e between external vertices.) Namely, we will have factors like

$$\sum_{T_\lambda^* \in \mathcal{T}_{n_\lambda^*}^{V_\lambda^*}} \prod_{e \in T_\lambda^*} |g_e|.$$

We bound these as follows. If $\# V_\lambda^* = 1$, we do nothing and define $T_{\lambda,1}^* = T_\lambda^*$. Otherwise, iteratively pick any edge on the path between any two external vertices and bound the factor $|g_e| \le 1$. Remove this edge from T_λ^*. Repeating this procedure $\# V_\lambda^* - 1$ many times results in $\# V_\lambda^*$ many trees all with exactly one external vertex. Label these as $T_{\lambda,1}^*, \ldots, T_{\lambda,\# V_\lambda^*}^*$. We then have the bound

$$\prod_{e \in T_\lambda^*} |g_e| \le \prod_{\nu=1}^{\# V_\lambda^*} \prod_{e \in T_{\lambda,\nu}^*} |g_e|.$$

Using this bound together with the Gram–Hadamard inequality as above, we get

$$\begin{aligned}
&\sum_{p=0}^{\infty} \frac{1}{p!} \int \cdots \int \mathrm{d}X_{[q+1,q+p]} |W_p^q| \rho^{(q+p)} \\
&\le \sum_{\kappa=1}^{q} \frac{1}{\kappa!} \sum_{\substack{(V_1^*,\ldots,V_\kappa^*) \\ \text{part. of } \{1,\ldots,q\} \\ V_\lambda^* \neq \emptyset}} \sum_{n_1^*,\ldots,n_\kappa^* \ge 0} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n_1,\ldots,n_k \ge 2} \frac{1}{\prod_{\lambda=1}^{\kappa} n_\lambda^*! \prod_{\ell=1}^{k} n_\ell!} \rho_0^{q + \sum_\lambda n_\lambda^* + \sum_\ell n_\ell} \\
&\quad \times \sum_{T_\lambda^* \in \mathcal{T}_{n_\lambda^*}^{V_\lambda^*}} \sum_{T_\ell \in \mathcal{T}_{n_\ell}} \left[\prod_{\lambda=1}^{\kappa} \prod_{\nu=1}^{\# V_\lambda^*} \int \cdots \int \prod_{e \in T_{\lambda,\nu}^*} |g_e| \right] \left[\prod_{\ell=1}^{k} \int \cdots \int \prod_{e \in T_\ell} |g_e| \right].
\end{aligned}$$

In the integrations, each tree $T_{\lambda,\nu}^*$ is integrated over all but the one external vertex and so gives a value $(\int |g|)^{\# T_{\lambda,\nu}^* - 1}$, and each tree T_ℓ is integrated over all coordinates giving the value $(\int |g|)^{n_\ell - 1} L^d$. Moreover, $\sum_\nu (\# T_{\lambda,\nu}^* - 1) = n_\lambda^*$. Thus, using additionally Cayley's formula (trivially extended to the setting with external vertices: $\# \mathcal{T}_{n_\lambda^*}^{V_\lambda^*} \le C^{n_\lambda^* + \# V_\lambda^*} (n_\lambda^* + \# V_\lambda^*)!$),

$$\begin{aligned}
&\le \sum_{\kappa=1}^{q} \frac{1}{\kappa!} \sum_{\substack{(V_1^*,\ldots,V_\kappa^*) \\ \text{part. of } \{1,\ldots,q\} \\ V_\lambda^* \neq \emptyset}} \sum_{n_1^*,\ldots,n_\kappa^* \ge 0} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n_1,\ldots,n_k \ge 2} \frac{1}{\prod_{\lambda=1}^{\kappa} n_\lambda^*! \prod_{\ell=1}^{k} n_\ell!} \rho_0^{q + \sum_\lambda n_\lambda^* + \sum_\ell n_\ell} \\
&\quad \times C^{q + \sum_\lambda n_\lambda^* + \sum_\ell n_\ell} \prod_{\lambda=1}^{\kappa} (n_\lambda^* + \# V_\lambda^*)! \prod_{\ell=1}^{k} n_\ell! \left(\int |g| \right)^{\sum_\lambda n_\lambda^* + \sum_\ell n_\ell - k} L^{dk}.
\end{aligned}$$

Next, we may bound the binomial coefficients as $(n+m)! \le 2^{n+m} n! m!$ so $\prod_{\lambda=1}^{\kappa} (n_\lambda^* + \# V_\lambda^*)! \le 2^{\sum_\lambda n_\lambda^* + q} \prod_\lambda n_\lambda^*! (\# V_\lambda^*)!$. Thus,

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$$\leq C^q \sum_{\kappa=1}^{q} \frac{1}{\kappa!} \sum_{\substack{(V_1^*,\ldots,V_\kappa^*) \\ \text{part. of } \{1,\ldots,q\} \\ V_\lambda^* \neq \emptyset}} \prod_{\lambda=1}^{\kappa} (\# V_\lambda^*)! \left[\sum_{n^*=0}^{\infty} (C\rho_0 \|g\|_{L^1})^{n_*} \right]^{\kappa}$$

$$\times \rho_0^q \sum_{k=0}^{\infty} \frac{1}{k!} \left[CL^d \rho_0 \sum_{n=2}^{\infty} (C\rho_0\|g\|_{L^1})^{n-1} \right]^k$$

$$\leq C_q \rho_0^q \exp\left(CL^d \rho_0 \|g\|_{L^1} \right) < \infty$$

if $\rho_0 \|g\|_{L^1}$ is small enough. □

4.4. *Calculation of* $\|g\|_{L^1}, \left\|\gamma^{(1)}\right\|_{L^1}$

In this section, we bound the quantities $\|g\|_{L^1}$ and $\left\|\gamma^{(1)}\right\|_{L^1} = \int \left|\gamma^{(1)}(x)\right| \mathrm{d}x$. We show (recall $\zeta = 1 + |\log z|$)

Lemma 4.6. *The quantities* $\|g\|_{L^1}$ *and* $\left\|\gamma^{(1)}\right\|_{L^1}$ *satisfy*

$$\|g\|_{L^1} \leq C a^d \log b/a, \qquad \left\|\gamma^{(1)}\right\|_{L^1} \leq C\zeta^{d/2}.$$

Note that these bounds are uniform in the volume L^d.

Proof. The bound $\|g\|_{L^1} \leq Ca^d \log b/a$ follows from Equation (3.6). For $\left\|\gamma^{(1)}\right\|_{L^1}$, we have for any (length) $\lambda > 0$,

$$\begin{aligned} \left\|\gamma^{(1)}\right\|_{L^1} &= \int_{[0,L]^d} \left|\gamma^{(1)}(x)\right| \\ &\leq \left(\int_{\mathbb{R}^d} \left|\gamma^{(1)}(x)\right|^2 (\lambda^2 + |x|^2)^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} \frac{1}{(\lambda^2 + |x|^2)^2} \, \mathrm{d}x \right)^{1/2} \\ &= C\lambda^{d/2-2} \left[\frac{1}{L^d} \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} \left| (\lambda^2 \widehat{+ |x|^2)\gamma^{(1)}}(k) \right|^2 \right]^{1/2}. \end{aligned}$$

Moreover, (with $\hat{\gamma}(k) = ze^{-\beta|k|^2}$ is as in Equation (3.12)),

$$\begin{aligned} (\lambda^2 \widehat{+ |x|^2)\gamma^{(1)}}(k) &= \left[\lambda^2 - \Delta_k\right] \hat{\gamma}^{(1)}(k) \\ &= \frac{\lambda^2 \hat{\gamma}(k)^3 + (2\lambda^2 - 4\beta^2|k|^2 - 2d\beta)\hat{\gamma}(k)^2 + (\lambda^2 + 4\beta^2|k|^2 - 2d\beta)\hat{\gamma}(k)}{(1 + \hat{\gamma}(k))^3}. \end{aligned}$$

Using Equation (3.12), we conclude that

$$\frac{1}{L^d} \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} \left| (\lambda^2 \widehat{+ |x|^2)\gamma^{(1)}}(k) \right|^2 \leq C\rho_0 \left(\lambda^4 + \beta^4 \rho_0^{4/d} + \beta^2 \right) \leq C\rho_0 \left(\lambda^4 + \zeta^2 \beta^2 \right).$$

Thus, for $\lambda = \beta^{1/2}\zeta^{1/2}$, we have $\left\|\gamma^{(1)}\right\|_{L^1} \leq C\zeta^{d/2}$. (Recall that $\beta \sim \zeta \rho_0^{-2/d}$ by Remark 3.7.) □

We conclude that $\rho_0 \|g\|_{L^1} \left\|\gamma^{(1)}\right\|_{L^1} \leq Ca^d \rho_0 \zeta^{d/2} \log b/a$. This concludes the proof of Theorem 4.3.

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5. Calculation of terms in Equation (3.5)

In this section, we compute and bound the different terms in Equation (3.5) and thereby prove Proposition 1.9.

5.1. Energy

The kinetic energy of the trial state Γ_J is

$$\begin{aligned}\langle \mathcal{H}\rangle_J &= \frac{1}{Z_J}\sum_{n=1}^{\infty}\int\cdots\int \left[(-\Delta_{X_n})[F_n(X_n)\Gamma_n(X_n,Y_n)F_n(Y_n)]\right]_{Y_n=X_n}\,\mathrm{d}X_n\\ &= \frac{1}{Z_J}\sum_{n=1}^{\infty}\int\cdots\int \Big(|\nabla_{X_n}F|^2\Gamma_n(X_n;X_n) - F_n^2(\Delta_{X_n}\Gamma_n)(X_n;X_n)\Big)\,\mathrm{d}X_n.\end{aligned}$$

The second term may be calculated as (recall that $\langle\cdot\rangle_J$ means expectation in the state Γ_J)

$$\begin{aligned}\frac{1}{Z_J}\sum_{n=1}^{\infty}\int\cdots\int F_n^2(-\Delta_{X_n}\Gamma_n)(X_n;X_n)\,\mathrm{d}X_n &= \frac{Z}{Z_J}\operatorname{Tr}[F^2\mathcal{H}\Gamma]\\ &= \frac{1}{Z_J}\operatorname{Tr}[F^2(-\partial_\beta(Z\Gamma)+\mu\mathcal{N}Z\Gamma)]\\ &= -\partial_\beta\log Z_J + \mu\langle\mathcal{N}\rangle_J.\end{aligned}$$

Here, we used that Γ is differentiable in β in the topology of trace-class operators. This may be easily verified. For the first term, we have that

$$|\nabla_{X_n}F_n|^2 = \left[2\sum_{j<k}\left|\frac{\nabla f_{jk}}{f_{jk}}\right|^2 + \sum_{\substack{i,j,k\\ \text{all distinct}}}\frac{\nabla f_{ij}\nabla f_{jk}}{f_{ij}f_{jk}}\right]F_n^2.$$

Thus, the full energy is

$$\begin{aligned}\langle\mathcal{H}-\mu\mathcal{N}+\mathcal{V}\rangle_J = -\partial_\beta\log Z_J &+ \iint\left[\left|\frac{\nabla f_{12}}{f_{12}}\right|^2+\frac{1}{2}v_{12}\right]\rho_J^{(2)}\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &+\iiint\frac{\nabla f_{12}\nabla f_{13}}{f_{12}f_{13}}\rho_J^{(3)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3.\end{aligned}\tag{5.1}$$

5.2. Entropy

We note that $\Gamma_J = \frac{Z}{Z_J}F\Gamma F$ is isospectral to $\frac{Z}{Z_J}\Gamma^{1/2}F^2\Gamma^{1/2}$. Moreover, since $F\le 1$, we have $\Gamma^{1/2}F^2\Gamma^{1/2}\le\Gamma$ as operators. Thus, by operator monotonicity of the logarithm,

$$\begin{aligned}\operatorname{Tr}[\Gamma_J\log\Gamma_J] &= \frac{Z}{Z_J}\operatorname{Tr}\left[\Gamma^{1/2}F^2\Gamma^{1/2}\left(\log\frac{Z}{Z_J}+\log\Gamma^{1/2}F^2\Gamma^{1/2}\right)\right]\\ &\le \log\frac{Z}{Z_J}+\frac{Z}{Z_J}\operatorname{Tr}\left[\Gamma^{1/2}F^2\Gamma^{1/2}\log\Gamma\right]\\ &= -\log Z_J - \beta\frac{Z}{Z_J}\operatorname{Tr}\left[F^2\Gamma(\mathcal{H}-\mu\mathcal{N})\right]\\ &= -\log Z_J + \frac{1}{Z_J}\beta\partial_\beta\operatorname{Tr}\left[F^2Z\Gamma\right]\\ &= -\log Z_J + \beta\partial_\beta\log Z_J.\end{aligned}$$

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We conclude the bound on the entropy

$$-\frac{1}{\beta}S(\Gamma_J) = \frac{1}{\beta}\operatorname{Tr}[\Gamma_J \log \Gamma_J] \le -\frac{1}{\beta}\log Z_J + \partial_\beta \log Z_J. \tag{5.2}$$

5.3. Pressure

Combining Equations (5.1) and (5.2), the terms $\pm\partial_\beta \log Z_j$ cancel, and we conclude the bound for the pressure

$$\begin{aligned} L^d P[\Gamma_J] &= -\langle \mathcal{H} - \mu\mathcal{N} + \mathcal{V}\rangle_J + \frac{1}{\beta}S(\Gamma_J) \\ &\ge \frac{1}{\beta}\log Z_J - \iint \left[\left|\frac{\nabla f_{12}}{f_{12}}\right|^2 + \frac{1}{2}v_{12}\right]\rho_J^{(2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 - \iiint \frac{\nabla f_{12}\nabla f_{13}}{f_{12}f_{13}}\rho_J^{(3)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3. \end{aligned}$$

Remark 5.1. The cancellation of the terms $\pm\partial_\beta \log Z_j$ is not essential. Namely, the energy of the trial state Γ_J is the energy of the free gas plus the relevant interaction term up to small errors. And the entropy of the trial state Γ_J is bounded from above by the entropy of the free gas up to small errors. To see this, write

$$-\partial_\beta \log Z_J = -\partial_\beta \log Z - \partial_\beta \log \frac{Z_J}{Z} = \langle \mathcal{H} - \mu\mathcal{N}\rangle_0 - \partial_\beta \log \frac{Z_J}{Z}.$$

One can show that $\partial_\beta \log \frac{Z_J}{Z}$ is small compared to the interaction of order $L^d a^d \rho_0^{2+2/d}$. Thus, the energy of the trial state Γ_J is

$$\langle \mathcal{H} - \mu\mathcal{N} + \mathcal{V}\rangle_J = \langle \mathcal{H} - \mu\mathcal{N}\rangle_0 + \iint \left[\left|\frac{\nabla f_{12}}{f_{12}}\right|^2 + \frac{1}{2}v_{12}\right]\rho_J^{(2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 + \text{small error}.$$

Similarly, for the entropy,

$$\begin{aligned} -\frac{1}{\beta}\log Z_J + \partial_\beta \log Z_J &= -\frac{1}{\beta}\log Z + \partial_\beta \log Z - \frac{1}{\beta}\log\frac{Z_J}{Z} + \partial_\beta \log \frac{Z_J}{Z} \\ &= -\frac{1}{\beta}S(\Gamma) - \frac{1}{\beta}\log\frac{Z_J}{Z} + \partial_\beta \log \frac{Z_J}{Z}. \end{aligned}$$

We show below that $\frac{1}{\beta}\log\frac{Z_J}{Z}$ is small compared to the interaction term of size $L^d a^d \rho_0^{2+2/d}$. Thus, the entropy of the trial state Γ_J may be bounded as

$$-\frac{1}{\beta}S(\Gamma_J) \le -\frac{1}{\beta}S(\Gamma) + \text{small error}.$$

The proof that $\partial_\beta \log \frac{Z_J}{Z}$ is small is somewhat analogous to the proof of Lemma 5.2 in Section 5.4. As we will not need it, we omit the details.

By Equation (4.2), we have for $a^d\rho_0\zeta^{d/2}\log b/a$ sufficiently small that

$$\rho_J^{(2)} = f_{12}^2\left[\rho^{(2)} + \sum_{p=1}^{\infty}\frac{1}{p!}\sum_{(\pi,G)\in\tilde{\mathcal{L}}_p^2}\Gamma_{\pi,G}^2\right].$$

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We may then write

$$\begin{aligned}
&L^d P[\Gamma_J] \\
&\geq \frac{1}{\beta}\log Z - \iint \left[|\nabla f_{12}|^2 + \frac{1}{2}v_{12}f_{12}^2\right]\rho^{(2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 + \underbrace{\frac{1}{\beta}\log\frac{Z_J}{Z}}_{\varepsilon_Z} \\
&\quad \underbrace{- \iint \left[|\nabla f_{12}|^2 + \frac{1}{2}v_{12}f_{12}^2\right]\sum_{p=1}^{\infty}\frac{1}{p!}\sum_{(\pi,G)\in\tilde{\mathcal{L}}_p^2}\Gamma_{\pi,G}^2\,\mathrm{d}x_1\,\mathrm{d}x_2}_{\varepsilon_2} \underbrace{- \iiint \frac{\nabla f_{12}\nabla f_{13}}{f_{12}f_{13}}\rho_J^{(3)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3}_{\varepsilon_3}.
\end{aligned} \tag{5.3}$$

The first term is the pressure of the free gas (times the volume), the second term leads to the leading order correction, and the remaining terms are error terms. We shall show in Section 5.4 below the following bounds. (Recall that $\zeta = 1 + |\log z|$.)

Lemma 5.2. *For $z \gtrsim 1$, there exists a constant $c > 0$ such that if $a^d\rho_0\zeta^{d/2}\left|\log a^d\rho_0\right| < c$, then, for sufficently large L, the error terms are bounded as*

$$\begin{aligned}
\frac{|\varepsilon_Z|}{L^d} &\leq Ca^d b^2\rho_0^{2+4/d}\zeta^{-1} + Ca^{2d}\rho_0^{3+2/d}\zeta^{d/2-1}(\log b/a)^2 \\
\frac{|\varepsilon_2|}{L^d} &\leq \begin{cases} Ca^{2d}\rho_0^{3+2/d}\log b/a + Ca^{4d-2}\rho_0^5\zeta^{3d/2}(\log b/a)^3 & d\geq 2, \\ Cab\rho_0^5\log b/a + Ca^2\rho_0^5\zeta^{3/2}(\log b/a)^3 & d=1,\end{cases} \\
\frac{|\varepsilon_3|}{L^d} &\leq \begin{cases} Ca^{2d}b^2\rho_0^{3+4/d} + Ca^{3d-2}\rho_0^4\zeta^{d/2}\log b/a & d\geq 2, \\ Ca^2\rho_0^5\zeta(\log b/a)^2 & d=1.\end{cases}
\end{aligned}$$

In particular, we have the bounds (recalling that $a \ll b \lesssim \rho_0^{-1/d}$)

$$\frac{|\varepsilon_Z| + |\varepsilon_2| + |\varepsilon_3|}{L^d} \leq \begin{cases} Ca^3b^2\rho_0^{10/3}\zeta^{-1} + Ca^6\rho_0^{11/3}\zeta^{1/2}(\log b/a)^2 + Ca^{10}\rho_0^5\zeta^{9/2}(\log b/a)^3 & d=3, \\ Ca^2b^2\rho_0^4\zeta^{-1} + Ca^4\rho_0^4\zeta\log b/a + Ca^6\rho_0^5\zeta^3(\log b/a)^3 & d=2, \\ Cab\rho_0^5\log b/a + Ca^2\rho_0^5\zeta^{3/2}(\log b/a)^3 & d=1.\end{cases} \tag{5.4}$$

Note that this is increasing in b. For the second term in Equation (5.3) above, we use Equations (3.7), (3.8) and (3.11); thus,

$$\begin{aligned}
&\iint \left[|\nabla f_{12}|^2 + \frac{1}{2}v_{12}f_{12}^2\right]\rho^{(2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= 2\pi\frac{-\mathrm{Li}_{d/2+1}(-e^{\beta\mu})}{(-\mathrm{Li}_{d/2}(-e^{\beta\mu}))^{1+2/d}}\rho_0^{2+2/d}L^d\int\left(|\nabla f|^2 + \frac{1}{2}vf^2\right)|x|^2\,\mathrm{d}x\left(1 + O(L^{-1}\zeta\rho_0^{-1/d})\right) \\
&\quad + O\left(L^d\rho_0^{2+4/d}\int\left(|\nabla f|^2 + \frac{1}{2}vf^2\right)|x|^4\,\mathrm{d}x\right) \\
&= 2\pi c_d\frac{-\mathrm{Li}_{d/2+1}(-e^{\beta\mu})}{(-\mathrm{Li}_{d/2}(-e^{\beta\mu}))^{1+2/d}}L^d a^d\rho_0^{2+2/d}\left(1 + O(a^d/b^d) + O\left(L^{-1}\zeta\rho_0^{-1/d}\right)\right) \\
&\quad + \begin{cases} O\left(L^d a^{d+2}\rho_0^{2+4/d}\log b/a\right) & d\geq 2 \\ O\left(La^2b\rho_0^6\right) & d=1,\end{cases}
\end{aligned} \tag{5.5}$$

where c_d is defined in Equation (1.3). Note that the first error term is decreasing in b. This competes with the other error terms and leads to the choice of b below. Combining Equations (5.3), (5.4) and (5.5), we thus conclude the bound

$$\begin{aligned}
&\psi(\beta,\mu) \\
&\quad \geq \limsup_{L\to\infty} P[\Gamma_J] \\
&\quad \geq \lim_{L\to\infty}\left[\frac{1}{L^d\beta}\log Z\right] - 2\pi c_d \frac{-\operatorname{Li}_{d/2+1}(-e^{\beta\mu})}{(-\operatorname{Li}_{d/2}(-e^{\beta\mu}))^{1+2/d}} a^d \rho_0^{2+2/d} \\
&\quad + \begin{cases} O\left(a^6 b^{-3}\rho_0^{8/3} + a^3 b^2 \rho_0^{10/3}\zeta^{-1} + a^6\rho_0^{11/3}\zeta^{1/2}(\log b/a)^2 + a^{10}\rho_0^5\zeta^{9/2}(\log b/a)^3\right) & d=3, \\ O\left(a^4 b^{-2}\rho_0^3 + a^2 b^2\rho_0^4\zeta^{-1} + a^4\rho_0^4\zeta\log b/a + a^6\rho_0^5\zeta^3(\log b/a)^3\right) & d=2, \\ O\left(a^2 b^{-1}\rho_0^4 + ab\rho_0^5\log b/a + a^2\rho_0^5\zeta^{3/2}(\log b/a)^3\right) & d=1. \end{cases}
\end{aligned}$$

Using that $\lim_{L\to\infty}\left[\frac{1}{L^d\beta}\log Z\right] = \psi_0(\beta,\mu)$ and optimizing in b, we find for the choices (recall that we require $b \lesssim \rho_0^{-1/d}$)

$$b = \begin{cases} \min\left\{a(a^3\rho_0)^{-2/15}\zeta^{1/5}, \rho_0^{-1/3}\right\} & d=3, \\ \min\left\{a(a^2\rho_0)^{-1/4}\zeta^{1/4}, \rho_0^{-1/2}\right\} & d=2, \\ a(a\rho_0)^{-1/2}|\log a\rho_0|^{-1/2} & d=1, \end{cases}$$

that

$$\psi(\beta,\mu) \geq \psi_0(\beta,\mu) - 2\pi c_d \frac{-\operatorname{Li}_{d/2+1}(-e^{\beta\mu})}{(-\operatorname{Li}_{d/2}(-e^{\beta\mu}))^{1+2/d}} a^d \rho_0^{2+2/d}[1+\delta_d],$$

where δ_d is as in Equation (1.13). The calculations above are valid as long as the conditions of Theorem 4.3 are satisfied – that is, if $a^d\rho_0\zeta^{d/2}\left|\log a^d\rho_0\right|$ is sufficiently small. This concludes the proof of Proposition 1.9. It remains to give the proof of Lemma 5.2.

5.4. Error terms (proof of Lemma 5.2)

In this section we give the following:

Proof of Lemma 5.2. To better illustrate where the different error terms come from, we will write them in terms of the quantities $\|g\|_{L^1}, \left\|\gamma^{(1)}\right\|_{L^1}$ and $\|| \cdot |^n g\|_{L^1} = \int_{\mathbb{R}^d} |x|^n |g(x)|\,\mathrm{d}x$, $n \geq 1$. By Lemma 4.6 and Equation (3.6), we have the bounds

$$\|g\|_{L^1} \leq Ca^d \log b/a, \quad \left\|\gamma^{(1)}\right\|_{L^1} \leq C\zeta^{d/2} = C(1+|\log z|)^{d/2}, \quad \|| \cdot |^n g\|_{L^1} \leq Ca^d b^n.$$

For the analysis of the error terms, we use the bounds [Lau23, Equation (4.13)] and [LS24b, Equations (4.10) and (4.22)]. To state these, we define for any diagram $(\pi, G) \in \tilde{\mathcal{L}}_p^m$ the numbers $k = k(G) = k(\pi, G)$ as the number of clusters (connected components of G; recall Definition 4.1) entirely with internal vertices (of sizes $n_1, \ldots, n_k$) and $\kappa = \kappa(G) = \kappa(\pi, G)$ as the number of clusters with each at least one external vertex (of sizes [meaning number of internal vertices] $n_1^*, \ldots, n_\kappa^*$). Define

$$\nu^* := \sum_{\lambda=1}^{\kappa} n_\lambda^*, \qquad \nu := \sum_{\ell=1}^{k} n_\ell - 2k.$$

As discussed around [Lau23, Equation (4.13)] and [LS24b, Equations (4.10) and (4.22)], the numbers ν^* and ν count the 'number of added vertices'. Concretely, a diagram with k clusters of only internal vertices has at least $2k$ internal vertices. Then ν^* is the number of additional internal vertices in clusters with external vertices, and ν is the number of additional internal vertices in clusters with only internal vertices.

The bounds [Lau23, Equation (4.13)], [LS24b, Equations (4.10) and (4.22)] (note that there bounds on $\|g\|_{L^1}, \|\gamma^{(1)}\|_{L^1}$ analogous to those of Lemma 4.6 are already used) then read for any k_0, ν_0

$$\frac{1}{p!}\left|\sum_{\substack{(\pi,G)\in\tilde{\mathcal{L}}_p^m \\ k(\pi,G)=k_0 \\ \nu(\pi,G)+\nu^*(\pi,G)=\nu_0}} \Gamma_{\pi,G}^m\right| \le \begin{cases} CL^d\rho_0(C\rho_0\|g\|_{L^1})^{\nu_0+k_0}\|\gamma^{(1)}\|_{L^1}^{k_0-1} & m=0, \\ C_m\rho_0^m(C\rho_0\|g\|_{L^1})^{\nu_0+k_0}\|\gamma^{(1)}\|_{L^1}^{k_0} & m>0, \end{cases} \qquad p=2k_0+\nu_0, \quad (5.6)$$

where the constants C, C_m depend only on m but not on ν_0 or k_0 (in particular, not on p). □

Remark 5.3. The case $m=0$ is not included in the statement in [Lau23, Equation (4.13)] and [LS24b, Equations (4.10) and (4.22)]. It follows from the analysis in [LS24b, Section 3.1.1] (see also [Lau23, Section 4.1]), however.

More precisely, the analysis in [LS24b, Section 3.1.1] consists of the following steps: (1), decompose the linked diagrams in $\mathcal{L}_p$ according to the connected components of the graphs, (2) use the tree-graph inequality [Uel18] to bound each sum over graphs by a sum over trees in each connected component of the graph, (3) use the Brydges–Battle–Federbush formula (see [GMR21, Appendix D]) to bound the truncated correlations, (4) compute the integrals, each being now an integral of either $|g|$ or $|\gamma^{(1)}|$.

In any of the equations in [LS24b, Section 3.1.1], the only effect of the p-summation is to eliminate the factor $\chi_{(\sum_\ell n_\ell = p)}$ present in the very first equation (where there is no p-summation). That is, not performing the p-summation, all equations in [LS24b, Section 3.1.1] remain valid, only with no p-summation on their left-hand sides and with an additional factor $\chi_{(\sum_\ell n_\ell = p)}$ on their right-hand sides. Thus, from the analysis in [LS24b, Section 3.1.1], modified by not performing the p-summation, we find the following modification of the final formula in [LS24b, Section 3.1.1]:

$$\frac{1}{p!}\left|\sum_{\substack{(\pi,G)\in\mathcal{L}_p \\ k(\pi,G)=k_0 \\ \nu(\pi,G)=\nu_0}} \Gamma_{\pi,G}\right| \le CN\|\gamma^{(1)}\|_{L^1}^{k_0-1} \sum_{\substack{n_1,\ldots,n_{k_0}\ge 2 \\ \sum_\ell n_\ell = 2k_0+\nu_0}} (\rho_0\|g\|_{L^1})^{\sum_\ell(n_\ell-1)}, \qquad p=2k_0+\nu_0,$$

from which Equation (5.6) in the case $m=0$ follows.

From this bound, the natural 'size' of a diagram $(\pi,G)\in\tilde{\mathcal{L}}_p^m$ is not p but rather $\nu+\nu^*+k$, since its value is (neglecting log's and dependence on z) $\lesssim \rho_0^m(a^d\rho_0)^{\nu+\nu^*+k}$. For the bounds of the terms $\varepsilon_2, \varepsilon_3, \varepsilon_Z$, we will bound sufficiently large diagrams by the bound in Equation (5.6) and do a more precise computation for small diagrams.

Additionally, we have the following:

Lemma 5.4. *The reduced densities $\rho^{(3)}$ and $\rho^{(4)}$ satisfy*

$$\rho^{(3)}(x_1,x_2,x_3) \le C\rho_0^{3+4/d}|x_1-x_2|^2|x_1-x_3|^2,$$
$$\rho^{(4)}(x_1,x_2,x_3,x_4) \le C\rho_0^{4+6/d}|x_1-x_2|^2|x_1-x_3|^2|x_1-x_4|^2.$$

Proof. Note that both $\rho^{(3)}$ and $\rho^{(4)}$ vanish whenever two particles are incident and are invariant under permutation of the particle positions. Thus, for fixed x_1 as functions of x_j, $j\ne 1$, they vanish

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quadratically around $x_j = x_1$. Writing $\rho^{(q)} = \det[\gamma^{(1)}_{ij}]_{1\le i,j\le q}$ using the Wick rule, Taylor expanding in x_j, $j \ne 1$ around $x_j = x_1$ and using Equation (3.12) to bound the derivatives, we conclude the proof of the lemma. □

We first bound ε_Z.

5.4.1. Bound of ε_Z

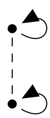
We have by Theorem 4.3

$$\varepsilon_Z = -\frac{1}{\beta}\log\frac{Z_J}{Z} = -\frac{1}{\beta}\sum_{p=2}^{\infty}\frac{1}{p!}\sum_{(\pi,G)\in\mathcal{L}_p}\Gamma_{\pi,G}.$$

We use the bound in Equation (5.6) above for $m = 0$ and for diagrams with $\nu + k \ge 2$. These are precisely the diagrams with $p \ge 3$ (note that $k \ge 1$ for any diagram $(\pi, G) \in \mathcal{L}_p$). Thus,

$$\begin{aligned}
\sum_{p=3}^{\infty}\frac{1}{p!}\left|\sum_{(\pi,G)\in\mathcal{L}_p}\Gamma_{\pi,G}\right| &\le CL^d\rho_0 \sum_{\substack{k_0\ge 1\\ \nu_0+k_0\ge 2}} (C\rho_0\|g\|_{L^1})^{\nu_0+k_0}\left\|\gamma^{(1)}\right\|_{L^1}^{k_0-1}\\
&= CL^d\rho_0\left[\sum_{\nu_0=1}^{\infty}(C\rho_0\|g\|_{L^1})^{\nu_0+1} + \sum_{k_0=2}^{\infty}\sum_{\nu_0=0}^{\infty}(C\rho_0\|g\|_{L^1})^{\nu_0+k_0}\left\|\gamma^{(1)}\right\|_{L^1}^{k_0-1}\right]\\
&\le CL^d\rho_0^3\|g\|_{L^1}^2\left(1+\left\|\gamma^{(1)}\right\|_{L^1}\right)
\end{aligned}$$

for sufficiently small $\rho_0\|g\|_{L^1}$ and $\rho_0\|g\|_{L^1}\left\|\gamma^{(1)}\right\|_{L^1}$. For the diagrams with $\nu + k = 1$, we do a more precise calculation. These are precisely the diagrams with $p = 2$. In particular, these diagrams have $\nu = 0$ and $k = 1$. We have then (recall that pictures of diagrams refer to their values)

$$\sum_{(\pi,G)\in\mathcal{L}_2}\Gamma_{\pi,G} =$$  $+$ 

$$\begin{aligned}
&= \iint \det\begin{bmatrix}\gamma^{(1)}(0) & \gamma^{(1)}(x-y)\\ \gamma^{(1)}(y-x) & \gamma^{(1)}(0)\end{bmatrix} g(x-y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint \rho^{(2)}(x,y)g(x-y)\,\mathrm{d}x\,\mathrm{d}y\\
&= O\left(L^d\left\||\cdot|^2 g\right\|_{L^1}\rho_0^{2+2/d}\right)
\end{aligned}$$

using Equation (3.11). Thus, using Lemma 4.6 and recalling that $\beta \sim \zeta\rho_0^{-2/d}$ from Remark 3.7, we conclude that

$$\begin{aligned}
\frac{1}{L^d}|\varepsilon_Z| = \frac{1}{\beta L^d}\left|\log\frac{Z_J}{Z}\right| &\le C\left\||\cdot|^2 g\right\|_{L^1}\rho_0^{2+4/d}\zeta^{-1} + C\|g\|_{L^1}^2\left(\left\|\gamma^{(1)}\right\|_{L^1}+1\right)\rho_0^{3+2/d}\zeta^{-1}\\
&\le Ca^d b^2\rho_0^{2+4/d}\zeta^{-1} + Ca^{2d}\rho_0^{3+2/d}\zeta^{d/2-1}(\log b/a)^2.
\end{aligned}$$

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5.4.2. Bound of ε_3

We have by Theorem 4.3

$$\rho_J^{(3)} = f_{12}^2 f_{13}^2 f_{23}^2 \left[\rho^{(3)} + \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\tilde{\mathcal{L}}_p^3} \Gamma_{\pi,G}^3 \right].$$

We use the bound in Lemma 5.4 to bound $\rho^{(3)}$ and the bound in Equation (5.6) on the remaining terms. (That is, a precise calculation for diagrams with $\nu + \nu^* + k = 0$ and the bound in Equation (5.6) for diagrams with $\nu + \nu^* + k \geq 1$.) Thus, by a similar computation as for ε_Z,

$$\begin{aligned} \left| \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{(\pi,G)\in\tilde{\mathcal{L}}_p^3} \Gamma_{\pi,G}^3 \right| &\leq C\rho_0^3 \left[\sum_{\nu_0=1}^{\infty} (C\rho_0 \|g\|_{L^1})^{\nu_0} + \sum_{k_0=1}^{\infty} \sum_{\nu_0=0}^{\infty} (C\rho_0\|g\|_{L^1})^{\nu_0+k_0} \left\|\gamma^{(1)}\right\|_{L^1}^{k_0} \right] \\ &\leq C\rho_0^4 \|g\|_{L^1} \left(1 + \left\|\gamma^{(1)}\right\|_{L^1}\right) \end{aligned}$$

for sufficiently small $\rho_0\|g\|_{L^1}$ and $\rho_0\|g\|_{L^1}\left\|\gamma^{(1)}\right\|_{L^1}$. Moreover, $f \leq 1$, and the support of ∇f is contained a ball of radius $\sim b$. Thus, by Equation (3.9) and Lemma 4.6,

$$\begin{aligned} |\varepsilon_3| &\leq CL^d \rho_0^{3+4/d} \left(\int f |\nabla f| |x|^2 \right)^2 + CL^d \|g\|_{L^1} \left(\left\|\gamma^{(1)}\right\|_{L^1} + 1 \right) \rho_0^4 \left(\int f|\nabla f| \right)^2 \\ &\leq CL^d a^{2d} b^2 \rho_0^{3+4/d} + CL^d a^{3d-2} \rho_0^4 \zeta^{d/2} \log b/a. \end{aligned}$$

Refined analysis in dimension $d = 1$.

In dimension $d = 1$, we need also to analyze diagrams with $k + \nu + \nu^* = 1$ in more detail. Intuitively, this follows by 'counting powers of ρ_0': the claimed leading term in Theorem 1.2 is of order $a\rho_0^4$. Thus, we need to compute precisely all diagrams for which the naive bound Equation (5.6) only gives a power ≤ 4 of ρ_0.

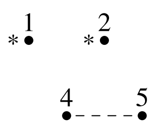
The diagrams with $k + \nu + \nu^* = 1$ have either $p = 1$, in which case $\nu^* = 1$, or $p = 2$, in which case $k = 1$. For the diagrams with $p = 1$ for any graph, any permutation makes each linked component have at least one external vertex, and thus, we get

$$\sum_{(\pi,G)\in\tilde{\mathcal{L}}_1^3} \Gamma_{\pi,G}^3 = \sum_{G\in\mathcal{G}_1^3} \int \rho^{(4)}(x_1,x_2,x_3,x_4) \prod_{e\in G} g_e \, \mathrm{d}x_4.$$

Bound all but one g-factor, by symmetry say g_{14}, by $|g_{ij}| \leq 1$, and bound $\rho^{(4)}$ using Lemma 5.4. We conclude

$$\begin{aligned} |\cdot| &\leq C\rho_0^{10} |x_1 - x_2|^2 |x_1 - x_3|^2 \int |g(z)||z|^2 \, \mathrm{d}z \\ &\leq Cab^2 \rho_0^{10} |x_1 - x_2|^2 |x_1 - x_3|^2. \end{aligned}$$

By Equation (3.9), this gives the contribution $La^3 b^4 \rho_0^{10}$ to ε_3. For $p = 2$, we have the graph (recall that $*$'s label external vertices)

$G =$ 

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The only π's for which $(\pi, G) \notin \tilde{\mathcal{L}}_2^3$ are those not connecting $\{4,5\}$ to $\{1,2,3\}$. Thus,

$$\sum_{(\pi,G)\in\tilde{\mathcal{L}}_2^3} \Gamma^3_{\pi,G} = \int \left[\rho^{(5)}(x_1,\ldots,x_5) - \rho^{(3)}(x_1,x_2,x_3)\rho^{(2)}(x_4,x_5)\right] g_{45}\,\mathrm{d}x_4\,\mathrm{d}x_5.$$

This vanishes (quadratically) whenever any x_i and x_j, $i,j = 1,2,3$ are incident. Thus, as with $\rho^{(3)}$ and $\rho^{(4)}$, we bound the derivatives and use Taylor's theorem. Denote the derivative w.r.t. x_j by ∂_{x_j}. We are thus interested in bounding $\partial_{x_2}^2\partial_{x_3}^2\Gamma^3_{\pi,G}$. By explicit computation (with the permutation denoted π^{-1} for convenience of notation), we have

$$\begin{aligned}
&\partial_{x_2}^2\partial_{x_3}^2\Gamma^3_{\pi^{-1},G} \\
&\quad= \partial_{x_2}^2\partial_{x_3}^2\left[(-1)^\pi \frac{1}{L^5}\sum_{k_1,\ldots,k_5}\hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_5)\iint e^{i(k_1-k_{\pi(1)})x_1}\cdots e^{i(k_5-k_{\pi(5)})x_5} g_{45}\,\mathrm{d}x_4\,\mathrm{d}x_5\right] \\
&\quad= -(-1)^\pi\frac{1}{L^4}\sum_{k_1,\ldots,k_5}(k_2-k_{\pi(2)})^2(k_3-k_{\pi(3)})^2\hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_5) \\
&\qquad\times e^{i(k_1-k_{\pi(1)})x_1}\cdots e^{i(k_3-k_{\pi(3)})x_3}\hat{g}(k_4-k_{\pi(4)})\chi_{(k_5-k_{\pi(5)}+k_4-k_{\pi(4)}=0)},
\end{aligned}$$

where χ denotes a characteristic function. Any permutation such that $(\pi,G) \in \tilde{\mathcal{L}}_3^2$ has $\pi(\{4,5\}) \neq \{4,5\}$. In particular, for the relevant permutations, the characteristic function is not identically one, and thus effectively it reduces the number of k-sums by 1. More precisely, we get for the permutations with $\pi(5),\pi(4) \neq 5$ (the others are similar)

$$\begin{aligned}
&= -(-1)^\pi\frac{1}{L^4}\sum_{k_1,\ldots,k_4}(k_2-k_{\pi(2)})^2(k_3-k_{\pi(3)})^2\hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_4)\hat{\gamma}^{(1)}(-k_4+k_{\pi(4)}+k_{\pi(5)}) \\
&\quad\times e^{i(k_1-k_{\pi(1)})x_1}\cdots e^{i(k_3-k_{\pi(3)})x_3}\hat{g}(k_4-k_{\pi(4)}).
\end{aligned}$$

Bounding $\left|\hat{\gamma}^{(1)}(-k_4+k_{\pi(4)}+k_{\pi(5)})\right| \leq 1$ and $|\hat{g}| \leq \|g\|_{L^1} \leq Ca\log b/a$, the k-sums are readily bounded by Equation (3.12). Thus, for any valid permutation π, we have

$$\left|\partial_{x_2}^2\partial_{x_3}^2\Gamma^3_{\pi,G}\right| \leq Ca\rho_0^{4+4}\log b/a.$$

By Taylor's theorem, we conclude that

$$\left|\Gamma^3_{\pi,G}\right| \leq Ca\rho_0^{4+4}\log b/a|x_1-x_2|^2|x_1-x_3|^2.$$

We thus get the contribution to ε_3 of $La^3b^2\rho_0^8\log b/a$ by Equation (3.9). Finally, using the bound in Equation (5.6) for diagrams with $k+\nu+\nu^* \geq 2$, we get (again for suffiently small $\rho_0\|g\|_{L^1}$ and $\rho_0\|g\|_{L^1}\|\gamma^{(1)}\|_{L^1}$)

$$\sum_{p=2}^\infty\frac{1}{p!}\left|\sum_{\substack{(\pi,G)\in\tilde{\mathcal{L}}_p^2 \\ (k+\nu+\nu^*)(\pi,G)\geq 2}}\Gamma^2_{\pi,G}\right| \leq Ca^3\rho_0^5\zeta(\log b/a)^2.$$

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By Equation (3.9), this gives a contribution to ε_3 of $La^2\rho_0^5(\log b/a)^2$. We conclude the bound

$$|\varepsilon_3| \le CL\Big(a^2b^4\rho_0^9 + a^3b^4\rho_0^{10} + a^3b^2\rho_0^8\log b/a + a^2\rho_0^5\zeta(\log b/a)^2\Big) \\ \le CLa^2\rho_0^5\zeta(\log b/a)^2$$

in dimension $d = 1$.

5.4.3. Bound of ε_2

We use the bound in Equation (5.6) for diagrams with $\nu + \nu^* + k \ge 3$ and a more precise analysis for the small diagrams. Write

$$\sum_{p=2}^{\infty}\frac{1}{p!}\sum_{(\pi,G)\in\tilde{\mathcal{L}}_p^2}\Gamma_{\pi,G}^2 = \xi_{=1} + \xi_{=2} + \xi_{\ge 3}, \tag{5.7}$$

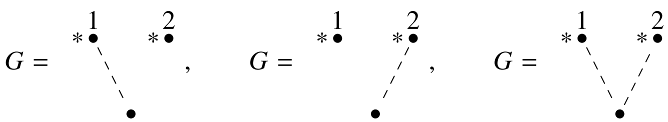
where $\xi_{=j}$ is the sum of the values of all diagrams with $\nu + \nu^* + k = j$ and $\xi_{\ge 3}$ is the sum of the values of all diagrams with $\nu + \nu^* + k \ge 3$.

For the large diagrams with $\nu+\nu^*+k \ge 3$, we have similarly as above for $\rho_0\|g\|_{L^1}$ and $\rho_0\|g\|_{L^1}\left\|\gamma^{(1)}\right\|_{L^1}$ sufficiently small

$$|\xi_{\ge 3}| = \left|\sum_{p=2}^{\infty}\frac{1}{p!}\sum_{\substack{(\pi,G)\in\tilde{\mathcal{L}}_p^2\\ (k+\nu+\nu^*)(\pi,G)\ge 2}}\Gamma_{\pi,G}^2\right| \le C\rho_0^5\|g\|_{L^1}^3\left(1 + \left\|\gamma^{(1)}\right\|_{L^1}^3\right). \tag{5.8}$$

Diagrams with $k + \nu + \nu^* = 1$.

For the diagrams with $p = 1$ and $p = 2$ with $k = 1$, we do a more precise calculation. For $p = 1$, there are three possible g-graphs: (Recall that $*$'s label the external vertices)



Any permutation makes any of these diagrams have at least one external vertex in each linked component, and thus,

$$\sum_{(\pi,G)\in\tilde{\mathcal{L}}_1^2}\Gamma_{\pi,G}^2 = \int\rho^{(3)}(x_1,x_2,x_3)[g_{13} + g_{23} + g_{13}g_{23}]\,\mathrm{d}x_3.$$

Bounding $|g_{13}g_{23}| \le |g_{13}|$ and recalling the bound $\rho^{(3)}(x_1,x_2,x_3) \le C\rho_0^{3+4/d}|x_1-x_2|^2|x_1-x_3|^2$ from Lemma 5.4, we get by symmetry

$$\left|\sum_{(\pi,G)\in\tilde{\mathcal{L}}_1^2}\Gamma_{\pi,G}^2\right| \le C\rho_0^{3+4/d}|x_1-x_2|^2\int|g(z)||z|^2\,\mathrm{d}z = C\left\||\cdot|^2g\right\|_{L^1}\rho_0^{3+4/d}|x_1-x_2|^2. \tag{5.9}$$

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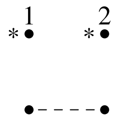
The diagrams with $p = 2$ and $k = 1$ have g-graph

$G =$  (5.10)

The only permutations π such that $(\pi, G) \notin \tilde{\mathcal{L}}_2^2$ are those connecting only external to external and internal to internal (i.e., those with either $\pi(3) = 3, \pi(4) = 4$ or $\pi(3) = 4, \pi(4) = 3$). Thus,

$$\sum_{\substack{(\pi,G)\in\tilde{\mathcal{L}}_2^2 \\ k(\pi,G)=1}} \Gamma^2_{\pi,G} = \iint \left[\rho^{(4)}(x_1,\ldots,x_4) - \rho^{(2)}(x_1,x_2)\rho^{(2)}(x_3,x_4)\right] g_{34}\,\mathrm{d}x_3\,\mathrm{d}x_4. \tag{5.11}$$

Clearly, this vanishes quadratically in $x_1 - x_2$ since both determinants do; thus, we bound it using Taylor's theorem, expanding in x_1 around $x_1 = x_2$ analogously to what we did for (some of the diagrams for) ε_3 above. We treat each diagram separately. (For convenience, we denote the permutation π^{-1}.) Denoting the derivative with respect to x_1^μ by $\partial^\mu_{x_1}$, we have

$$\begin{aligned}
\partial^\mu_{x_1}\partial^\nu_{x_1}\Gamma^2_{\pi^{-1},G} &= -\frac{1}{L^{4d}}\sum_{k_1,\ldots,k_4}\left(k_1^\mu - k^\mu_{\pi(1)}\right)\left(k_1^\nu - k^\nu_{\pi(1)}\right)\hat{\gamma}^{(1)}(k_1)\hat{\gamma}^{(1)}(k_2)\hat{\gamma}^{(1)}(k_3)\hat{\gamma}^{(1)}(k_4) \\
&\qquad \times e^{i(k_1-k_{\pi(1)})x_1}e^{i(k_2-k_{\pi(2)})x_2}\iint e^{i(k_3-k_{\pi(3)})x_3}e^{i(k_4-k_{\pi(4)})x_4}g(x_3-x_4)\,\mathrm{d}x_3\,\mathrm{d}x_4 \\
&= -\frac{1}{L^{3d}}\sum_{k_1,\ldots,k_4}\left(k_1^\mu - k^\mu_{\pi(1)}\right)\left(k_1^\nu - k^\nu_{\pi(1)}\right)\hat{\gamma}^{(1)}(k_1)\hat{\gamma}^{(1)}(k_2)\hat{\gamma}^{(1)}(k_3)\hat{\gamma}^{(1)}(k_4) \\
&\qquad \times \hat{g}(k_{\pi(3)} - k_3)\chi_{(k_4-k_{\pi(4)}=k_{\pi(3)}-k_3)}.
\end{aligned}$$

The only permutations for which the characteristic function is identically 1 are those with either $\pi(3) = 3, \pi(4) = 4$ or $\pi(3) = 4, \pi(4) = 3$. These are exactly the permutations that do not appear in Equation (5.11) above. Thus, similarly as for (some of the diagrams for) ε_3 above, the characteristic function effectively reduces the number of k-sums by 1. Bounding $|\hat{g}| \le \|g\|_{L^1}$, $\hat{\gamma}^{(1)} \le 1$ for one of the $\gamma^{(1)}$-factors, and using Equation (3.12) to bound the k-sums, we have for any diagram $(\pi, G) \in \tilde{\mathcal{L}}_2^2$ with G as in Equation (5.10)

$$\left|\partial^\mu_{x_1}\partial^\nu_{x_1}\Gamma^2_{\pi,G}\right| \le C\|g\|_{L^1}\rho_0^{3+2/d}.$$

We conclude the bound

$$\left|\sum_{\substack{(\pi,G)\in\tilde{\mathcal{L}}_2^2 \\ k(\pi,G)=1}} \Gamma^2_{\pi,G}\right| \le C\|g\|_{L^1}\rho_0^{3+2/d}|x_1 - x_2|^2. \tag{5.12}$$

In particular, by combining Equations (5.9) and (5.12), we have

$$|\xi_{=1}| \le C\|g\|_{L^1}\rho_0^{3+2/d}|x_1 - x_2|^2. \tag{5.13}$$

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Diagrams with $k + \nu + \nu^* = 2$.
Finally, consider all diagrams with $k + \nu + \nu^* = 2$ more precisely. We split these into three groups.

(i) $\nu^* = 2$
(ii) $\nu^* = 1$ and vertices $\{1\}$ and $\{2\}$ are connected
(iii) Remaining diagrams

We will use a Taylor expansion to bound the values of the diagrams in group (iii). Write

$$\xi_{=2} = \xi_{\text{(i)}} + \xi_{\text{(ii)}} + \xi_{\text{(iii)}}.$$

Then as $\rho_J^{(2)}(x_2; x_2) = 0$, we get from Equation (5.7)

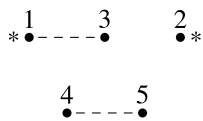
$$\left|\xi_{\text{(iii)}}(x_2, x_2)\right| \le \left|\xi_{\text{(i)}}(x_2, x_2)\right| + \left|\xi_{\text{(ii)}}(x_2, x_2)\right| + |\xi_{=1}(x_2, x_2)| + |\xi_{\ge 3}(x_2, x_2)|.$$

Moreover, $\xi_{\text{(iii)}}$ is symmetric in exchange of x_1 and x_2, so the first order vanishes. We conclude by Taylor's theorem that

$$\begin{aligned} \left|\xi_{\text{(iii)}}(x_1, x_2)\right| \le{}& \left|\xi_{\text{(i)}}(x_2, x_2)\right| + \left|\xi_{\text{(ii)}}(x_2, x_2)\right| + |\xi_{=1}(x_2, x_2)| + |\xi_{\ge 3}(x_2, x_2)| \\ &+ C \sup_{\mu,\nu} \sup_{z_1, z_2} \left|\partial_{x_1}^\mu \partial_{x_1}^\nu \xi_{\text{(iii)}}(z_1, z_2)\right| |x_1 - x_2|^2, \end{aligned} \tag{5.14}$$

where again $\partial_{x_1}^\mu$ denotes the derivative w.r.t. x_1^μ. Bounding $\partial_{x_1}^\mu \partial_{x_1}^\nu \xi_{\text{(iii)}}$ is analogous to the argument in [LS24b, Proof of Lemmas 4.1 and 4.8]: For diagrams with an internal vertex connected to $\{1\}$ with a g-edge, we do a precise calculation as in [LS24b, Proof of Lemma 4.8]. For the remaining diagrams where $\{1\}$ has no incident g-edges, we modify the proof of the absolute convergence of the GGR expansion as in [LS24b, Proof of Lemma 4.1].

First, the diagrams in group (iii) with an internal vertex connected to $\{1\}$ with a g-edge all have g-graph

$G =$  (5.15)

since $\nu^* = 1$ and $k + \nu + \nu^* = 2$. Then

$$\begin{aligned} \Gamma^2_{\pi^{-1},G} &= (-1)^\pi \frac{1}{L^{5d}} \sum_{k_1,\ldots,k_5} \hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_5) \iiint e^{i(k_1 - k_{\pi(1)})x_1} \cdots e^{i(k_5 - k_{\pi(5)})x_5} g_{13} g_{45}\, \mathrm{d}x_3\, \mathrm{d}x_4\, \mathrm{d}x_5 \\ &= (-1)^\pi \frac{1}{L^{4d}} \sum_{k_1,\ldots,k_5} \hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_5) e^{i(k_1 - k_{\pi(1)} + k_3 - k_{\pi(3)})x_1} e^{i(k_2 - k_{\pi(2)})x_2} \\ &\quad \times \hat{g}(k_3 - k_{\pi(3)}) \hat{g}(k_5 - k_{\pi(5)}) \chi_{(k_4 - k_{\pi(4)} + k_5 - k_{\pi(5)} = 0)}. \end{aligned}$$

The characteristic function χ is identically 1 only if $\pi(\{4,5\}) = \{4,5\}$, but then $(\pi, G) \notin \tilde{\mathcal{L}}_3^2$ so these permutations do not appear in $\xi_{\text{(iii)}}$. Taking the derivative, bounding $|\hat{g}| \le \|g\|_{L^1}$ and using Equation (3.12) to bound the k-sums, we conclude as above that

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$$\left|\partial_{x_1}^{\mu}\partial_{x_1}^{\nu}\Gamma^2_{\pi^{-1},G}\right| \le C\rho_0^{4+2/d}\|g\|_{L^1}^2$$

for all diagrams $(\pi, G) \in \tilde{\mathcal{L}}_3^2$ with G as in Equation (5.15).

Next, for the diagrams with no g-edges connected to $\{1\}$, the argument is as for the bound of $\partial_{x_1}^{\mu}\partial_{x_1}^{\nu}\xi_0$ in [LS24b, Proof of Lemma 4.1]. Analogously to [LS24b, Equations (4.19) and (4.20)], we conclude the bound (the term 1 in the factor $\|\gamma^{(1)}\|_{L^1} + 1$ arises similarly as in the bounds above from the value of diagrams with $k = 1$)

$$\left|\partial_{x_1}^2 \sum_{\substack{(\pi,G)\in\tilde{\mathcal{L}}_3^2\\ \text{no} g\text{-edges incident to } \{1\}}} \Gamma^2_{\pi^{-1},G}\right| \le C\rho_0^4\|g\|_{L^1}^2\left(\left\|\gamma^{(1)}\right\|_{L^1} + 1\right)\left[\rho_0^{2/d}\left\|\gamma^{(1)}\right\|_{L^1} + \rho_0^{1/d}\left\|\partial\gamma^{(1)}\right\|_{L^1} + \left\|\partial^2\gamma^{(1)}\right\|_{L^1}\right],$$

where with a similar abuse of notation,

$$\left\|\partial\gamma^{(1)}\right\|_{L^1} = \max_{\mu}\int_{[0,L]^d}\left|\partial^{\mu}\gamma^{(1)}\right|\mathrm{d}x, \qquad \left\|\partial^2\gamma^{(1)}\right\|_{L^1} = \max_{\mu,\nu}\int_{[0,L]^d}\left|\partial^{\mu}\partial^{\nu}\gamma^{(1)}\right|\mathrm{d}x.$$

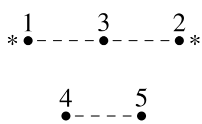
Recall that $\|\gamma^{(1)}\|_{L^1} \le C\zeta^{d/2}$ by Lemma 4.6. By a simple modification of the proof of Lemma 4.6, we may bound $\|\partial\gamma^{(1)}\|_{L^1} \le C\zeta^{d/2}\rho_0^{1/d}$ and $\|\partial^2\gamma^{(1)}\|_{L^1} \le C\zeta^{d/2}\rho_0^{2/d}$. Thus,

$$\left|\partial_{x_1}^2\xi_{(\mathrm{iii})}(z_1, z_2)\right| \le C\rho_0^{4+2/d}\|g\|_{L^1}^2\zeta^d. \tag{5.16}$$

Next, we bound $\xi_{(\mathrm{i})}$. For the diagrams with $\nu^* = 2$, if G is any graph with $\nu^*(G) = 2$, then for any permutation $\pi \in \mathcal{S}_4$, we have $(\pi, G) \in \tilde{\mathcal{L}}_2^2$. Thus, using Lemma 5.4 to bound $\rho^{(4)}$ and bounding some g-factors by 1, we get similarly to Equation (5.9)

$$\begin{aligned}\xi_{(\mathrm{i})} &= \sum_{\substack{G\in\mathcal{G}_2^2\\ \nu^*(G)=2}} \iint \rho^{(4)}\prod_{e\in G} g_e \,\mathrm{d}x_3\,\mathrm{d}x_4\\ \left|\xi_{(\mathrm{i})}\right| &\le C\rho_0^{4+6/d}|x_1 - x_2|^2\iint |g(z_1)|^2|g(z_2)|^2(|z_1|^2 + |z_2|^2 + |x_1 - x_2|^2)^2\,\mathrm{d}z_1\,\mathrm{d}z_2\\ &\le C\|g\|_{L^1}^2\rho_0^{4+6/d}|x_1 - x_2|^2(b^2 + |x_1 - x_2|^2)^2.\end{aligned} \tag{5.17}$$

Finally, we bound $\xi_{(\mathrm{ii})}$. All diagrams with $\nu^* = 1$ and $\{1\}$ and $\{2\}$ connected have g-graph

$G_0 =$  (5.18)

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For convenience of notation, we denote the permutation in the diagram π^{-1}. Then

$$
\begin{aligned}
&\Gamma^2_{\pi^{-1},G_0}\\
&= (-1)^\pi \frac{1}{L^{5d}} \sum_{k_1,\ldots,k_5} \hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_5)\\
&\quad \times \iiint e^{i(k_1-k_{\pi(1)})x_1}\cdots e^{i(k_5-k_{\pi(5)})x_5} g(x_1-x_3)g(x_2-x_3)g(x_4-x_5)\,\mathrm{d}x_3\,\mathrm{d}x_4\,\mathrm{d}x_5\\
&= (-1)^\pi \frac{1}{L^{5d}} \sum_{k_1,\ldots,k_5} \hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_5) e^{i\left(k_1-k_{\pi(1)}-\frac{k_3-k_{\pi(3)}}{2}\right)x_1} e^{i\left(k_2-k_{\pi(2)}-\frac{k_3-k_{\pi(3)}}{2}\right)x_2}\\
&\quad \times \int e^{i(k_3-k_{\pi(3)})(x_3-\frac{x_1+x_2}{2})} g\Big(\frac{x_1-x_2}{2}+\frac{x_1+x_2}{2}-x_3\Big) g\Big(-\frac{x_1-x_2}{2}+\frac{x_1+x_2}{2}-x_3\Big)\,\mathrm{d}x_3\\
&\quad \times \iint g(x_4-x_5) e^{i(k_4-k_{\pi(4)})(x_4-x_5)} e^{i(k_5-k_{\pi(5)}+k_4-k_{\pi(4)})x_5}\,\mathrm{d}x_4\,\mathrm{d}x_5\\
&= (-1)^\pi \frac{1}{L^{4d}} \sum_{k_1,\ldots,k_5} \hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_5)\\
&\quad \times e^{i\left(k_1-k_{\pi(1)}-\frac{k_3-k_{\pi(3)}}{2}\right)x_1} e^{i\left(k_2-k_{\pi(2)}-\frac{k_3-k_{\pi(3)}}{2}\right)x_2} \hat{G}_1(k_3-k_{\pi(3)})\hat{g}(k_{\pi(4)}-k_4)\chi_{(k_5-k_{\pi(5)}+k_4-k_{\pi(4)}=0)},
\end{aligned}
$$

where

$$
\hat{G}_1(k) := \int e^{-ikz} g\Big(\frac{x_1-x_2}{2}+z\Big) g\Big(-\frac{x_1-x_2}{2}+z\Big)\,\mathrm{d}z.
$$

We group together pairs of diagrams π and (using cycle notation) $\pi\cdot(4\,5) = (\pi(4)\,\pi(5))\cdot\pi$, meaning where $\pi(4)$ and $\pi(5)$ are swapped. These have opposite signs. Thus,

$$
\begin{aligned}
&\Gamma^2_{\pi^{-1},G_0} + \Gamma^2_{(\pi(4\,5))^{-1},G_0}\\
&\quad = (-1)^\pi \frac{1}{L^{4d}} \sum_{k_1,\ldots,k_5} \hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_5) e^{i\left(k_1-k_{\pi(1)}-\frac{k_3-k_{\pi(3)}}{2}\right)x_1} e^{i\left(k_2-k_{\pi(2)}-\frac{k_3-k_{\pi(3)}}{2}\right)x_2}\\
&\qquad \times \hat{G}_1(k_3-k_{\pi(3)})\chi_{(k_5-k_{\pi(5)}+k_4-k_{\pi(4)}=0)}\big[\hat{g}(k_{\pi(4)}-k_4)-\hat{g}(k_{\pi(5)}-k_4)\big].
\end{aligned}
$$

We Taylor expand $\hat{g}(k_{\pi(5)}-k_4)$ in $k_{\pi(5)}$ around $k_{\pi(5)} = k_{\pi(4)}$. That is,

$$
\hat{g}(k_{\pi(5)}-k_4) = \hat{g}(k_{\pi(4)}-k_4) + O(\nabla\hat{g})\big|k_{\pi(4)}-k_{\pi(5)}\big|,
$$

where $O(\nabla\hat{g})$ should be interpreted as being bounded by $|\nabla\hat{g}(k)| \le \int |x||g(x)| = \||\cdot|g\|_{L^1}$ uniformly in $k_{\pi(4)}-k_{\pi(5)}$. Moreover, $\big|\hat{G}_1\big| \le \|g\|_{L^1}$. Thus,

$$
\begin{aligned}
&\Big|\Gamma^2_{\pi^{-1},G_0} + \Gamma^2_{(\pi(4\,5))^{-1},G_0}\Big|\\
&\quad \le C\|g\|_{L^1}\||\cdot|g\|_{L^1} \times \frac{1}{L^{4d}} \sum_{k_1,\ldots,k_5} \hat{\gamma}^{(1)}(k_1)\cdots\hat{\gamma}^{(1)}(k_5)\big|k_{\pi(4)}-k_{\pi(5)}\big|\chi_{(k_5-k_{\pi(5)}+k_4-k_{\pi(4)}=0)}.
\end{aligned}
$$

The characteristic function is not identically 1 for linked diagrams. Indeed, if $\pi(\{4,5\}) = \{4,5\}$, then the diagram would not be linked. Thus, the characteristic function effectively reduces the number of

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k-sums by 1. Bounding similarly as above $\hat{\gamma}^{(1)} \leq 1$ and using finally Equation (3.12) to bound the k-sums, we conclude for any permutation π such that $(\pi, G_0) \in \tilde{\mathcal{L}}_3^2$ that

$$\left|\Gamma^2_{\pi^{-1},G_0} + \Gamma^2_{(\pi(4\,5))^{-1},G_0}\right| \leq C\rho_0^{4+1/d} ||| \cdot |g\|_{L^1}\|g\|_{L^1}.$$

Since π and $\pi(4\,5)$ either both give rise to linked diagrams or neither do, we conclude that

$$\left|\xi_{\text{(ii)}}\right| = \frac{1}{3!}\left|\sum_{(\pi,G_0)\in\tilde{\mathcal{L}}_3^2} \Gamma_{\pi,G_0}\right| \leq C\rho_0^{4+1/d} ||| \cdot |g\|_{L^1}\|g\|_{L^1}. \tag{5.19}$$

Combining then Equations (5.8), (5.13), (5.14), (5.16), (5.17) and (5.19) and using Lemma 4.6, we conclude the bound

$$\left|\xi_{\text{(iii)}}\right| \leq Ca^{2d}b\rho_0^{4+1/d}\log b/a + Ca^{3d}\rho_0^5\zeta^{3d/2}(\log b/a)^3 + Ca^{2d}\rho_0^{4+2/d}\zeta^d(\log b/a)^2|x_1-x_2|^2.$$

We conclude the bound

$$\begin{aligned}\left|\sum_{p=1}^{\infty}\frac{1}{p!}\sum_{(\pi,G)\in\tilde{\mathcal{L}}_p^2}\Gamma^2_{\pi,G}\right| &\leq Ca^d\rho_0^{3+2/d}\log b/a|x_1-x_2|^2 + a^db^2\rho_0^{3+4/d}|x_1-x_2|^2 \\ &\quad + Ca^d\rho_0^{4+6/d}|x_1-x_2|^2(b^2+|x_1-x_2|^2)^2(\log b/a)^2 + Ca^{2d}b\rho_0^{4+1/d}\log b/a \\ &\quad + Ca^{2d}\rho_0^{4+2/d}\zeta^d(\log b/a)^2|x_1-x_2|^2 + Ca^{3d}\rho_0^5\zeta^{3d/2}(\log b/a)^3.\end{aligned}$$

Thus, using Lemma 3.4, we get

$$\begin{aligned}\frac{|\varepsilon_2|}{L^d} &\leq Ca^{2d}\rho_0^{3+2/d}\log b/a + Ca^{2d}b^2\rho_0^{3+4/d} + Ca^{4d}b^{4-d}\rho_0^{4+6/d}(\log b/a)^2 + Ca^{3d-2}b\rho_0^{4+1/d}\log b/a \\ &\quad + Ca^{3d}\rho_0^{4+2/d}\zeta^d(\log b/a)^2 + Ca^{4d-2}\rho_0^5\zeta^{3d/2}(\log b/a)^3. \\ &\leq \begin{cases} Ca^{2d}\rho_0^{3+2/d}\log b/a + Ca^{4d-2}\rho_0^5\zeta^{3d/2}(\log b/a)^3 & d\geq 2, \\ Cab\rho_0^5\log b/a + Ca^2\rho_0^5\zeta^{3/2}(\log b/a)^3 & d=1.\end{cases}\end{aligned}$$

This concludes the proof of Lemma 5.2.

Remark 5.5 (Necessity of precise analysis of diagrams with $k+\nu+\nu^*=2$). For the bound of ε_2, we give here a precise analysis of the diagrams with $k+\nu+\nu^*=2$. In general, one should not expect this to be needed in dimensions $d=2,3$. More precisely, by just considering powers of ρ_0, one would expect that diagrams with $k+\nu+\nu^*\geq 1$ are all subleading as they carry a higher power of ρ_0 (using Equation (5.6)) than the claimed leading term, with exponent $2+2/d$.

The reason we need a precise analysis here is the temperature dependence of our bounds: for some regime of temperatures, the bound one would get by using Equation (5.6) is not good enough.

Remark 5.6 (Optimality of the error bounds). One should not expect the bound given in Equation (5.19) to be optimal. More precisely, in Equation (5.19), we only took into account the cancellations of pairs of diagrams. However, one should expect much more cancellations. We have

$$\xi_{\text{(ii)}} = \frac{1}{3!}\iiint\left[\rho^{(5)}(x_1,\ldots,x_5) - \rho^{(3)}(x_1,x_2,x_3)\rho^{(2)}(x_4,x_5)\right]g_{13}g_{23}g_{45}\,\mathrm{d}x_3\,\mathrm{d}x_4\,\mathrm{d}x_5.$$

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Naively, just using that $\rho^{(5)}(x_1,\dots,x_5) - \rho^{(3)}(x_1,x_2,x_3)\rho^{(2)}(x_4,x_5)$ vanishes whenever any two of the particles 1, 2, 3 or the particles 4, 5 are incident, we get by Taylor expansion

$$\left|\rho^{(5)}(x_1,\dots,x_5) - \rho^{(3)}(x_1,x_2,x_3)\rho^{(2)}(x_4,x_5)\right| \le C\rho_0^{5+6/d}|x_1-x_2|^2|x_1-x_3|^2|x_4-x_5|^2. \tag{5.20}$$

Using this bound and bounding $|g_{23}| \le 1$, we get

$$\left|\xi_{(\mathrm{ii})}\right| \le \rho_0^{5+6/d}a^{2d}b^4L^d|x_1-x_2|^2. \tag{5.21}$$

This bound is too large by a volume factor. (This arises since we 'forget' that the relevant diagrams are linked when we do the Taylor expansion.) It, however, illustrates how many more cancellations between the different permutations are present than what we used in the bound Equation (5.19) — it carries a higher power of ρ_0. Using these cancellations but losing the information that diagrams are linked is what we did in [LS24b].

If one could somehow see these cancellations, while still keeping the information that the diagrams have to be linked, one might be able to improve upon the bound Equation (5.19). In one dimension, this error term is actually (for some regime of temperatures) the dominant error term. Thus, by improving the analysis of these diagrams, one might improve the error term in Proposition 1.9 in $d=1$.

A. Particle density of the trial state

In this section, we give the following:

Proof of Equation (3.3). We calculate $\langle\mathcal{N}\rangle_J$ and compare it to $\langle\mathcal{N}\rangle_0 = \rho_0L^d$. We have by Equation (4.2)

$$\langle\mathcal{N}\rangle_J = \int \rho_J^{(1)}(x)\,\mathrm{d}x = L^d\left[\rho^{(1)} + \sum_{p=1}^\infty \frac{1}{p!}\sum_{(\pi,G)\in\mathcal{L}_p^1}\Gamma_{\pi,G}^1\right] = \langle\mathcal{N}\rangle_0 + L^d\sum_{p=1}^\infty\frac{1}{p!}\sum_{(\pi,G)\in\mathcal{L}_p^1}\Gamma_{\pi,G}^1.$$

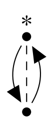
Next, we bound $\sum_{p=1}^\infty \frac{1}{p!}\sum_{(\pi,G)\in\mathcal{L}_p^1}\Gamma_{\pi,G}^1$. We use the bound in Equation (5.6) for diagrams with $k+\nu+\nu^*\ge 2$, i.e., for $p=2$ with $k=0$, $\nu^*=2$ and for $p\ge 3$. That is,

$$\frac{1}{2!}\left|\sum_{\substack{(\pi,G)\in\mathcal{L}_2^1\\ k(\pi,G)=0}}\Gamma_{\pi,G}^1\right| \le C\|g\|_{L^1}^2\rho_0^3, \qquad \sum_{p=3}^\infty\frac{1}{p!}\left|\sum_{(\pi,G)\in\mathcal{L}_p^1}\Gamma_{\pi,G}^1\right| \le C\|g\|_{L^1}^2\left(1+\left\|\gamma^{(1)}\right\|_{L^1}^2\right)\rho_0^3$$

for sufficiently small $\rho_0\|g\|_{L^1}$ and $\rho_0\|g\|_{L^1}\left\|\gamma^{(1)}\right\|_{L^1}$. Thus, we get

$$\sum_{p=1}^\infty\frac{1}{p!}\sum_{(\pi,G)\in\mathcal{L}_p^1}\Gamma_{\pi,G}^1 = \sum_{(\pi,G)\in\mathcal{L}_1^1}\Gamma_{\pi,G}^1 + \frac{1}{2}\sum_{\substack{(\pi,G)\in\mathcal{L}_2^1\\ k(\pi,G)=1}}\Gamma_{\pi,G}^1 + O\left(\|g\|_{L^1}^2\left(\left\|\gamma^{(1)}\right\|_{L^1}^2+1\right)\rho_0^3\right).$$

For the $p=1$-term, there are two diagrams. Thus (where $*$ labels the external vertex),

$$\sum_{(\pi,G)\in\mathcal{L}_1^1}\Gamma_{\pi,G}^1 =$$  +  $$= \int \det\begin{bmatrix}\gamma^{(1)}(0) & \gamma^{(1)}(x)\\ \gamma^{(1)}(x) & \gamma^{(1)}(0)\end{bmatrix} g(x)\,\mathrm{d}x = O\left(\left\||\cdot|^2 g\right\|_{L^1}\rho_0^{2+2/d}\right).$$

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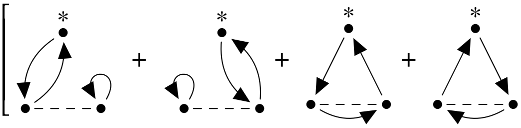
For the $p = 2$-term with $k = 1$, there are 4 diagrams. Thus,

$$\frac{1}{2} \sum_{\substack{(\pi,G)\in\mathcal{L}_2^1 \\ k(\pi,G)=1}} \Gamma^1_{\pi,G} = \frac{1}{2}\Big[\ \cdots + \cdots + \cdots + \cdots\ \Big]$$



$$\begin{aligned} &= \frac{1}{L^{3d}} \sum_{k_1,k_2,k_3} \iint \mathrm{d}x_2\, \mathrm{d}x_3\, \hat\gamma^{(1)}(k_1)\hat\gamma^{(1)}(k_2)\hat\gamma^{(1)}(k_3) g(x_2 - x_3) \\ &\quad \times \left[e^{i(k_1-k_2)(x_1-x_2)} - e^{ik_1(x_1-x_2)} e^{ik_2(x_2-x_3)} e^{ik_3(x_3-x_1)} \right] \\ &= \frac{1}{L^{2d}} \sum_{k_1,k_2,k_3} \hat\gamma^{(1)}(k_1)\hat\gamma^{(1)}(k_2)\hat\gamma^{(1)}(k_3)\hat g(k_1-k_2)\left[\chi_{(k_1=k_2)} - \chi_{(k_1=k_3)}\right] \\ &= \frac{1}{L^{2d}} \sum_{k,\ell} \hat\gamma^{(1)}(k)^2 \hat\gamma^{(1)}(\ell)[\hat g(0) - \hat g(k-\ell)]. \end{aligned}$$

Taylor expanding $\hat g$ and using that $\int x g(x) = 0$ so $\nabla \hat g(0) = 0$, we get

$$\left| \frac{1}{2} \sum_{\substack{(\pi,G)\in\mathcal{L}_2^1 \\ k(\pi,G)=1}} \Gamma^1_{\pi,G} \right| \le C \big\| |\cdot|^2 g \big\|_{L^1} \rho_0^{2+2/d}.$$

Thus, by Lemma 4.6,

$$\begin{aligned} \left| \sum_{p=1}^\infty \frac{1}{p!} \sum_{(\pi,G)\in\mathcal{L}_p^1} \Gamma^1_{\pi,G} \right| &\le C \big\| |\cdot|^2 g \big\|_{L^1} \rho_0^{2+2/d} + C\|g\|_{L^1}^2 \left\|\gamma^{(1)}\right\|_{L^1}^2 \rho_0^3 + C\|g\|_{L^1}^2 \rho_0^3 \\ &\le C a^d b^2 \rho_0^{2+2/d} + C a^{2d} \rho_0^3 \zeta^d (\log b/a)^2. \end{aligned}$$

That is, Equation (3.3) is satisfied. □

Competing interest. The authors have no competing interest to declare.

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