Canad. Math. Bull. Vol. 58 (4), 2015 pp. 835–845
http://dx.doi.org/10.4153/CMB-2015-039-7

Real Hypersurfaces in Complex Two-Plane Grassmannians with GTW Harmonic Curvature

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Abstract. We prove the non-existence of Hopf real hypersurfaces in complex two-plane Grassmannians with harmonic curvature with respect to the generalized Tanaka–Webster connection if they satisfy some further conditions.

1 Introduction

The generalized Tanaka–Webster connection (GTW connection) for contact metric manifolds was introduced by Tanno [12] as a generalization of the connection defined by Tanaka in [11] and, independently, by Webster in [13]. The Tanaka–Webster connection is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. A real hypersurface M in a Kähler manifold has an (integrable) CR-structure associated with the almost contact structure (ϕ, ξ, η, g) induced on M by the Kähler structure, but, in general, this CR-structure is not guaranteed to be pseudo-Hermitian. Cho defined the GTW connection for a real hypersurface of a Kähler manifold (see [4, 5]) by

$$\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.1}$$

for any X, Y tangent to M, where ∇ denotes the Levi–Civita connection on M, A is the shape operator on M, and k is a non-zero real number. In particular, if the real hypersurface satisfies $A\phi + \phi A = 2k\phi$, then the GTW connection $\widehat{\nabla}^{(k)}$ coincides with the Tanaka–Webster connection (see [4]).

Let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. It is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see Berndt and Suh [2]). In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible Kähler, quaternionic Kähler manifold that is not a hyper-Kähler manifold.

Received by the editors August 29, 2014; revised February 3, 2015.
Published electronically July 14, 2015.
This work was supported by grant Proj. No. NRF-2015-R1A2A1A-01002459 from the National Research Foundation of Korea. The first author is partially supported by MCT-FEDER Grant MTM2010-18099, the second author by Grant Proj. No. NRF-2012-R1A2A2A-01043023, and the third author by Fostering Core Leaders No. NRF-2013-H1A8A1004325.
AMS subject classification: **53C40**, 53C15.
Keywords: real hypersurfaces, complex two-plane Grassmannians, Hopf hypersurface, generalized Tanaka–Webster connection, harmonic curvature.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ and let N be a local normal unit vector field on M. Also, let A be the shape operator of M associated with N. Then we define the structure vector field of M by $\xi = -JN$. Moreover, if $\{J_1, J_2, J_3\}$ is a local basis of $\mathfrak{J}$, we define $\xi_i = -J_i N$, $i = 1, 2, 3$. We will call $\mathfrak{D}^\perp = \operatorname{Span}\{\xi_1, \xi_2, \xi_3\}$.

M is called *Hopf* if ξ is principal, that is, $A\xi = \alpha\xi$. Berndt and Suh [2] proved that if $m \geq 3$, a real hypersurface M of $G_2(\mathbb{C}^{m+2})$ for which both $[\xi]$ and $\mathfrak{D}^\perp$ are A-invariant must be an open part of either (A) a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or (B) a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$. In this second case $m = 2n$.

Let S denote the Ricci tensor of the real hypersurface M. In [7] we proved the non-existence of Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel Ricci tensor, that is $\nabla S = 0$, if the Ricci tensor commutes with the structure tensor ϕ.

This result was improved by Suh [9] who proved that the second condition is redundant.

Recently, in [8], as a generalization of the notion of the parallelism of the Ricci tensor we have studied real hypersurfaces in a complex two-plane Grassmannian with GTW connection, obtaining the following non-existence theorem.

Theorem 1.1 *There do not exist connected, orientable, Hopf, real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose Ricci tensor is parallel with respect to the GTW connection.*

The tensor field T of type (1,1) on M is called of *Codazzi type* if $(\nabla_X T)Y = (\nabla_Y T)X$ for any X, Y tangent to M. In the case of the Ricci tensor S, if it is of Codazzi type, M is said to have harmonic curvature. Suh [10] has recently proved the following theorem.

Theorem 1.2 *Let M be a Hopf real hypersurface of harmonic curvature with constant scalar and mean curvatures. If the shape operator commutes with the structure tensor ϕ on the distribution $\mathfrak{D}^\perp$, then M is locally congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius r, $\cot^2(\sqrt{2}r) = \frac{4}{3}(m-1)$.*

In this paper we deal with the same conditions considering the GTW on M. We will say that M has GTW harmonic curvature if $(\widehat{\nabla}^{(k)}_X S)Y = (\widehat{\nabla}^{(k)}_Y S)X$ for any X, Y tangent to M. To prove this result, we need two geometric notions, mean and scalar curvature. Mean curvature h is the trace of the shape operator $h = \operatorname{Tr}(A)$ and scalar curvature r is defined by the trace of the Ricci tensor *i.e.,* $r = \operatorname{Tr}(S)$. Thus, we will prove the following theorem.

Theorem 1.3 *There do not exist Hopf real hypersurfaces of GTW harmonic curvature with constant scalar and mean curvatures in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, if the shape operator commutes with the structure tensor ϕ on the distribution $\mathfrak{D}^\perp$.*

2 Preliminaries

For the study of the Riemannian geometry of $G_2(\mathbb{C}^{m+2})$, see [1]. All the notation we will use from now on are from [2, 3]. We will suppose that the metric g of $G_2(\mathbb{C}^{m+2})$ is normalized for the maximal sectional curvature of the manifold to be eight. Then the Riemannian curvature tensor $\overline{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\overline{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &\quad - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + \sum_{\nu=1}^{3}\left\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\right\} \\ &\quad + \sum_{\nu=1}^{3}\left\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\right\},\end{aligned}$$

where $\{J_1,J_2,J_3\}$ is any canonical local basis of $\mathfrak{J}$.

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ,ξ,η,g). More explicitly, we can define a tensor field ϕ of type (1,1), a vector field ξ and its dual 1-form η on M by $g(\phi X,Y) = g(JX,Y)$ and $\eta(X) = g(\xi,X)$ for any tangent vector fields X and Y on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \text{and} \quad \eta(\xi) = 1$$

for any tangent vector field X on M. Furthermore, let $\{J_1,J_2,J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M in such a way that a tensor field ϕ_ν of type (1,1), a vector field ξ_ν and its dual 1-form η_ν on M are defined by $g(\phi_\nu X,Y) = g(J_\nu X,Y)$ and $\eta_\nu(X) = g(\xi_\nu,X)$ for any tangent vector fields X and Y on M, respectively. Then they also satisfy

$$\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \phi_\nu\xi_\nu = 0, \quad \eta_\nu(\phi_\nu X) = 0, \quad \text{and} \quad \eta_\nu(\xi_\nu) = 1$$

for any tangent vector field X on M and $\nu = 1,2,3$. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\overline{\nabla}$ of $(G_2(\mathbb{C}^{m+2}),g)$, for any canonical local basis $\{J_1,J_2,J_3\}$ of $\mathfrak{J}$ there exist three local 1-forms q_1,q_2,q_3 such that

$$\overline{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$$

for any X tangent to $G_2(\mathbb{C}^{m+2})$, where subindices are taken modulo 3.

From the expression of the curvature tensor of $G_2(\mathbb{C}^{m+2})$ the Gauss equation is given by

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y \\ &\quad + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &\quad + \sum_{\nu=1}^{3}\left\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\right\}\end{aligned} \tag{2.1}$$

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$$+\sum_{\nu=1}^{3}\left\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X-g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\right\}$$
$$-\sum_{\nu=1}^{3}\left\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X-\eta(X)\eta_\nu(Z)\phi_\nu\phi Y\right\}$$
$$-\sum_{\nu=1}^{3}\left\{\eta(X)g(\phi_\nu\phi Y,Z)-\eta(Y)g(\phi_\nu\phi X,Z)\right\}\xi_\nu$$
$$+g(AY,Z)AX-g(AX,Z)AY,$$

for any X, Y, Z tangent to M. The Codazzi equation is also given by

$$\begin{aligned}(\nabla_X A)Y-(\nabla_Y A)X&=\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\\&\quad+\sum_{\nu=1}^{3}\left\{\eta_\nu(X)\phi_\nu Y-\eta_\nu(Y)\phi_\nu X-2g(\phi_\nu X,Y)\xi_\nu\right\}\\&\quad+\sum_{\nu=1}^{3}\left\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\right\}\\&\quad+\sum_{\nu=1}^{3}\left\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\right\}\xi_\nu\end{aligned}$$

for any X, Y tangent to M. The derivatives of the structure tensor ϕ and the Reeb vector field ξ in almost contact structure (ϕ,ξ,η,g) of M in $G_2(\mathbb{C}^{m+2})$ can be respectively given by

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi\quad\text{and}\quad\nabla_X\xi=\phi AX.$$

Moreover, the derivatives of the structure tensor ϕ_ν and the structure vector fields ξ_ν, $\nu=1,2,3$ in almost contact metric 3-structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ of M in $G_2(\mathbb{C}^{m+2})$ are respectively given by

$$(\nabla_X\phi_\nu)Y=-q_{\nu+1}(X)\phi_{\nu+2}Y+q_{\nu+2}(X)\phi_{\nu+1}Y+\eta_\nu(Y)AX-g(AX,Y)\xi_\nu,$$
$$\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX.$$

From (2.1) the Ricci tensor S of M in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned}SX&=\sum_{i=1}^{4m-1}R(X,e_i)e_i\\&=(4m+7)X-3\eta(X)\xi+hAX-A^2X\\&\quad+\sum_{\nu=1}^{3}\left\{-3\eta_\nu(X)\xi_\nu+\eta_\nu(\xi)\phi_\nu\phi X-\eta(\phi_\nu X)\phi_\nu\xi-\eta(X)\eta_\nu(\xi)\xi_\nu\right\},\end{aligned}\tag{2.2}$$

for any X tangent to M, where h denotes $\mathrm{Tr}(A)$.

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From (2.2) we can compute the derivative of the Ricci tensor S as follows (see [7]):

(2.3)
$$\begin{aligned}
(\nabla_X S)Y = &-3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX \\
&- 3\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+1}(X)\eta_{\nu+2}(Y) + g(\phi_\nu AX, Y)\Big\}\xi_\nu \\
&- 3\sum_{\nu=1}^{3}\eta_\nu(Y)\Big\{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX\Big\} \\
&+ \sum_{\nu=1}^{3}\Big\{X(\eta_\nu(\xi))\phi_\nu\phi Y + \eta_\nu(\xi)\{-q_{\nu+1}(X)\phi_{\nu+2}\phi Y \\
&\quad + q_{\nu+2}(X)\phi_{\nu+1}\phi Y + \eta_\nu(\phi Y)AX - g(AX, \phi Y)\xi_\nu\} \\
&\quad + \eta_\nu(\xi)\{\eta(Y)\phi_\nu AX - g(AX, Y)\phi_\nu\xi\} - g(\phi AX, \phi_\nu Y)\phi_\nu\xi \\
&\quad + \Big\{q_{\nu+1}(X)\eta(\phi_{\nu+2}Y) - q_{\nu+2}(X)\eta(\phi_{\nu+1}Y) \\
&\qquad - \eta_\nu(Y)\eta(AX) + \eta(\xi_\nu)g(AY, X)\Big\}\phi_\nu\xi \\
&\quad - \eta(\phi_\nu Y)\{q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi \\
&\qquad + \phi_\nu\phi AX - \eta(AX)\xi_\nu + \eta(\xi_\nu)AX\} \\
&\quad - g(\phi AY, X)\eta_\nu(\xi)\xi_\nu - \eta(Y)X(\eta_\nu(\xi))\xi_\nu - \eta(Y)\eta_\nu(\xi)\nabla_X\xi_\nu\Big\} \\
&+ (Xh)AY + h(\nabla_X A)Y - (\nabla_X A^2)Y
\end{aligned}$$

for any X, Y tangent to M, where the subindices are taken modulo 3.

For a real hypersurface of type (A) (resp., (B)), we recall two propositions due to Berndt and Suh [2] as follows.

Proposition A *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\}, \\
T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2,\ \xi_3\}, \\
T_\lambda &= \{X \mid X \perp \mathbb{H}\xi,\ JX = J_1X\}, \\
T_\mu &= \{X \mid X \perp \mathbb{H}\xi,\ JX = -J_1X\},
\end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$, and $\mathbb{H}\xi$ denote the real, complex, and quaternionic spans of the structure vector field ξ, respectively, and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Proposition B *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \operatorname{Span}\{\xi\}, \\ &T_\beta = \mathfrak{J}J\xi = \operatorname{Span}\{\xi_\nu \mid \nu = 1,2,3\}, \\ &T_\gamma = \mathfrak{J}\xi = \operatorname{Span}\{\phi_\nu\xi \mid \nu = 1,2,3\}, \\ &T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^\perp$ is the orthogonal complement of $\mathbb{H}\mathbb{C}\xi$, where

$$\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

3 Proof of the Theorem 1.3

The GTW parallel Ricci tensor is defined by

$$\begin{aligned} (\widehat{\nabla}^{(k)}_X S)Y &= \widehat{\nabla}^{(k)}_X(SY) - S\widehat{\nabla}^{(k)}_X Y \\ &= \nabla_X(SY) + g(\phi AX, SY)\xi - \eta(SY)\phi AX - k\eta(X)\phi SY \\ &\quad - S\nabla_X Y - g(\phi AX, Y)S\xi + \eta(Y)S\phi AX + k\eta(X)S\phi Y. \end{aligned}$$

And from (1.1), as we suppose that M has GTW harmonic curvature, we have

$$\begin{aligned} (\nabla_X S)Y - (\nabla_Y S)X = &-g(\phi AX, SY)\xi + \eta(SY)\phi AX + k\eta(X)\phi SY \\ &+ g(\phi AX, Y)S\xi - \eta(Y)S\phi AX - k\eta(X)S\phi Y \\ &+ g(\phi AY, SX)\xi - \eta(SX)\phi AY - k\eta(Y)\phi SX \\ &- g(\phi AY, X)S\xi + \eta(X)S\phi AY + k\eta(Y)S\phi X \end{aligned} \tag{3.1}$$

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for any X, Y tangent to M. Thus, by using (2.3), (3.1) can be written as follows:

(3.2)
$$
\begin{aligned}
&-3g(\phi AX,Y)\xi-3\eta(Y)\phi AX\\
&-3\sum_{\nu=1}^{3}\{q_{\nu+2}(X)\eta_{\nu+1}(Y)-q_{\nu+1}(X)\eta_{\nu+2}(Y)+g(\phi_\nu AX,Y)\}\xi_\nu\\
&-3\sum_{\nu=1}^{3}\eta_\nu(Y)\{q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX\}\\
&+\sum_{\nu=1}^{3}\{X(\eta_\nu(\xi)\phi_\nu\phi Y+\eta_\nu(\xi)\{-q_{\nu+1}(X)\phi_{\nu+2}\phi Y\\
&+q_{\nu+2}(X)\phi_{\nu+1}\phi Y+\eta_\nu(\phi Y)AX-g(AX,\phi Y)\xi_\nu\}\\
&+\eta_\nu(\xi)\{\eta(Y)\phi_\nu AX-g(AX,Y)\phi_\nu\xi\}-g(\phi AX,\phi_\nu Y)\phi_\nu\xi\\
&+\{q_{\nu+1}(X)\eta(\phi_{\nu+2}Y)-q_{\nu+2}(X)\eta(\phi_{\nu+1}Y)-\eta_\nu(Y)\eta(AX)\\
&+\eta(\xi_\nu)g(AX,Y)\}\phi_\nu\xi-\eta(\phi_\nu Y)\{q_{\nu+2}(X)\phi_{\nu+1}\xi\\
&-q_{\nu+1}(X)\phi_{\nu+2}\xi+\phi_\nu\phi AX-\eta(AX)\xi_\nu+\eta(\xi_\nu)AX\}\\
&-g(\phi AX,Y)\eta_\nu(\xi)\xi_\nu-\eta(Y)X(\eta_\nu(\xi))\xi_\nu-\eta(Y)\eta_\nu(\xi)\nabla_X\xi_\nu\}\\
&+X(h)AY+h(\nabla_XA)Y-(\nabla_XA^2)Y\\
&+3g(\phi AY,X)\xi+3\eta(X)\phi AY\\
&+3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(X)-q_{\nu+1}(Y)\eta_{\nu+2}(X)+g(\phi_\nu AY,X)\}\xi_\nu\\
&+3\sum_{\nu=1}^{3}\eta_\nu(X)\{q_{\nu+2}(Y)\xi_{\nu+1}-q_{\nu+1}(Y)\xi_{\nu+2}+\phi_\nu AY\}\\
&-\sum_{\nu=1}^{3}\{Y(\eta_\nu(\xi)\phi_\nu\phi X+\eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi X\\
&+q_{\nu+2}(Y)\phi_{\nu+1}\phi X+\eta_\nu(\phi X)AY-g(AY,\phi X)\xi_\nu\}\\
&+\eta_\nu(\xi)\{\eta(X)\phi_\nu AY-g(AY,X)\phi_\nu\xi\}-g(\phi AY,\phi_\nu X)\phi_\nu\xi\\
&+\{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X)-q_{\nu+2}(Y)\eta(\phi_{\nu+1}X)-\eta_\nu(X)\eta(AY)\\
&+\eta(\xi_\nu)g(AY,X)\}\phi_\nu\xi-\eta(\phi_\nu X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi\\
&-q_{\nu+1}(Y)\phi_{\nu+2}\xi+\phi_\nu\phi AY-\eta(AY)\xi_\nu+\eta(\xi_\nu)AY\}\\
&-g(\phi AY,X)\eta_\nu(\xi)\xi_\nu-\eta(X)Y(\eta_\nu(\xi))\xi_\nu-\eta(X)\eta_\nu(\xi)\nabla_Y\xi_\nu\}\\
&-Y(h)AX-h(\nabla_YA)X+(\nabla_YA^2)X\\
=&-g(\phi AX,SY)\xi+\eta(SY)\phi AX+k\eta(X)\phi SY\\
&+g(\phi AX,Y)S\xi-\eta(Y)S\phi AX-k\eta(X)S\phi Y+g(\phi AY,SX)\xi-\eta(SX)\phi AY\\
&-k\eta(Y)\phi SX-g(\phi AY,X)S\xi+\eta(X)S\phi AY+k\eta(Y)S\phi X
\end{aligned}
$$

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for any X, Y tangent to M.

We can write $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$, where X_0 is a unit vector field in $\mathfrak{D}$. Suppose that $A\xi = \alpha\xi$ and that $\eta(X_0)\eta(\xi_1) \neq 0$.

Bearing in mind that

$$\begin{aligned} \eta_\nu(\phi X_0) &= 0 \quad \text{for} \quad \nu = 1,2,3,\\ S\xi &= (4m+4+h\alpha-\alpha^2)\xi - 4\eta(\xi_1)\xi_1,\\ \eta(\phi_1\phi X_0) &= \eta(X_0)\eta(\xi_1),\\ \eta(\phi_\nu\phi X_0) &= 0 \quad \text{for} \quad \nu = 2,3,\\ \xi(\eta_1(\xi)) &= g(\xi, \nabla_\xi \xi_1), \end{aligned}$$

by putting $X = \xi$, $Y = \phi X_0$ in (3.2) and taking scalar product of (3.2) with ξ, we get

$$4(\alpha-k)\eta^2(\xi_1)\eta(X_0) - 16\eta(\xi_1)g(A\phi X_0, \phi_1\xi) + (\phi X_0)(\alpha^2 - \alpha h) = 0. \tag{3.3}$$

By using

$$\phi_1\xi = -\frac{\eta(X_0)}{\eta(\xi_1)}\phi X_0 \quad \text{and} \quad A\phi X_0 = -\frac{\eta(\xi_1)}{\eta(X_0)}A\phi_1\xi,$$

(3.3) becomes

$$4(\alpha-k)\eta^2(\xi_1)\eta(X_0) + 16\eta^2(\xi_1)g(A\phi\xi_1, \phi\xi_1) + \eta(\xi_1)(\phi\xi_1)(\alpha h - \alpha^2) = 0.$$

Since the shape operator A commutes with the structure tensor ϕ on the distribution $\mathfrak{D}^\perp$, we have

$$g(A\phi\xi_1, \phi\xi_1) = g(\phi A\xi_1, \phi\xi_1) = g(A\xi_1, \xi_1) - \alpha\eta^2(\xi_1).$$

Thus, we arrive at

$$\begin{aligned} 4(\alpha-k)\eta^2(\xi_1)\eta^2(X_0) &+ 16\eta^2(\xi_1)g(A\xi_1, \xi_1)\\ &- 16\alpha\eta^4(\xi_1) + \eta(\xi_1)(\phi\xi_1)(\alpha h - \alpha^2) = 0. \end{aligned}$$

Let $\{E_i\}_{i=1,\dots,4m-1}$ be an orthonormal basis of eigenvectors of M. If we develop a contracted formula $\sum_{i=1}^{4m-1} g((\nabla_{E_i}S)Y - (\nabla_Y S)E_i, E_i)$ in (3.1), the left side of the equality (3.2) yields for $Y = \xi$ (see [10, (5.4)]),

$$\begin{aligned} -3\sum_{\nu=1}^{3} g(\phi_\nu A\xi_\nu, \xi) + \alpha\xi(h) - \xi(h)h + h\big(\xi(\alpha) - \xi(h) - tr(A\phi A)\big)&\\ - (\xi(\alpha^2) - \mathrm{Tr}(A^2\phi A) - \xi(\mathrm{Tr}(A^2)))&. \end{aligned}$$

On the other hand, the contracted formula in the right side of (3.1), bearing in mind that $g(\phi SY, \xi) = 0$, $g(\phi AE_i, E_i) = 0$, because E_i is principal and $g(\xi, \phi SY) = 0$, gives

$$-g(A\phi Y, S\xi) - \eta(Y)\sum_{i=1}^{4m-1} g(S\phi AE_i, E_i) - kg(\xi, S\phi Y) + 2k\eta(Y)\sum_{i=1}^{4m-1} g(SE_i, \phi E_i).$$

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Now we get

$$g(S\xi, A\phi Y) = -4\eta(\xi_1)g(\xi_1, A\phi Y),$$
$$\sum_{i=1}^{4m-1} g(\phi AE_i, SE_i) = 4\sum_{\nu=1}^{3} g(\xi_\nu, A\phi\xi_\nu),$$
$$g(S\xi, \phi Y) = -4\eta(\xi_1)g(\xi_1, \phi Y).$$

From this, together with inserting $Y = \xi$ in above formula, the right side of the contracted formula becomes $-4\sum_{\nu=1}^{3} g(\xi_\nu, A\phi\xi_\nu)$. Then both sides of the contracted formula in (3.2) can be given by

$$\text{(3.4)}\qquad h\xi(\alpha) - \xi(\alpha^2) + \xi(\mathrm{Tr}(A^2)) = -7\sum_{\nu=1}^{3} g(\xi_\nu, A\phi\xi_\nu) = 0,$$

where we have applied that h is constant and ϕ and A commute on $\mathfrak{D}^\perp$.

If r denotes the scalar curvature of M,

$$r = \sum_{i=1}^{4m-1} g(SE_i, E_i) = 16m^2 + 24m - 19 + h^2 - h_2,$$

where $h_2 = \mathrm{Tr}(A^2)$; see [10]. As r is constant, h_2 is also constant. Thus, (3.4) yields $\xi(\alpha h - \alpha^2) = 0$. This gives us either $\xi(\alpha) = 0$ or $h = 2\alpha$.

If $h = 2\alpha$ and α is constant, then from Berndt and Suh [2] we may use the following

$$Y(\alpha) = \xi(\alpha)\eta(Y) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any $Y \in TM$. This yields $\phi_1\xi = 0$, which gives a contradiction.

Suppose now that $\xi(\alpha) = 0$. As above, $Y(\alpha) = 4\eta_1(\xi)g(Y, \phi\xi_1)$. Thus,

$$(\phi\xi_1)(\alpha) = 4\eta(\xi_1)\big(1 - \eta^2(\xi_1)\big) = 4\eta(\xi_1)\eta^2(X_0)$$

and

$$(\phi\xi_1)(\alpha h - \alpha^2) = (h - 2\alpha)(\phi\xi_1)(\alpha) = 4(h - 2\alpha)\eta(\xi_1)\eta^2(X_0).$$

From (3.4) we obtain

$$\text{(3.5)}\qquad \begin{aligned} 0 &= 4(h - k - \alpha)\eta^2(X_0) + 16g(A\xi_1, \xi_1) - 16\alpha\eta^2(\xi_1) \\ &= 4(h - k - \alpha)\eta^2(X_0) + 16\alpha\eta^2(X_0) \\ &= 4(h - k + 3\alpha)\eta^2(X_0). \end{aligned}$$

We also have $g(\nabla_X \mathrm{grad}(\alpha), Y) = g(\nabla_Y \mathrm{grad}(\alpha), X)$ for any X, Y tangent to M. Bearing in mind that $\mathrm{grad}(\alpha) = 4\eta_1(\xi)\phi_1\xi$ and taking $X = \xi$, we get $g((\nabla_\xi\phi)\xi_1, Y) + g(\phi\nabla_\xi\xi_1, Y) = g((\nabla_Y\phi)\xi_1, \xi)$, where we have applied that $\eta(\xi_1) = \eta_1(\xi) \neq 0$. If we apply the formulas in Section 2 for $Y = \xi_1$, we obtain $g(A\xi_1, \xi_1) = \alpha$. Introducing this in (3.5) we have $4(h - k + 3\alpha) = 0$. Thus, α is constant, and, as above, we arrive to a contradiction.

Thus, we have obtained that either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.

If $\xi \in \mathfrak{D}$, M is locally congruent ([6]) to a type (B) real hypersurface. If we bear in mind the principal curvatures of such a real hypersurface in order our conditions to be satisfied, we should have $A\phi\xi_2 = 0 = \phi A\xi_2 = 2\cot(2r)\phi\xi_2$. This yields $2\cot(2r) = \cot(r) - \tan(r) = 0$. Thus, $r = \frac{\pi}{4}$, which is impossible.

Now we suppose $\xi \in \mathfrak{D}^\perp$ and write $\xi = \xi_1$. If we take the scalar product of (3.2) with ξ, we get

$$\begin{aligned}(3.6)\qquad & -3g\big((A\phi+\phi A)X,Y\big)-3g\big((\phi_1 A+A\phi_1)X,Y\big)-6\eta_2(Y)\eta_3(AX)\\ & +6\eta_3(Y)\eta_2(AX)-4\eta_2(X)\eta_3(AY)+4\eta_3(X)\eta_2(AY)\\ & +2(h-\alpha)\big\{2\eta_2(X)\eta_3(Y)-2\eta_2(Y)\eta_3(X)-g(\phi_1 X,Y)-g(\phi X,Y)\big\}\\ & -2\alpha g(A\phi AX,Y)+g((A\phi+\phi A)AX,AY)\\ & \quad =-g\big((S\phi A+\phi AS)X,Y\big)+g(S\xi,\xi)g\big((\phi A+A\phi)X,Y\big),\end{aligned}$$

for any X, Y tangent to M.

If we change X and Y in (3.6) and add the result to (3.6), we obtain

$$\begin{aligned}(3.7)\quad & -10\eta_2(Y)\eta_3(AX)+10\eta_3(Y)\eta_2(AX)-10\eta_2(X)\eta_3(AY)+10\eta_3(X)\eta_2(AY)\\ & \qquad =-g\big((S\phi A+\phi AS)X,Y\big)+g\big((A\phi S+SA\phi)X,Y\big),\end{aligned}$$

for any X, Y tangent to M.

Taking $Y=\xi_2$, $X\in\mathfrak{D}$ in (3.7) we have

$$\begin{aligned}-10\eta_3(AX) &= -g((S\phi A+\phi AS)X,\xi_2)+g((A\phi S+SA\phi)X,\xi_2)\\ &= -g(\phi AX,S\xi_2)+g(A\phi X,S\xi_2)=0,\end{aligned}$$

due to the fact that $A\phi X=\phi AX$ for any tangent vector field X. Thus, $\eta_3(AX)=0$ for any $X\in\mathfrak{D}$, and analogously for $Y=\xi_3$, we obtain $\eta_2(AX)=0$. From these facts, we conclude that M is locally congruent to a type (A) real hypersurface. Bearing in mind that these real hypersurfaces have constant principal curvatures and that $A\phi=\phi A$ on them, they have constant mean and scalar curvatures.

Taking $X=\xi_3$ and $Y=\xi_2$ in (3.6) we obtain

$$\begin{aligned}(3.8)\qquad -4(h-\alpha)-2\beta^2(\alpha-\beta)-2\beta &= 2\beta(4m+h\alpha-\alpha^2)\\ &\quad -\beta g\big(\xi_2,(4m+6)\xi_2+hA\xi_2-A^2\xi_2\big)\\ &\quad -\beta g\big(\xi_3,(4m+6)\xi_3+hA\xi_3-A^2\xi_3\big),\end{aligned}$$

where we have applied that

$$\begin{cases}S\xi=(4m+h\alpha-\alpha^2)\xi\\ S\xi_2=(4m+6)\xi_2+hA\xi_2-A^2\xi_2\\ S\xi_3=(4m+6)\xi_3+hA\xi_3-A^2\xi_3.\end{cases}$$

As $A\xi_2=\beta\xi_2$ and $A\xi_3=\beta\xi_3$, (3.8) becomes

$$\beta(h\alpha-\alpha^2-5)+(h-\alpha)(2-\beta^2)=0.$$

From this it follows that

$$(3.9)\qquad (h-\alpha)(2-\beta^2+\alpha\beta)-5\beta=0.$$

Bearing in mind the values of α and β, we have $\alpha-\beta=\sqrt{8}\cot(\sqrt{8}r)-\sqrt{2}\cot(\sqrt{2}r)$ and $\sqrt{8}\cot(\sqrt{8}r)=\sqrt{2}\big(\cot(\sqrt{2}r)-\tan(\sqrt{2}r)\big)$. So it follows that $\beta(\alpha-\beta)=-2$. From this, together with (3.9), we conclude that $\beta=0$, which is impossible. This completes the proof of our main theorem in the introduction.

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