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RESEARCH ARTICLE

Relative Severi inequality for fibrations of maximal Albanese dimension over curves

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Abstract
Let $f : X \longrightarrow B$ be a relatively minimal fibration of maximal Albanese dimension from a variety X of dimension $n \geq 2$ to a curve B defined over an algebraically closed field of characteristic zero. We prove that $K_{X/B}^n \geq 2n!\chi_f$. It verifies a conjectural formulation of Barja in [2]. Via the strategy outlined in [4], it also leads to a new proof of the Severi inequality for varieties of maximal Albanese dimension. Moreover, when the equality holds and $\chi_f > 0$, we prove that the general fibre F of f has to satisfy the Severi equality that $K_F^{n-1} = 2(n-1)!\chi(F, \omega_F)$. We also prove some sharper results of the same type under extra assumptions.

Contents

1. Introduction

The Severi inequality states that

$$K_X^n \geq 2n!\chi(X, \omega_X)$$

for an n-dimensional minimal variety X of general type and maximal Albanese dimension. It was originally stated for surfaces by Severi [21] and was proved by Pardini [20]. Later, it was generalised to arbitrary dimension by Barja [2] as well as the second author [25]. From now on, we refer to this inequality as the *absolute* Severi inequality in order to distinguish from the result in the current paper.

The goal of this paper is to establish a *relative* version of the absolute Severi inequality. More precisely, we prove that

$$K_{X/B}^n \geq 2n!\chi_f$$

for a relatively minimal fibration $f : X \longrightarrow B$ of maximal Albanese dimension from an n-dimensional variety X to a curve B. This inequality was conjecturally formulated by Barja in [2, §1]. The $g(B) = 0$ case of this relative inequality can be applied to give a new proof of the above absolute Severi inequality. Moreover, the above relative inequality is sharp, and if $K_{X/B}^n = 2n!\chi_f > 0$, we prove that the general fibre F of f has to satisfy the absolute Severi equality that

$$K_F^{n-1} = 2(n-1)!\chi(F, \omega_F).$$

We also use our method to deduce some sharper relative results of the same type under extra assumptions. As an upshot, the corresponding $g(B) = 0$ case implies the recent geographical results of absolute Severi type obtained by Barja, Pardini and Stoppino [6].

Throughout this paper, we work over an arbitrary algebraically closed field k of characteristic zero. All varieties are assumed to be projective.

1.1. Albanese dimension of fibrations and χ_f

We start with some notation. In the study of irregular varieties, a major tool is to consider the Albanese map. For an irregular variety X, the so-called Albanese dimension $\mathrm{albdim}(X)$ of X is one of the most important invariants of X. In the following, we consider its relative version.

Let $f : X \longrightarrow Y$ be a fibration between two normal varieties X and Y with a general fibre F. Let $a : X \longrightarrow \mathrm{Alb}(X)$ be the Albanese map of X.

Definition 1.1. The *Albanese dimension of f*, denoted by $\mathrm{albdim}(f)$, is defined to be $\dim a(F)$, namely the dimension of the image of F under the Albanese map of X. We say that f is *of maximal Albanese dimension*, if $\mathrm{albdim}(f) = \dim F$.

It is easy to check that the following properties hold:

(1) When f is the structural morphism (i.e., $Y = \mathrm{Spec}(k)$), then

$$\mathrm{albdim}(f) = \mathrm{albdim}(X).$$

Thus the Albanese dimension for fibrations is indeed a generalisation of that for varieties.

(2) In general, we have

$$\mathrm{albdim}(f) \leq \mathrm{albdim}(X) - \mathrm{albdim}(Y).$$

In particular, if f is the Stein factorisation of the Albanese map of X, then $\mathrm{albdim}(f) = 0$.

(3) If both Y and f are of maximal Albanese dimension, so is X.

Another important invariant associated to f is the relative Euler characteristic

$$\chi_f := \chi(X, \omega_X) - \chi(Y, \omega_Y)\chi(F, \omega_F).$$

Regarding this invariant, the first interesting case is when $f : X \longrightarrow Y$ is a surface fibration: that is, X is a smooth surface and Y is a curve. In this case, it is well known that

$$\chi_f = \deg f_*\omega_{X/Y}.$$

In particular, by [11, Main Theorem], we know that $\chi_f \geq 0$. There are a number of important results related to χ_f, such as the Arakelov inequality [1] (see [22] for a survey together with generalisations),

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the slope inequality of Cornalba-Harris [9] and Xiao [23], and the geography of irregular surfaces (see [17] for a detailed survey). The study of these results as well as their refinements and generalisations has been active throughout recent decades.

Another interesting case, which is more related to this paper, is when f is a fibration of maximal Albanese dimension and Y is a curve. In this case, by the work of Hacon and Pardini [12, Theorem 2.4] (see Proposition 4.1 for a slightly generalised version adapting to the setting of this paper), we know that

$$\chi_f = \deg f_*(\omega_{X/Y} \otimes \mathcal{P}),$$

where $\mathcal{P}$ is a general torsion element in $\mathrm{Pic}^0(X)$. Moreover, Hacon and Pardini showed that $\chi_f \geq 0$ still holds in this case.

1.2. Main results

Now we state the first main theorem of this paper.

Theorem 1.2 (Relative Severi inequality). *Let $f : X \to B$ be a relatively minimal fibration from a variety X of dimension $n \geq 2$ to a smooth curve B. Suppose that f is of maximal Albanese dimension. Then we have the following sharp inequality:*

$$K_{X/B}^n \geq 2n!\chi_f. \tag{1.1}$$

We call the inequality in equation (1.1) a *relative Severi inequality* because it literally replaces the absolute invariants K_X^n and $\chi(X, \omega_X)$ in the absolute Severi inequality by the relative invariants $K_{X/B}^n$ and χ_f.

Let us put Theorem 1.2 into perspective. When $n = 2$, it is already known by Xiao [23, Corollary 1]. More precisely, Xiao proved that for a relatively minimal surface fibration $f : X \to B$ with a general fibre of genus $g \geq 2$, the inequality in equation (1.1) holds, provided that $h^1(X, \mathcal{O}_X) > g(B)$. Note that this assumption is equivalent to f being of maximal Albanese dimension, as the fibre in this case is just a curve.

For general $n > 2$, the problem with finding such inequalities has already been addressed by Mendes Lopes and Pardini [17, §5.3], whose purpose was to generalise, using Pardini's original approach in [20], the Severi inequality for surfaces to higher dimensions. To our knowledge, the precise version of equation (1.1) was first formulated conjecturally by Barja in [2, §1, Page 545]. Barja also observed that equation (1.1) is in fact a consequence of the f-positivity conjecture [4, Conjecture 1] of himself and Stoppino.[1] Another interesting observation, which probably motivates the formulation in equation (1.1), is that when X is of maximal Albanese dimension, one can indeed deduce the absolute Severi inequality by combining Pardini's approach and equation (1.1) for $g(B) = 0$ (see [4, Proposition 4.4] for details).

When $g(B) = 1$, it is easy to see that equation (1.1) coincides with the absolute Severi inequality. In addition to this and prior to our result, Barja proved equation (1.1) for $g(B) = 0$ under the extra assumptions that X is of maximal Albanese dimension and K_X is nef. Barja also obtained a weaker version of equation (1.1) when $g(B) \geq 2$. See [2, Corollary C] as well as its proof for details.

Our Theorem 1.2 verifies completely the conjectural formulation of Barja for the base curve B of arbitrary genus. Moreover, if $g(B) = 0$, our assumption that f is of maximal Albanese dimension is strictly weaker than X itself being of maximal Albanese dimension. As mentioned before, Theorem 1.2 for $g(B) = 0$ can be applied to give an alternative proof of the absolute Severi inequality that is different from that in [2] or [25].[2]

[1]This conjecture was recently studied by the authors in [13], where it is shown that counterexamples to this conjecture do exist for any $n > 2$.

[2]Since a detailed strategy has been carried out in [4, Proposition 4.4], we will not repeat this proof in this paper and refer the reader to [4, Proposition 4.4] for details.

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Since equation (1.1) is sharp, a new question naturally arises: can one characterise the equality case? In this paper, we also consider this problem. We prove the following result.

Theorem 1.3. *In Theorem 1.2, if the equality in equation (1.1) holds and* $\chi_f > 0$*, then*

(1) *the Albanese map of X maps a general fibre of f onto an abelian variety of dimension $n-1$. In particular,*

$$h^1(X, \mathcal{O}_X) - g(B) = n - 1;$$

(2) *the general fibre F of f satisfies the absolute Severi equality: that is,*

$$K_F^{n-1} = 2(n-1)!\chi(F, \omega_F).$$

Previously, (1) was known only when $n = 2$ due to Xiao [23, Theorem 3]. This paper mainly concerns the higher-dimensional case, and our result shows that (1) holds for any $n \geq 2$. The much more interesting and stronger part comes from (2): unlike (1) or the absolute Severi inequality, (2) is trivial when $n = 2$ – that is, when the fibre is a curve – which says that $\deg K_F = 2\chi(F, \omega_F)$. It holds true for any surface fibration, not necessarily of maximal Albanese dimension. However, for $n > 2$, (2) was completely unknown before, and it reveals a new connection between the geometry of a family of higher-dimensional varieties and the geometry of a general member in this family.

Recall that for a surface fibration $f : X \to B$, the relative irregularity is defined as $q_f := h^1(X, \mathcal{O}_X) - g(B)$. Recently, Pardini proposed a problem [8, Problem 2] to study various notions of relative irregularity for families of higher-dimensional varieties. The result (1) also sheds some light on this problem, suggesting that the number $h^1(X, \mathcal{O}_X) - g(B)$ may also serve as the relative irregularity for higher-dimensional fibrations over curves.

When $\dim F \geq 2$, by a very recent result of Barja, Pardini and Stoppino [3, Theorem 1.2] characterising the variety satisfying the absolute Severi equality (see also [5, 16] when $\dim F = 2$), we know that (2) actually implies (1). However, our proof of (1) is independent of (2).

1.3. Related results

If more assumptions are imposed on the Albanese map of X, we obtain sharper results. For example, we prove the following theorem.

Theorem 1.4. *Let $f : X \to B$ be a relatively minimal fibration from a variety X of dimension $n \geq 3$ to a smooth curve B. Denote by F a general fibre of f. Suppose that f is of maximal Albanese dimension and $a : X \to \mathrm{Alb}(X)$ is the Albanese map of X.*

(1) *If $a|_F$ is birational, then*

$$K_{X/B}^n \geq \frac{5n!}{2}\chi_f.$$

(2) *If $a|_F$ is not composed with an involution, then*

$$K_{X/B}^n \geq \frac{9n!}{4}\chi_f.$$

Combining Theorem 1.4 in the $g(B) = 0$ case with the method in [4, Proposition 14], it is easy to get the following conclusion, which was recently obtained by Barja, Pardini and Stoppino in [6, §1].

Corollary 1.5. *Let X be a minimal variety of general type of dimension $n \geq 3$. Suppose that X is of maximal Albanese dimension.*

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(1) *If the Albanese map of X is birational onto its image, then*

$$K_X^n \geq \frac{5n!}{2}\chi(X, \omega_X).$$

(2) *If the Albanese map of X is not composed with an involution, then*

$$K_X^n \geq \frac{9n!}{4}\chi(X, \omega_X).$$

In the same spirit as before, we may view Theorem 1.4 as a relative version of Corollary 1.5.

In [6], Barja, Pardini and Stoppino consider a more general map $a : X \longrightarrow A$ such that $a^* : \mathrm{Pic}^0(A) \longrightarrow \mathrm{Pic}^0(X)$ is injective (which they call strongly generating) and prove Corollary 1.5 when a is birational or when a is not composed with an involution. In fact, by the universal property of the Albanese map, we see that if a is birational or is not composed with an involution, so is the Albanese map of X.

Furthermore, we would like to mention that the proof of the absolute Severi type inequalities by Barja, Pardini and Stoppino in [6] relies on their study of the continuous rank function. More precisely, they deduce these absolute results by integrating the derivative of the so-called continuous rank function. From the viewpoint of our paper, those absolute inequalities are consequences of their corresponding relative counterparts. To summarise, we have seen again, as in the work of Pardini [20], that the study of the *relative geography*, namely the relation among relative birational invariants (such as the relative canonical volume, the relative Euler characteristic, etc.) plays a crucial role in understanding the geography of algebraic varieties in the classical sense.

Notation and conventions

In this paper, a fibration always means a surjective morphism with connected fibres.

Let $f : X \longrightarrow B$ be a fibration over a curve B. We say that f is *relatively minimal* if X is normal with at worst terminal singularities and K_X is f-nef. The assumption implies that a general fibre F of f is also normal with at worst terminal singularities by the adjunction. Moreover, if a general fibre of f is of maximal Albanese dimension (which is exactly under the setting of Theorem 1.2), then the relative minimality also ensures that $K_{X/B}$ is nef.[3]

For divisors, we always use $\sim$ to denote the linear equivalence and $\equiv$ to denote the numerical equivalence. Let D_1 and D_2 be two $\mathbb{Q}$-divisors on a variety V. The notation $D_1 \geq D_2$ means $D_1 - D_2$ is effective. Let D be a $\mathbb{Q}$-divisor on V. We use $\lfloor D \rfloor$ to denote its integral part. The volume of D is defined as

$$\mathrm{Vol}(D) := \limsup_{m\to\infty} \frac{h^0(V, \lfloor mD \rfloor)}{m^{\dim V}/(\dim V)!}.$$

2. A Clifford type inequality

In this section, we recall a Clifford type result in [24] that will be used later. All results in this section hold also in positive characteristics.

2.1. ε for divisors

Let V be a smooth variety of dimension $n > 0$, and let L be a $\mathbb{Q}$-divisor on V. For any big divisor M on V with $|M|$ base point free, take the smallest integer $\lambda_M > 0$ so that the divisor $\lambda_M M - L$ is

[3] In fact, Fujino [10, Theorem 1.1] proved that in this case, the general fibre has a good minimal model. Thus, by a result of Nakayama [18, Theorem 5], K_X is f-semi-ample. Using the argument as in the proof of [19, Theorem 1.4], we deduce that $K_{X/B}$ is nef.

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pseudo-effective. When $n \geq 2$, we define

$$\varepsilon(V, L, M) := (\lambda_M + 1)^{n-1} M^n.$$

When $n = 1$, we simply set

$$\varepsilon(V, L, M) = 1.$$

For any $n > 0$, define

$$\varepsilon(V, L) := \inf_M \varepsilon(V, L, M),$$

where the infimum is taken over all divisors M on V chosen as above. In particular, when $n = 1$, we have

$$\varepsilon(V, L) = 1.$$

It is straightforward to check that

Proposition 2.1. *The above ε satisfies the following properties:*

(1) *If $L' \geq L$, then $\varepsilon(V, L', M) \geq \varepsilon(V, L, M)$ for any M chosen as above. In particular, $\varepsilon(V, L') \geq \varepsilon(V, L)$.*
(2) *Let $\sigma : V' \to V$ be a birational morphism. Then $\varepsilon(V', \sigma^* L) \leq \varepsilon(V, L)$.*

2.2. A Clifford type inequality

The main result in this section is the following one, which will be used later in the proof of Theorem 1.2.

Theorem 2.2. *Let V be a smooth variety of dimension $n > 0$. Suppose that L is a $\mathbb{Q}$-divisor on V such that $K_V - L$ is pseudo-effective. Then*

$$h^0(V, \lfloor L \rfloor) \leq \frac{1}{2n!} \mathrm{Vol}(L) + n\varepsilon(V, L).$$

Proof. By [24, Theorem 1.2], which was stated only for integral divisors, we have

$$h^0(V, \lfloor L \rfloor) \leq \frac{1}{2n!} \mathrm{Vol}(\lfloor L \rfloor) + n\varepsilon(V, \lfloor L \rfloor).$$

Note that $\mathrm{Vol}(\lfloor L \rfloor) \leq \mathrm{Vol}(L)$, and by Proposition 2.1, $\varepsilon(V, \lfloor L \rfloor) \leq \varepsilon(V, L)$. Thus the result follows easily. □

Remark 2.3. As is explained in [24], Theorem 2.2 is a natural generalisation of the classical Clifford inequality.

3. Sharper estimate under extra assumptions

To prove Theorem 1.3, we need some estimates on the dimension of $H^0(V, L)$ similar to Theorem 2.2 but stronger. All the sharper bounds in this section are inspired by the work of Barja, Pardini, and Stoppino in [6], where they proved the so-called 'continuous' estimates. However, under our setting, we need explicit results instead, and the method we will employ is based on [24, 25, 26].

3.1. A filtration for nef divisors

Let $f : V \to B$ be a fibration from a smooth variety V of dimension n to a smooth curve B with a general fibre F. Let L be a nef divisor on V. We first recall the following theorem.

Theorem 3.1 ([26, Theorem 4.1]). *Let $f : V \longrightarrow B$, F and L be as above. Then there is a birational morphism $\sigma : V_L \longrightarrow V$ and a sequence of triples*

$$\{(L_i, Z_i, a_i) | i = 0, 1, \cdots, N\}$$

on V_L with the following properties:

- $(L_0, Z_0, a_0) = (\sigma^* L, 0, \mathrm{int}_{f_L}(L_0))$, *where $f_L : V_L \xrightarrow{\sigma} V \xrightarrow{f} B$ is the induced fibration.*
- *For any $i = 0, \cdots, N-1$, there is a decomposition*

$$|L_i - a_i F_L| = |L_{i+1}| + Z_{i+1}$$

  such that $Z_{i+1} \geq 0$ is the fixed part of $|L_i - a_i F_L|$ and the movable part $|L_{i+1}|$ of $|L_i - a_i F_L|$ is base point free. Here $F_L = \sigma^* F$ denotes a general fibre of f_L, and $a_i = \mathrm{int}_{f_L}(L_i)$.
- *We have $h^0(V_L, L_N - a_N F_L) = 0$.*

In the above theorem, for any $0 \leq i \leq N$, the number $\mathrm{int}_{f_L}(L_i)$ is defined by

$$\mathrm{int}_{f_L}(L_i) := \min\{a \in \mathbb{Z} | L_i - aF_L \text{ is not nef}\}.$$

Thus via Theorem 3.1, we obtain a filtration

$$\sigma^* L = L_0 > L_1 > \cdots > L_N \geq 0$$

of nef divisors on a birational model V_L of V. For simplicity, we denote by F a general fibre of $f_L : V_L \longrightarrow B$ in the rest of this section.

Proposition 3.2 ([24, Proposition 2.2]). *We have the following two inequalities:*

$$h^0(V, L) \leq \sum_{i=0}^{N} a_i h^0(F, L_i|_F);$$

$$L^n \geq n \sum_{i=0}^{N} a_i (L_i|_F)^{n-1} - n(L_0|_F)^{n-1}.$$

Proposition 3.3 ([24, Lemma 2.3]). *We have*

$$L_0^n \geq \left(\sum_{i=0}^{N} a_i - 1 \right) (L_0|_F)^{n-1}.$$

3.2. Sharper bound involving subcanonicity

Let V be a smooth variety of dimension $n > 0$ with the Kodaira dimension $\kappa(V) \geq 0$, and let L be a $\mathbb{Q}$-divisor on V. Let M be a big divisor on V such that $|M|$ is base point free. We recall that the numerical subcanonicity of L with respect to M is defined in [6, Definition 5.1] as follows:

$$r(L, M) := \frac{LM^{n-1}}{K_V M^{n-1}}.$$

When $n = 1$, set $r(L, M) = \frac{\deg L}{\deg K_V}$. When $K_V M^{n-1} = 0$, we have $\kappa(V) = 0$. In this case, we set $r(L, M) = +\infty$. Define a function δ as follows:

$$\delta(x) = \begin{cases} 2, & x \leq 1; \\ \frac{2x}{2x-1}, & x > 1. \end{cases}$$

Theorem 3.4. *Let L and M be as above, and write $r = r(L, M)$. Then*

$$h^0(V, \lfloor L \rfloor) \le \frac{1}{\delta(r)n!}\mathrm{Vol}(L) + n\varepsilon(V, L, M).$$

Proof. The proof is by induction, and we present it in several steps.

Notice that the required inequality holds trivially if $h^0(V, \lfloor L \rfloor) = 0$. We may make assumption $h^0(V, \lfloor L \rfloor) > 0$ from now on.

Step 1: Reduce to the case when L is nef.

In fact, by replacing V by an appropriate blowing up, we may assume that

$$L = L' + Z,$$

where L' is the movable part of $|\lfloor L \rfloor|$ and Z is its fixed part. It is clear that

$$r(L, M) \ge r(L', M), \quad \mathrm{Vol}(L) \ge \mathrm{Vol}(L'), \quad \varepsilon(V, L, M) \ge \varepsilon(V, L', M).$$

Thus it suffices to prove Theorem 3.4 for L'.

From now on, we assume that L is a nef divisor.

Step 2: The $n = 1$ case.

When $n = 1$, Theorem 3.4 is straightforward. If $h^1(V, L) \neq 0$, the classical Clifford inequality implies Theorem 3.4. Otherwise, by the Riemann-Roch theorem,

$$h^0(V, L) = \deg L - \frac{1}{2}\deg K_V = \left(1 - \frac{1}{2r}\right)\deg L.$$

Thus the proof is completed.

Step 3: The proof when $L^n > 0$.

Now we assume that Theorem 3.4 holds for dimension $k < n$. Choose a general pencil in $|M|$, and blow up the indeterminacies of this pencil, denoted by $\pi : V_0 \to V$. We get a fibration

$$f : V_0 \to \mathbb{P}^1$$

such that the general fibre F of f is isomorphic to a general member of the chosen pencil. By the adjunction, $\kappa(F) \ge 0$. Write $M_0 = \pi^* M$ and $L_0 = \pi^* L$. It follows that

$$r(L, M) = \frac{L_0 M_0^{n-2} F}{(\pi^* K_V) M_0^{n-2} F} \ge r(L_0|_F, M_0|_F),$$

where the last inequality follows from the adjunction.

Apply Theorem 3.1 to f and L_0. Replacing V_0 by a further blowing up if necessary, we get triples

$$(L_i, Z_i, a_i) \quad (i = 0, \ldots, N)$$

on V_0, and L_i and a_i satisfy the inequalities in Proposition 3.2 and 3.3. Note that by the definition of $r(L, M)$, we see that

$$r(L_i|_F, M_0|_F) \le r(L_0|_F, M_0|_F) = r.$$

By induction and using the fact that the function δ is nonincreasing, we have

$$h^0(F, L_i|_F) \le \frac{1}{\delta(r)(n-1)!}(L_i|_F)^{n-1} + (n-1)\varepsilon(F, L_i|_F, M_0|_F).$$

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Combine this with Proposition 3.2. It follows that

$$h^0(V_0, L_0) - \frac{1}{\delta(r)n!}L_0^n \le (n-1)\sum_{i=0}^{N} a_i\varepsilon(F, L_i|_F, M_0|_F) + \frac{1}{(n-1)!}(L_0|_F)^{n-1}.$$

To estimate the right-hand side of the above inequality, let λ be the smallest integer such that $\lambda M - L$ is pseudo-effective. Note that $L^n > 0$.

(1) It implies that $L^n \le \lambda L^{n-1}M = \lambda(L_0|_F)^{n-1}$. In particular, $(L_0|_F)^{n-1} > 0$. Thus, by Proposition 3.3,

$$\sum_{i=0}^{N} a_i \le \frac{L_0^n}{(L_0|_F)^n} + 1 \le \lambda + 1.$$

(2) By Proposition 2.1 (1),

$$\varepsilon(F, L_i|_F, M_0|_F) \le \varepsilon(F, L_0|_F, M_0|_F).$$

Moreover, since $\lambda M_0|_F - L_0|_F$ is also pseudo-effective, we have

$$\varepsilon(F, L_0|_F, M_0|_F) \le (\lambda+1)^{n-2}(M_0|_F)^{n-1} = (\lambda+1)^{n-2}M^n.$$

(3) We have

$$(L_0|_F)^{n-1} = L^{n-1}M \le \lambda L^{n-2}M^2 \le \cdots \le \lambda^{n-1}M^n.$$

Combining all of the above inequalities, it follows that

$$\begin{aligned} h^0(V_0, L_0) - \frac{1}{\delta(r)n!}L_0^n &\le (n-1)(\lambda+1)^{n-1}M^n + \frac{1}{(n-1)!}\lambda^{n-1}M^n \\ &\le n\varepsilon(V, L, M). \end{aligned}$$

Thus the proof in this case is completed.

Step 4. The proof when $L^n = 0$.

In this case, the proof is easier. Since L is not big, we know that

$$h^0(V, L - M) = 0.$$

Take W to be a general member in $|M|$, and we have

$$h^0(V, L) \le h^0(W, L|_W).$$

Therefore, by induction, we deduce that

$$h^0(V, L) \le \frac{1}{(n-1)!}(L|_W)^{n-1} + (n-1)\varepsilon(W, L|_W, M|_W).$$

Let λ be the smallest integer such that $\lambda M - L$ is pseudo-effective. Similar to Step 3, we have

(1) $(L|_W)^{n-1} = L^{n-1}M \le \lambda^{n-1}M^n$;
(2) $\varepsilon(W, L|_W, M|_W) \le (\lambda+1)^{n-2}M^n$.

Combining the above inequalities, it follows that

$$h^0(V, L) \le \frac{1}{(n-1)!}\lambda^{n-1}M^n + (n-1)(\lambda+1)^{n-2}M^n \le n\varepsilon(V, L, M).$$

Thus the whole proof is completed. □

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3.3. Sharper bound involving the mapping degree

Let V be a smooth variety of dimension $n \geq 2$, and let L be a $\mathbb{Q}$-divisor on V such that $K_V - L$ is pseudo-effective. Instead of the subcanonicity, we suppose that

$$a : V \to \Sigma$$

is a generically finite morphism onto a (possibly singular) variety Σ. Let H be a sufficiently ample divisor on Σ, and write $M = a^*H$. The assumption will be used until the end of this section.

3.3.1. Preparation

We first assume that V is a surface and $|L|$ is base point free. Although this assumption looks simple, all the results we need can be reduced to this setting.

Lemma 3.5. *If* $h^0(V, L - M) = 0$*, then*

$$h^0(L) \leq \frac{1}{2}LM + 1 \leq \varepsilon(V, L, M).$$

Proof. Choose a general curve $C \in a^*|H|$. By Bertini's theorem, we may assume that C is smooth. The assumption $h^0(V, L - M) = 0$ just tells us that $h^0(V, L) \leq h^0(C, L|_C)$. Thus the first inequality is just a combination of the Clifford inequality and the Riemann-Roch theorem again.

The second inequality is directly from the definition of ε. Let λ be the smallest integer such that $\lambda M - L$ is pseudo-effective. Then

$$\frac{1}{2}LM + 1 \leq \frac{\lambda}{2}M^2 + 1 \leq (\lambda + 1)M^2 = \varepsilon(V, L, M).$$

The proof is completed. □

Now suppose that $h^0(V, L - M) > 0$. Let

$$\gamma := \max\{i \in \mathbb{Z} | h^0(V, L - iM) > 0\}.$$

Obviously, $\gamma \geq 1$.

Lemma 3.6. *If* $h^0(V, L - M) > 0$*, then*

$$L^2 \geq h^0(V, 2L) - h^0(V, L) - 1.$$

Proof. Take a general member $D \in |L|$. By assumption, D is big. Thus we may assume that D is smooth and irreducible. Consider the following exact sequence:

$$0 \to H^0(V, L) \to H^0(V, 2L) \to H^0(D, 2L|_D).$$

Since $K_V - L$ is pseudo-effective, we know that $\deg(2L|_D) \leq \deg(K_V|_D + L|_D) = \deg K_D$: that is, $K_D - 2L|_D$ is pseudo-effective. Apply the Clifford inequality (when $h^1(D, 2L|_D) > 0$) or the Riemann-Roch theorem (when $h^1(D, 2L|_D) = 0$) for $2L|_D$, and it follows that

$$L^2 = \frac{1}{2}\deg(2L|_D) \geq h^0(D, 2L|_D) - 1 \geq h^0(V, 2L) - h^0(V, L) - 1.$$

The proof is completed. □

Let $C \in a^*|H|$ be a general member, hence smooth. Consider the two restriction maps

$$\mathrm{res}_{1,i} : H^0(V, L - iM) \to H^0(C, L|_C - iM|_C)$$

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and

$$\mathrm{res}_{2,j} : H^0(V, 2L - jM) \to H^0(C, 2L|_C - jM|_C).$$

The kernels of the above two maps are $H^0(V, L - (i+1)M)$ and $H^0(V, 2L - (j+1)M)$, respectively.

Let $V_{1,i}$ (respectively, $V_{2,j}$) denote the image of $H^0(V, L - iM)$ (respectively, $H^0(V, 2L - jM)$) under $\mathrm{res}_{1,i}$ (respectively, $\mathrm{res}_{2,j}$).

Lemma 3.7. *We have*

$$h^0(V, L) = \sum_{i=0}^{\gamma-1} \dim V_{1,i} + h^0(V, L - \gamma M)$$
$$h^0(V, 2L) = \sum_{j=0}^{2\gamma-1} \dim V_{2,j} + h^0(V, 2L - 2\gamma M) \geq 2\sum_{i=0}^{\gamma-1} \dim V_{2,2i} - \dim V_{2,0}.$$

Proof. The two equalities are obvious. The last inequality in the second formula holds simply because $h^0(V, 2L - 2\gamma M) > 0$ and $\dim V_{2,2i-1} \geq \dim V_{2,2i}$ for any $1 \leq i \leq \gamma - 1$. □

Let $|N_i|$ denote the movable part of $|L - iM|$. Note that the base locus of $|N_i|$ is either empty or of dimension zero. We deduce that N_i is nef. Also, we have

$$\dim V_{1,i} = \dim |N_i||_C + 1.$$

Lemma 3.8. *For* $0 \leq i \leq \gamma$, *we have*

$$\dim V_{2,2i} \geq 2\dim V_{1,i} - 1.$$

If, moreover, the linear system $|N_i||_C$ *induces a birational map on C, then*

$$\dim V_{2,2i} \geq 3(\dim V_{1,i} - 1).$$

Proof. This is just [6, Lemma 5.3] for $k = 2$. □

In the following, we will apply the above results to deduce more inequalities subject to the degree of the map a. The notation here will be used frequently.

3.3.2. $\deg a = 1$

We first consider the case when a is birational.

Theorem 3.9. *Suppose that* $\deg a = 1$, *and* $K_V - L$ *is pseudo-effective. Then we have*

$$h^0(V, \lfloor L \rfloor) \leq \frac{2}{5n!}\mathrm{Vol}(L) + n\varepsilon(V, L, M).$$

Similar to the proof of Theorem 3.4, we may assume that L is nef. We may even assume that $|L|$ is base point free. Moreover, we only need to prove Theorem 3.9 when $n = 2$ (i.e., Lemma 3.10), and the general result follows by an inductive argument almost identical to Step 3 and Step 4 in the proof of Theorem 3.4.

One little difference is that, instead of choosing a general pencil in $|M|$ as in Step 3 of the proof of Theorem 3.4, here we choose a general pencil in the sublinear system $a^*|H| \subseteq |M|$. Since $a^*|H|$ is also base point free, the smoothness of a general member in it is guaranteed by Bertini's theorem. This adjustment will be used until the end of this section. Note that the restriction of a on a general member of $a^*|H|$ has degree one. This is the key point for us to use the induction.

With this adjustment and by Lemma 3.5, we eventually reduce Theorem 3.9 to the following lemma.

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Lemma 3.10. *Theorem 3.9 holds when $n = 2$, $|L|$ is base point free and $h^0(V, L - M) > 0$.*

Proof. We claim that

$$h^0(V, 2L) - 6h^0(V, L) \geq -8LM - 7. \tag{3.1}$$

Suppose the claim holds. Together with Lemma 3.6, we deduce that

$$h^0(V, L) \leq \frac{1}{5}L^2 + \frac{8}{5}(LM + 1),$$

and the proof will be completed by noting that

$$\frac{8}{5}(LM + 1) < 2\varepsilon(V, L, M)$$

just as in the proof of Lemma 3.5.

To prove the claim, let C, γ, $V_{1,i}$, $V_{2,j}$ be the same as in Section 3.3.1. For $0 \leq i \leq \gamma - 1$, $|M|$ is a sublinear system of $|L - iM|$, which means $|M||_C$ is a sublinear system of $V_{1,i}$. Note that $|M||_C$ induces a birational map from C. We deduce that the map induced by $V_{1,i}$ $(0 \leq i \leq \gamma - 1)$ is birational. Thus it follows from Lemma 3.7 and the second inequality in Lemma 3.8 that

$$h^0(V, 2L) - 6h^0(V, L) \geq -6\Big(\gamma + h^0(V, L - \gamma M)\Big) - \dim V_{2,0}.$$

Let us estimate the right-hand side of the above inequality.

(1) Since $h^0(V, L - (\gamma + 1)M) = 0$, by Lemma 3.5, we have

$$h^0(V, L - \gamma M) \leq \frac{1}{2}(LM - \gamma M^2) + 1 \leq LM - \gamma M^2 + 1.$$

In particular,

$$h^0(V, L - \gamma M) + \gamma \leq LM + 1.$$

(2) Note that $\dim V_{2,0} \leq h^0(C, 2L|_C)$. By the Clifford inequality and the Riemann-Roch theorem, similar to before, we simply deduce that

$$\dim V_{2,0} \leq h^0(C, 2L|_C) \leq \deg(2L|_C) + 1 = 2LM + 1.$$

Combining the above two inequalities, we prove the claim. □

3.3.3. a is not composed with an involution

Second, we consider the case when a is not composed with an involution. That is, there is no generically finite map $V \dashrightarrow V'$ of degree two through which a factors birationally.

Theorem 3.11. *Suppose that a is not composed with an involution and $K_V - L$ is pseudo-effective. Then we have*

$$h^0(V, \lfloor L \rfloor) \leq \frac{4}{9n!}\mathrm{Vol}(L) + n\varepsilon(V, L, M).$$

Similar to what we did for Theorem 3.9, we may assume that $n = 2$, $|L|$ is base point free and $h^0(V, L - M) > 0$. For general n, we use the induction. Note that by our assumption, the restriction of a on a general member of $a^*|H|$ is not composed with an involution either. See [6, Proposition 2.8], for example. This guarantees that the inductive argument also works in this situation. Therefore, Theorem 3.11 boils down to the following lemma.

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Lemma 3.12. *Theorem 3.11 holds when $n = 2$, $|L|$ is base point free and $h^0(V, L - M) > 0$.*

Proof. We sketch the proof here since it is similar to that of Lemma 3.10.

Let C, γ, $V_{1,i}$, $V_{2,j}$, N_i be identical to those in Section 3.3.1. Let

$$i_0 = \min\{0 \le i \le \gamma - 1 | V_{1,i} \text{ does not induce a birational map on } C\}.$$

With this notation, using the same strategy as for proving equation (3.1), we deduce that

$$h^0(V, 2L) - 6h^0(V, L) \ge -8LM - 7 - 2\sum_{i=i_0}^{\gamma-1} \dim V_{1,i}.$$

Comparing to the proof of equation (3.1), the only modification we make here is that, for $i \ge i_0$, we have to use the first inequality in Lemma 3.8 to compare $\dim V_{2,2i}$ with $\dim V_{1,i}$, which is the reason for having an extra term $-2\sum_{i=i_0}^{\gamma-1} \dim V_{1,i}$ on the right-hand side.

Combining this inequality with Lemma 3.6, it follows that

$$L^2 \ge 5h^0(V, L) - 2\sum_{i=i_0}^{\gamma-1} \dim V_{1,i} - 8(LM + 1). \tag{3.2}$$

On the other hand, recall that for any $0 \le i \le \gamma - 1$, N_i is nef and

$$\dim V_{1,i} = \dim |N_i||_C + 1.$$

Note that in the current setting, $N_0 = L$ and $|N_{i+1}|$ is also the movable part of $|N_i - M|$.

For any $i > 0$, we have

$$N_{i-1}^2 - N_i^2 \ge (N_{i-1} + N_i)M \ge 2N_iM \ge 4 \dim V_{1,i} - 4, \tag{3.3}$$

where the last inequality follows from the fact that $K_C - (L|_C - iM|_C)$ is pseudo-effective. When $i \ge i_0$, $V_{1,i}$ induces a map on C of degree at least three. Otherwise, the map $\phi_{|L-iM|}$ induced by the linear system $|L - iM|$ would factor through a degree two map from V, and a would factor through $\phi_{|L-iM|}$, which is a contradiction. Let

$$\phi_i : C \to C_i'$$

be the morphism induced by the movable part of $V_{1,i}$. Then $\deg \phi_i \ge 3$. Since ϕ_i factor through the normalisation of C_i', we may assume that the curve C_i' is normal, hence smooth. Then

$$|N_i||_C = \phi_i^*|L_i'| + Z_i',$$

where L_i' and Z_i' are effective divisors on C'. Since

$$\dim V_{1,i} \le h^0(C_i', L_i') \le \deg L_i' + 1 \le \frac{1}{\deg \phi_i} N_iM + 1,$$

similar to equation (3.3), we deduce that for $i \ge \max\{1, i_0\}$,

$$N_{i-1}^2 - N_i^2 \ge 2N_iM \ge 6 \dim V_{1,i} - 6. \tag{3.4}$$

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Note that we also have

$$\dim V_{1,0} \le \begin{cases} \frac{1}{2}LM + 1, \ i_0 > 0; \\ \frac{1}{3}LM + 1, \ i_0 = 0. \end{cases}$$

Together with equations (3.3) and (3.4) for all $i > 0$, we deduce that

$$\begin{aligned} L^2 &= \sum_{i=1}^{\gamma-1}(N_{i-1}^2 - N_i^2) + N_{\gamma-1}^2 \\ &\ge 4\sum_{i=0}^{i_0-1} \dim V_{1,i} + 6\sum_{i=i_0}^{\gamma-1} \dim V_{1,i} - 2LM - 6\gamma + N_{\gamma-1}^2 \\ &\ge 4h^0(V, L) + 2\sum_{i=i_0}^{\gamma-1} \dim V_{1,i} - 4h^0(V, L - \gamma M) - 2LM - 6\gamma \\ &\ge 4h^0(V, L) + 2\sum_{i=i_0}^{\gamma-1} \dim V_{1,i} - 4LM - 6\gamma - 4. \end{aligned} \tag{3.5}$$

The third inequality here is due to Lemma 3.7. For the last inequality, by Lemma 3.5 and the definition of γ, we have

$$h^0(V, L - \gamma M) \le \frac{1}{2}(LM - \gamma M^2) + 1.$$

Then it is easy to deduce that

$$4h^0(V, L - \gamma M) \le 2LM - 2\gamma M^2 + 4.$$

Thus equation (3.5) is verified.

Now, adding equations (3.2) and (3.5), it follows that

$$2L^2 \ge 9h^0(V, L) - 12LM - 6\gamma - 12,$$

that is,

$$h^0(V, L) \le \frac{2}{9}L^2 + \frac{4}{3}LM + \frac{2}{3}\gamma + \frac{4}{3}.$$

Finally, let λ be the smallest integer such that $\lambda M - L$ is pseudo-effective. Noting that $\gamma \le \lambda$, we deduce that

$$\frac{4}{3}LM + \frac{2}{3}\gamma + \frac{4}{3} \le \frac{4}{3}\lambda M^2 + \frac{2}{3}\lambda + \frac{4}{3} \le 2(\lambda + 1)M^2 = 2\varepsilon(V, L, M).$$

Thus the whole proof of this lemma is completed. □

3.3.4. a is composed with an involution and $\kappa(\Sigma) > 0$

Finally, we consider the case when a is composed with an involution and Σ is birational to a smooth projective variety of positive Kodaira dimension. Let $\pi : \Sigma' \longrightarrow \Sigma$ be a resolution of singularities of Σ.

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Then $\kappa(\Sigma') > 0$. Set

$$r'(L, M, \Sigma') := \frac{LM^{n-1}}{2K_{\Sigma'}(\pi^* H)^{n-1}}.$$

By the assumption, $K_{\Sigma'}(\pi^* H)^{n-1} > 0$. Thus $r'(L, M, \Sigma') < \infty$.

Theorem 3.13. *Let the notation be as above. Write $r' = r'(L, M, \Sigma')$. Suppose that $K_V - L$ is pseudo-effective. Then we have*

$$h^0(V, \lfloor L \rfloor) \leq \frac{2\delta(r') - 1}{(5\delta(r') - 3)n!}\mathrm{Vol}(L) + n\varepsilon(V, L, M).$$

Moreover, for any $\mathbb{Q}$-divisor $L_1 \leq L$, we have

$$h^0(V, \lfloor L_1 \rfloor) \leq \frac{2\delta(r') - 1}{(5\delta(r') - 3)n!}\mathrm{Vol}(L_1) + n\varepsilon(V, L, M).$$

Here the function $\delta(x)$ is the same as that in Theorem 3.4. Note that under this setting, $\delta(r') > 1$. Moreover, since $r'_1 := r'(L_1, M, \Sigma') \leq r'$, we have $\delta(r'_1) \geq \delta(r')$ and $\frac{2\delta(r'_1)-1}{5\delta(r'_1)-3} \leq \frac{2\delta(r')-1}{5\delta(r')-3}$. Therefore, the second inequality in Theorem 3.13 can be deduced from the first one for L_1.

Note that the restriction of a on a general member of $a^*|H|$ is composed with an involution. Furthermore, by the adjunction, a smooth model of a general member of $|H|$ has positive Kodaira dimension. Thus the induction method works here, and Theorem 3.13 is finally reduced to the following result.

Lemma 3.14. *Theorem 3.13 holds when $n = 2$, $|L|$ is base point free and $h^0(V, L - M) > 0$.*

Proof. The proof is just a modification of the proof of Lemma 3.12. We sketch it and leave the details to the interested reader.

Let C, γ, $V_{1,i}$, $V_{2,j}$, N_i and i_0 be identical to those in the proof of Lemma 3.12. Then it is easy to see that equation (3.2) still holds here: that is,

$$L^2 \geq 5h^0(V, L) - 2\sum_{i=i_0}^{\gamma-1} \dim V_{1,i} - 8(LM + 1). \tag{3.6}$$

For any $1 \leq i \leq \gamma - 1$, equation (3.3) also holds here: that is,

$$\dim V_{1,i} \leq \frac{1}{4}(N_{i-1}^2 - N_i^2) + 1. \tag{3.7}$$

The major modification is a replacement of equation (3.4). For $i_0 \leq i \leq \gamma - 1$, $V_{1,i}$ induces a map on C of degree at least two. Let $\phi_i : C \longrightarrow C'_i$, L'_i and Z'_i be as in the proof of Lemma 3.12. We may further assume that the curve C'_i is normal. By Theorem 3.4 and the fact that $\deg \phi_i \geq 2$, we deduce that

$$\dim V_{1,i} \ \leq \ h^0(C'_i, L'_i) \leq \frac{1}{\delta(r'_i)} \deg L'_i + 1 \leq \frac{1}{2\delta(r'_i)} N_i M + 1,$$

where $r'_i = \frac{\deg L'_i}{\deg K_{C'_i}}$. Now we claim that

$$\delta(r'_i) \geq \delta(r')$$

for any $i \geq i_0$ as above. With this claim, we deduce that for $i \geq \max\{1, i_0\}$,

$$\dim V_{1,i} \leq \frac{1}{4\delta(r')}(N_{i-1}^2 - N_i^2) + 1. \tag{3.8}$$

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To prove the claim, we only need to prove that $r'_i \le r'$. Since we already have $\deg L'_i \le \frac{1}{2}LM$ as above, it suffices to prove that $\deg K_{C'_i} \ge K_{\Sigma'}(\pi^*H)$. This is rather obvious. The key is to note that $a|_C$ factors through ϕ_i. Via this factorisation, C'_i maps to a general curve in $|H|$ on Σ. Since $\pi^*|H|$ is base point free, by Bertini's theorem, a general member of $\pi^*|H|$ is smooth. Moreover, the aforementioned map on C'_i lifts to a map from C'_i to a general member $C'' \in \pi^*|H|$. Therefore, by the Hurwitz formula and the adjunction formula,

$$\deg K_{C'_i} \ge \deg K_{C''} = K_{\Sigma'}(\pi^*H) + (\pi^*H)^2 > K_{\Sigma'}(\pi^*H).$$

Thus the claim is verified, and equation (3.8) is established.

Having the above modification, we can proceed the proof as before. Sum up equations (3.7) and (3.8) over all the above $i > 0$. Note that

$$\dim V_{1,0} \le \begin{cases} \frac{1}{2}LM + 1, & i_0 > 0; \\ \frac{1}{2\delta(r')}LM + 1, & i_0 = 0. \end{cases}$$

It follows that

$$\begin{aligned} L^2 &\ge 4\sum_{i=0}^{i_0-1} \dim V_{1,i} + 4\delta(r')\sum_{i=i_0}^{\gamma-1} \dim V_{1,i} - 2LM - 4\delta(r')\gamma + N_{\gamma-1}^2 \\ &\ge 4h^0(V,L) + 4(\delta(r')-1)\sum_{i=i_0}^{\gamma-1} \dim V_{1,i} - 4h^0(V, L-\gamma M) \\ &\quad - 2LM - 4\delta(r')\gamma. \end{aligned}$$

Using the argument for proving equation (3.5), we can similarly deduce that

$$4h^0(V, L-\gamma M) + 2LM + 4\delta(r')\gamma \le 4LM + 4\delta(r')\gamma + 4.$$

The above two inequalities imply that

$$L^2 \ge 4h^0(V,L) + 4(\delta(r')-1)\sum_{i=i_0}^{\gamma-1} \dim V_{1,i} - 4LM - 4\delta(r')\gamma - 4. \tag{3.9}$$

For simplicity, we write $\delta = \delta(r')$. As before, we use equations (3.6) and (3.9) to eliminate $\sum_{i=i_0}^{\gamma-1} \dim V_{1,i}$. It follows that

$$(2\delta-1)L^2 \ge (10\delta-6)h^0(V,L) - (16\delta-12)LM - 4\delta\gamma - (16\delta-12),$$

that is,

$$h^0(V,L) \le \frac{2\delta-1}{10\delta-6}L^2 + \frac{8\delta-6}{5\delta-3}LM + \frac{2\delta}{5\delta-3}\gamma + \frac{8\delta-6}{5\delta-3}.$$

Since $1 < \delta \le 2$, it is straightforward to check that the above inequality implies that

$$h^0(V,L) \le \frac{2\delta-1}{10\delta-6}L^2 + \frac{10}{7}LM + \gamma + \frac{10}{7}. \tag{3.10}$$

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Once again, let λ be the smallest integer such that $\lambda M - L$ is pseudo-effective. Since $M^2 = (\deg a)H^2 \geq 2$ and $\gamma \leq \lambda$, we deduce that

$$\frac{10}{7}LM + \gamma + \frac{10}{7} \leq \frac{10}{7}\lambda M^2 + \frac{1}{2}\lambda M^2 + \frac{10}{7} < 2(\lambda+1)M^2 = 2\varepsilon(V, L, M).$$

Thus the whole proof is completed. □

4. Some results about χ_f

Let $f : X \to B$ be a fibration from a smooth variety X to a smooth curve B of genus g, with a general fibre F. Recall that

$$\chi_f := \chi(X, \omega_X) - \chi(B, \omega_B)\chi(F, \omega_F).$$

The goal of this section is to list some results about this relative invariant. We always assume that f is of maximal Albanese dimension. Denote by

$$a : X \to A$$

the Albanese map of X. Let $q = \dim A = h^1(X, \mathcal{O}_X)$. The above notation will be used throughout this section.

4.1. χ_f *equals the degree of a twisted Hodge bundle*

The following result relates χ_f to the degree of a twisted Hodge bundle.

Proposition 4.1. *With the above notation, we have*

$$\chi_f = \deg f_*(\omega_{X/B} \otimes \mathcal{P}),$$

where $\mathcal{P}$ is a general torsion element in $\mathrm{Pic}^0(X)$.[4]

Proof. This result has been proved by Hacon and Pardini [12, Theorem 2.4] assuming $g(B) \geq 2$. In fact, this assumption can be removed. Here we give a slightly different proof that works for any curve B.

By the assumption, $a|_F : F \to A$ is generically finite onto its image. Let $\mathcal{P} \in \mathrm{Pic}^0(X)$ be a general torsion element. Applying exactly the proof of [12, Corollary 2.3], we conclude that $f_*(\omega_{X/B} \otimes \mathcal{P})$ is torsion free, hence a locally free sheaf on B of rank $r = \chi(F, \omega_F)$. Still, by [12, Corollary 2.3], for any $i > 0$,

$$R^i f_*(\omega_{X/B} \otimes \mathcal{P}) = 0.$$

Together with the Leray spectral sequence, we know that for any $i \geq 0$,

$$h^i(X, \omega_X \otimes \mathcal{P}) = h^i(B, f_*(\omega_X \otimes \mathcal{P})).$$

In particular,

$$\chi(X, \omega_X) = \chi(X, \omega_X \otimes \mathcal{P}) = \chi(B, f_*(\omega_X \otimes \mathcal{P})).$$

Combine all of the above, and apply the Riemann-Roch theorem for $f_*(\omega_X \otimes \mathcal{P})$. It follows that

$$\begin{aligned}\deg f_*(\omega_{X/B} \otimes \mathcal{P}) &= \deg f_*(\omega_X \otimes \mathcal{P}) - 2\chi(F, \omega_F)\chi(B, \omega_B)\\ &= \chi(B, f_*(\omega_X \otimes \mathcal{P})) - \chi(F, \omega_F)\chi(B, \omega_B)\end{aligned}$$

[4]Here, being general means $\mathcal{P}$ is not contained in a certain proper subvariety (usually called the cohomological jumping loci) of $\mathrm{Pic}^0(X)$.

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$$\begin{aligned} &= \chi(X, \omega_X) - \chi(F, \omega_F)\chi(B, \omega_B) \\ &= \chi_f . \end{aligned}$$

Thus the proof is completed. $\square$

4.2. *The degree of the Hodge bundle under étale covers*

In this subsection, we assume that $g > 0$. Thus X itself is of maximal Albanese dimension.

Let $\mu_m : A \to A$ be the multiplication-by-m map of A. Let $X_m = X \times_{\mu_m} A$. Since a is the Albanese map, X_m is irreducible. Let $J(B)$ be the Jacobian variety of B. By the abuse of notation, let $\mu_m : J(B) \to J(B)$ also denote the multiplication-by-m map of $J(B)$, and let $B_m = B \times_{\mu_m} J(B)$. Thus we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[d] \arrow[rrr, bend left, "\mu_m"] & X_m \arrow[l, "a_m"'] \arrow[r, "\nu_m"'] \arrow[d, "f_m"'] & X \arrow[r, "a"'] \arrow[d, "f"'] & A \arrow[d, "h"'] \\ J(B) \arrow[rrr, bend right, "\mu_m"'] & B_m \arrow[l] \arrow[r, "\sigma_m"'] & B \arrow[r] & J(B). \end{tikzcd}$$

Now we claim that if m is a sufficiently large prime number, the morphism

$$f_m : X_m \to B_m$$

is always a fibration: that is, it has connected fibres. To see this, let $A_F = \ker h$, which is also an abelian variety. We may assume that up to a translation by a point in $J(B)$, $a(F)$ generates A_F. Thus the kernel of the map $(a|_F)^* : \mathrm{Pic}^0(A_F) \to \mathrm{Pic}^0(F)$ is finite. Thus for any integer m coprime to the cardinality of this kernel, the general fibre of f_m is irreducible.

Proposition 4.2. *With the above notation, we have*

$$\lim_{m \text{ prime}, m \to \infty} \frac{\deg f_{m*}\omega_{X_m/B_m}}{m^{2q}} = \chi_f .$$

Proof. From the above construction, we know that for any $m > 0$, the morphism $\sigma_m : B_m \to B$ is étale. By the projection formula,

$$\sigma_{m*}\mathcal{O}_{B_m} = \bigoplus_{\mathcal{P} \in T_m(B)} \mathcal{P},$$

where $T_m(B) \subset \mathrm{Pic}^0(B)$ is the subgroup of all m-torsion line bundles on B. There is a natural injective group homomorphism

$$f^* : T_m(B) \to T_m(X)$$

given by the pull-back of f, where $T_m(X) \subset \mathrm{Pic}^0(X)$ is the subgroup of all m-torsion line bundles on X. Let m be a sufficiently large prime number, and let $X'_m = X \times_B B_m$. Then we have the following commutative diagram:

$$\begin{tikzcd} X_m \arrow[dr, "\nu'_m"] \arrow[drr, bend left, "\nu_m"] \arrow[ddr, bend right, "f_m"'] & & \\ & X'_m \arrow[r, "\sigma'_m"] \arrow[d, "f'_m"'] & X \arrow[d, "f"] \\ & B_m \arrow[r, "\sigma_m"] & B. \end{tikzcd}$$

It is clear that $\nu'_m : X_m \to X'_m$ is a Galois cover with $\mathrm{Gal}(\nu'_m) \simeq \frac{T_m(X)}{f^*T_m(B)}$. Thus, by the projection formula,

$$\nu'_{m*}\omega_{X_m/B_m} = \bigoplus_{\mathcal{Q}+f^*T_m(B)} \omega_{X'_m/B_m} \otimes (\sigma'_m)^*\mathcal{Q}.$$

Here the summation runs over all cosets of $f^*T_m(B)$ in $T_m(X)$ (whose cardinality equals m^{2q-2g}), and $\mathcal{Q}$ is any representative in each corresponding coset. Thus we have the following splitting:

$$f_{m*}\omega_{X_m/B_m} = f'_{m*}\left(\nu'_{m*}\omega_{X_m/B_m}\right) = \bigoplus_{\mathcal{Q}+f^*T_m(B)} \sigma_m^*(f_*(\omega_{X/B} \otimes \mathcal{Q})).$$

All the above imply particularly that

$$\begin{aligned} \deg f_{m*}\omega_{X_m/B_m} &= \deg \sigma_{m*}\left(f_{m*}\omega_{X_m/B_m}\right) \\ &= \deg\left(\sigma_{m*}\left(f_{m*}\omega_{X_m}\right) \otimes \omega_B^{-1}\right) \\ &= \deg\left(f_*\left(\nu_{m*}\omega_{X_m}\right) \otimes \omega_B^{-1}\right). \end{aligned}$$

On the other hand, by the projection formula,

$$\nu_{m*}\omega_{X_m} = \bigoplus_{\mathcal{P}\in T_m(X)} \omega_X \otimes \mathcal{P}.$$

Thus it follows that

$$\deg f_{m*}\omega_{X_m/B_m} = \sum_{\mathcal{P}\in T_m(X)} \deg f_*(\omega_{X/B} \otimes \mathcal{P}).$$

Let $S_m(X) = \{\mathcal{P} \in T_m(X) | \deg f_*(\omega_{X/B} \otimes \mathcal{P}) = \chi_f\}$ be the subset of $T_m(X)$. By Proposition 4.1, we know that the set

$$\bigcup_{m\in\mathbb{Z}} (T_m(X) \backslash S_m(X))$$

is contained in a proper subvariety of $\mathrm{Pic}^0(X)$. In particular,

$$\lim_{m\to\infty} \frac{\#S_m(X)}{\#T_m(X)} = \lim_{m\to\infty} \frac{\#S_m(X)}{m^{2q}} = 1.$$

Note that $\deg f_*(\omega_{X/B} \otimes \mathcal{P})$ is always nonnegative (e.g., see [12]) and bounded from above independent of m. We deduce that

$$\lim_{m\to\infty} \frac{\deg f_{m*}\omega_{X_m/B_m}}{m^{2q}} = \chi_f.$$

Thus the proof is completed. □

5. Slope inequalities for fibrations over curves

In this section, we prove a slope inequality for fibrations over curves whose general fibre is a smooth variety of general type. Throughout this section, we always assume that

$$f : X \to B$$

is a fibration from a smooth variety X of dimension $n \geq 2$ to a smooth curve B. Denote by F a general fibre of f.

5.1. *Xiao's method*

Here we review Xiao's method and list some inequalities deduced from it. Most of the following facts can be found in [23] when $n = 2$ and in [19, 15, 4] for general $n \geq 2$.

Let L be a nef $\mathbb{Q}$-divisor on X. Let

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_m = f_*\mathcal{O}_X(\lfloor L \rfloor)$$

be the Harder-Narasimhan filtration of $f_*\mathcal{O}_X(\lfloor L \rfloor)$. For any $0 \leq i \leq m$, set

$$r_i = \mathrm{rank}\mathcal{E}_i, \quad \mu_i = \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{\mathrm{rank}(\mathcal{E}_i/\mathcal{E}_{i-1})}.$$

Then we have

$$\mu_1 > \mu_2 > \cdots > \mu_m$$

as well as

$$\deg \mathcal{E}_k = \sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) + r_k\mu_k \tag{5.1}$$

for each $1 \leq k \leq m$.

For each $1 \leq i \leq m$, consider the rational map $\phi_i : X \dashrightarrow \mathbb{P}_B(\mathcal{E}_i)$ associated to the evaluation morphism $f^*\mathcal{E}_i \to \mathcal{O}_X(\lfloor L \rfloor)$. We may choose a common blowing up $\sigma : Y \to X$, which resolves all indeterminacies of ϕ_i. Denote by F_1 a general fibre of $f \circ \sigma : Y \to B$. Applying Xiao's method, we obtain a sequence of nef $\mathbb{Q}$-Cartier divisors

$$N_1 \leq N_2 \leq \cdots \leq N_m \leq N_{m+1} := \sigma^*L$$

on Y. Here $N_i = (\phi_i \circ \sigma)^*H_{\mathcal{E}_i} - \mu_i F_1$, where $H_{\mathcal{E}_i}$ is a hyperplane section of $\mathbb{P}_B(\mathcal{E}_i)$. For each $1 \leq i \leq m$, $N_i|_{F_1}$ is Cartier, $h^0(F_1, N_i|_{F_1}) = r_i$,

$$N_{i+1} \geq N_i + (\mu_i - \mu_{i+1})F_1,$$

and

$$\sigma^*L \geq N_i + \mu_i F_1.$$

In particular, $\sigma^*L - \mu_1 F_1$ is pseudo-effective, and for $1 \leq i \leq m-1$, we have

$$N_{i+1}^n \geq (N_i + (\mu_i - \mu_{i+1})F_1)^n \geq N_i^n + n(\mu_i - \mu_{i+1})(N_i|_{F_1})^{n-1}.$$

Thus the following lemma follows easily by induction.

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Lemma 5.1. *Keep the same notation as above. Suppose that for some* $1 \le i \le m$*, we have* $\mu_i \ge 0$*. Let* $k := \max\{i |\ 1 \le i \le m \text{ and } \mu_i \ge 0\}$*. Then we have*

$$L^n \ge n\sum_{i=1}^{k-1}(\mu_i - \mu_{i+1})(N_i|_{F_1})^{n-1} + n\mu_k(N_k|_{F_1})^{n-1}.$$

Proof. Inductively using the above estimate, we have

$$N_k^n \ge N_1^n + n\sum_{i=1}^{k-1}(\mu_i - \mu_{i+1})(N_i|_{F_1})^{n-1} \ge n\sum_{i=1}^{k-1}(\mu_i - \mu_{i+1})(N_i|_{F_1})^{n-1}.$$

The last inequality holds since N_1 is nef. Notice that $\sigma^*L \ge N_k + \mu_k F_1$ and $\mu_k \ge 0$. We have

$$L^n \ge (N_k + \mu_k F_1)^n = N_k^n + n\mu_k(N_k|_{F_1})^{n-1}.$$

Thus the proof is completed by combining the above estimates. □

5.2. *A basic slope inequality*

We have the following result.

Proposition 5.2. *Let* $f : X \to B$ *and* F *be as before. Suppose that* L *is a nef* $\mathbb{Q}$*-divisor on* X *such that* $L|_F$ *is big and* $K_F - L|_F$ *is pseudo-effective. Then we have*

$$\left(1 + \frac{2n!(n-1)\varepsilon(F, L|_F)}{(L|_F)^{n-1}}\right)L^n \ge 2n!\deg f_*\mathcal{O}_X(\lfloor L\rfloor).$$

Proof. The inequality holds trivially when $\deg f_*\mathcal{O}_X(\lfloor L\rfloor) \le 0$. Thus we may assume that $\deg f_*\mathcal{O}_X(\lfloor L\rfloor) > 0$.

Let

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_m = f_*\mathcal{O}_X(\lfloor L\rfloor)$$

be the Harder-Narasimhan filtration of $f_*\mathcal{O}_X(\lfloor L\rfloor)$. Keep the same notation as in Section 5.1. Since $\deg f_*\mathcal{O}_X(\lfloor L\rfloor) > 0$, we have $\mu_i > 0$ for some $1 \le i \le m$. Let $k := \max\{i |\ 1 \le i \le m \text{ and } \mu_i \ge 0\}$. We have

$$\deg \mathcal{E}_k \ge \deg f_*\mathcal{O}_X(\lfloor L\rfloor). \tag{5.2}$$

By equation (5.1) and Lemma 5.1, we have the following two inequalities:

$$L^n \ge n\sum_{i=1}^{k-1}(\mu_i - \mu_{i+1})(N_i|_{F_1})^{n-1} + n\mu_k(N_k|_{F_1})^{n-1},$$

$$\deg \mathcal{E}_k = \sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) + r_k\mu_k.$$

On the other hand, note that $N_i|_{F_1} \le \sigma^*L|_{F_1}$ for any $1 \le i \le m$ and $K_{F_1} - \sigma^*L|_{F_1} \ge \sigma^*(K_F - L|_F)$ is pseudo-effective. By Theorem 2.2 and Proposition 2.1, we have

$$r_i \le \frac{1}{2(n-1)!}(N_i|_{F_1})^{n-1} + (n-1)\varepsilon(F, L|_F). \tag{5.3}$$

Combine the above three (in)equalities. We deduce that

$$\begin{aligned}L^n &\geq 2n!\deg \mathcal{E}_k - 2n!(n-1)\varepsilon(F, L|_F)(\sum_{i=1}^{k-1}(\mu_i - \mu_{i+1}) + \mu_k)\\ &= 2n!\deg \mathcal{E}_k - 2n!(n-1)\varepsilon(F, L|_F)\mu_1\\ &\geq 2n!\deg f_*\mathcal{O}_X(\lfloor L\rfloor) - 2n!(n-1)\varepsilon(F, L|_F)\mu_1,\end{aligned}$$

where the last inequality follows by equation (5.2).

What is left to us is to estimate μ_1. Note that $\sigma^*L - \mu_1 F_1$ is pseudo-effective. Thus

$$L^n = (\sigma^*L)^n \geq \mu_1(\sigma^*L|_{F_1})^{n-1} = \mu_1(L|_F)^{n-1}.$$

As a result, we deduce that

$$\left(1 + \frac{2n!(n-1)\varepsilon(F, L|_F)}{(L|_F)^{n-1}}\right)L^n \geq 2n!\deg f_*\mathcal{O}_X(\lfloor L\rfloor).$$

Thus the proof is completed. □

Before going further, we would like to remark that the inequality in Proposition 5.2 is *by no means* sharp. For example, when $n = 2$, f is a relatively minimal fibration by curves of genus $g \geq 2$, and $L = K_{X/B}$ (in this case $\varepsilon(F, L|_F) = 1$), Proposition 5.2 yields

$$K_{X/B}^2 \geq \left(\frac{4g-4}{g+1}\right)\deg f_*\omega_{X/B},$$

which is weaker than the optimal slope inequality with the slope $\frac{4g-4}{g}$. This is because our estimate is not as delicate as Xiao's original version in [23], which also considers the intersection number contributed by the horizontal part $N_i|_{F_1} - N_{i+1}|_{F_1}$. See the proof of [23, Lemma 2] for details. In other words, we have not employed Xiao's method in its full strength. However, Proposition 5.2 is already enough to deduce Theorem 1.2. Moreover, instead of using Theorem 1.2, Proposition 5.2 is sufficient for us to run the argument as in [4, Proposition 4.4] to deduce the absolute Severi inequality.

5.3. Sharper slope inequalities

In the following, we assume that

$$a : F \to \Sigma$$

is a generically finite map onto a projective variety Σ. Let H be a sufficiently ample divisor on Σ. Let $M = a^*H$.

Proposition 5.3. *Let $f : X \to B$ and F be as before. Suppose that L is a nef $\mathbb{Q}$-divisor on X such that $L|_F$ is big and $K_F - L|_F$ is pseudo-effective.*

(1) *If a is birational, then*

$$\left(1 + \frac{5n!(n-1)\varepsilon(F, L|_F, M)}{2(L|_F)^{n-1}}\right)L^n \geq \frac{5n!}{2}\deg f_*\mathcal{O}_X(\lfloor L\rfloor).$$

(2) *If a is not composed with an involution, then*

$$\left(1 + \frac{9n!(n-1)\varepsilon(F, L|_F, M)}{4(L|_F)^{n-1}}\right)L^n \geq \frac{9n!}{4}\deg f_*\mathcal{O}_X(\lfloor L\rfloor).$$

(3) *If a is composed with an involution and Σ has a smooth model of positive Kodaira dimension, then*

$$\left(1+\frac{(5\delta(r')-3)n!(n-1)\varepsilon(F,L|_F,M)}{(2\delta(r')-1)(L|_F)^{n-1}}\right)L^n \geq \frac{(5\delta(r')-3)n!}{2\delta(r')-1}\deg f_*\mathcal{O}_X(\lfloor L\rfloor).$$

Here r' and δ are the same as in Theorem 3.13.

Proof. The proof is almost identical to Proposition 5.2. We only need to replace equation (5.3) by the inequalities in Theorems 3.9 and 3.11 and the second inequality in Theorem 3.13, respectively. Then the results will follow. We leave the details to the interested reader. □

6. Proof of the main theorems

In the final section, we prove the main theorems of this paper. We always assume that $f : X \to B$ is a relatively minimal fibration from a variety X of dimension $n \geq 2$ to a smooth curve B with a general fibre F and f is of maximal Albanese dimension. Let

$$a : X \to A$$

be the Albanese map of X. Write $q = h^1(X, \mathcal{O}_X) = \dim A$.

6.1. *Preparation when* $g(B) > 0$

Before proving the results, we list some notations that will be used throughout the section. We first assume that $g(B) > 0$. Note that in this case, X itself is of maximal Albanese dimension.

Let $\pi : Y \to X$ be a resolution of singularities of X. Thus Y is also of maximal Albanese dimension. Let

$$f' := f \circ \pi : Y \to B$$

be the induced fibration with a general fibre F', and let

$$b : Y \to A$$

be the Albanese map of Y.

Let m be a sufficiently large prime number. Similar to Section 4.2 but adding Y to it, we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[d] \arrow[rrr, bend left, "\mu_m"] & Y_m \arrow[l, "b_m"'] \arrow[r, "\nu_m"] \arrow[d, "\pi_m"] \arrow[dd, bend left, "f'_m"] & Y \arrow[r, "b"] \arrow[d, "\pi"] \arrow[dd, bend right, "f'"'] & A \arrow[d] \\
 & X_m \arrow[ul, "a_m"] \arrow[d, "f_m"] \arrow[r, bend right, "\nu_m"'] & X \arrow[ur, "a"'] \arrow[d, "f"] & \\
J(B) \arrow[rrr, bend right, "\mu_m"'] & B_m \arrow[l] \arrow[r, "\sigma_m"'] & B \arrow[r] & J(B).
\end{tikzcd}$$

Here μ_m still denotes the multiplication-by-m map of A or $J(B)$, the Jacobian variety of B, X_m and f_m are identical to those in Section 4.2, $Y_m = Y \times_{\mu_m} A$ and

$$f'_m : Y_m \to B_m$$

is the Stein factorisation of the morphism $Y_m \to Y \to B$. Clearly, X_m has at worst terminal singularities, and $\pi_m : Y_m \to X_m$ is also a resolution of singularities of X_m. Denote by F'_m a general fibre of f'_m. Moreover, we will fix a sufficiently ample divisor H on A. By [7, Proposition 2.3.5],

$$m^2 H \equiv \mu_m^* H. \tag{6.1}$$

6.2. Proof of Theorem 1.2

We divide the proof into two cases.

6.2.1. Case I: $g(B) > 0$

We first prove Theorem 1.2 when $g := g(B) > 0$.

If F is not of general type, neither is F'. In this case, for a general torsion element $\mathcal{P} \in \mathrm{Pic}^0(Y)$, $f_*(\omega_{Y/B} \otimes \mathcal{P})$ is of rank $\chi(F', \omega_{F'}) = 0$. We deduce that $f_*(\omega_{Y/B} \otimes \mathcal{P}) = 0$. By Proposition 4.1, $\chi_f = \chi_{f'} = 0$. Thus equation (1.1) holds trivially.

From now on, we will always assume that F is of general type. Set

$$L := \pi^* K_{X/B}, \quad L_m := \nu_m^* L = \pi_m^* K_{X_m/B_m}.$$

Clearly, L_m is nef, and $L_m|_{F'_m}$ is big. Since X has at worst terminal singularities, $K_Y - \pi^* K_X$ is effective. Thus $K_{F'_m} - L_m|_{F'_m}$ is pseudo-effective. Moreover, since

$$f'_{m*}\mathcal{O}_{Y_m}(\lfloor L_m \rfloor) = f_{m*}\omega_{X_m/B_m} = f'_{m*}\omega_{Y_m/B_m},$$

by [11, Main Theorem], we deduce that $f'_{m*}\mathcal{O}_{Y_m}(\lfloor L_m \rfloor)$ is semi-positive.

Since $\deg \nu_m = m^{2q}$, we have

$$L_m^n = m^{2q} L^n. \tag{6.2}$$

There is a natural restriction morphism

$$\nu_m|_{F'_m} : F'_m \to F'.$$

It is an étale morphism and $\deg \nu_m|_{F'_m} = m^{2q-2g}$. Therefore, we deduce that

$$(L_m|_{F'_m})^{n-1} = m^{2q-2g}(L|_{F'})^{n-1} = m^{2q-2g} K_F^{n-1}. \tag{6.3}$$

Moreover, we claim that

$$\varepsilon(F'_m, L_m|_{F'_m}) \sim O(m^{2q-2g-2}). \tag{6.4}$$

In fact, we may assume that $b^*H - L$ is pseudo-effective. By equation (6.1), $m^2(b_m^* H) - L_m$ is also pseudo-effective. Thus

$$\begin{aligned} \varepsilon(F'_m, L_m|_{F'_m}, (b_m^* H)|_{F'_m}) &\le (m^2+1)^{n-2}\big((b_m^* H)|_{F'_m}\big)^{n-1} \\ &\le \frac{2^{n-2}}{m^2}\big((b_m^*(\mu_m^* H))|_{F'_m}\big)^{n-1} \\ &= \frac{2^{n-2}}{m^2}\big((\nu_m^*(b^* H))|_{F'_m}\big)^{n-1} \\ &= 2^{n-2} m^{2q-2g-2}((b^* H)|_{F'})^{n-1}. \end{aligned}$$

Thus the claim is verified.

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Now, applying Proposition 5.2 to f'_m and L_m, we deduce that

$$\left(1+\frac{2n!(n-1)\varepsilon(F'_m, L_m|_{F'_m})}{(L_m|_{F'_m})^{n-1}}\right)L_m^n \geq 2n!\deg f'_{m*}\mathcal{O}_{Y_m}(\lfloor L_m \rfloor). \tag{6.5}$$

Recall that

$$f'_{m*}\mathcal{O}_{Y_m}(\lfloor L_m \rfloor) = f'_{m*}\omega_{Y_m/B_m}.$$

Together with equations (6.2), (6.3) and (6.4), the above inequality in equation (6.5) implies that

$$\left(1+O(m^{-2})\right)K_{X/B}^n \geq 2n!\left(\frac{\deg f'_{m*}\omega_{Y_m/B_m}}{m^{2q}}\right). \tag{6.6}$$

Let $m \to \infty$. The left-hand side of equation (6.6) clearly tends to $K_{X/B}^n$. By Proposition 4.2, the right-hand side tends to $\chi_{f'} = \chi(Y,\omega_Y) - \chi(F',\omega_{F'})\chi(B,\omega_B)$, which is nothing but χ_f. Thus the proof for $g > 0$ is completed.

6.2.2. Case II: $g(B) = 0$

Now we prove Theorem 1.2 when $g(B) = 0$. It is easy to see that the argument for $g(B) > 0$ does not directly apply here. However, we can reduce this case to the previous one via a base change.

Choose four general distinct closed points $P_1, \ldots, P_4$ on B. Let $\sigma : C \to B$ be a double cover branched along $P_1, \ldots, P_4$. By the Hurwitz formula, $g(C) = 1$. Let $Y = X \times_B C$ and

$$f' : Y \to C$$

be the induced fibration. Thus we have the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "\pi"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ C \arrow[r, "\sigma"'] & B. \end{tikzcd}$$

Since f is relatively of maximal Albanese dimension, so is f'. As $g(C) = 1$, Y itself is of maximal Albanese dimension. Since $P_1, \ldots, P_4$ are general, we deduce that Y is normal. Moreover, we claim that Y has at worst terminal singularities. In fact, let $\mu : X' \to X$ be a resolution of singularities of X. Then $X' \times_B C \to Y$ is just a resolution of singularities of Y, and the claim is an easy consequence of the adjunction.

Since $K_{Y/C} = \pi^* K_{X/B}$, f' is also relatively minimal, and we have

$$K_{Y/C}^n = 2K_{X/B}^n. \tag{6.7}$$

We also have

$$\pi_*\omega_Y = \omega_X \oplus (\omega_X \otimes \mathcal{O}_X(2F))$$

from the above double cover. Thus, from the adjunction formula, we deduce that

$$\begin{aligned} \chi_{f'} &= \chi(Y,\omega_Y) - \chi(C,\omega_C)\chi(F,\omega_F) \\ &= \chi(X,\omega_X) + \chi(X,\omega_X \otimes \mathcal{O}_X(2F)) \\ &= 2\chi(X,\omega_X) + 2\chi(F,\omega_F) \\ &= 2\chi_f. \end{aligned} \tag{6.8}$$

Now that $g(C) = 1 > 0$, we have

$$K_{Y/C}^n \geq 2n!\chi_{f'}$$

as in Section 6.2.1. Together with equations (6.7) and (6.8), it implies that

$$K_{X/B}^n \geq 2n!\chi_f.$$

Thus the whole proof of Theorem 1.2 is completed.

Remark 6.1. With this framework, it is easy to see that in order to get inequalities of the same type as equation (1.1) with various slopes, we only need to (up to a base change to the $g(B) > 0$ case) replace equation (6.5) by a corresponding explicit estimate with the same slope, and the same argument will give rise to the desired results. This is a crucial observation to us.

6.3. Sharper inequalities

As an example of the above remark, we can easily obtain the following result.

Theorem 6.2 (Theorem 1.4). *Let $f : X \to B$ be a relatively minimal fibration from a variety X of dimension $n \geq 3$ to a smooth curve B. Denote by F a general fibre of f. Suppose that f is of maximal Albanese dimension and $a : X \to \mathrm{Alb}(X)$ is the Albanese map of X.*

(1) *If $a|_F$ is birational, then*

$$K_{X/B}^n \geq \frac{5n!}{2}\chi_f.$$

(2) *If $a|_F$ is not composed with an involution, then*

$$K_{X/B}^n \geq \frac{9n!}{4}\chi_f.$$

Proof. Remark 6.1 allows us to assume that $g > 0$. In the following, we adopt the notation in Section 6.1.

To prove (1), note that now $B \to J(B)$ is an embedding. It implies that a separates any two distinct fibres of f. In particular, a is birational. Thus for every sufficiently large prime number $m > 0$, b_m is birational. So is $b_m|_{F'_m}$. Then we simply replace the estimate equation (6.5) in the proof of Theorem 1.2 by the inequality in Proposition 5.3 (1) for f'_m and L_m, and the conclusion will follow by letting $m \to \infty$.

The proof of (2) is similar. In this case, we know that $b|_{F'}$ is not composed with an involution. Let $d = \deg a|_F = \deg b|_{F'}$. By the following Lemma 6.3, $b_m|_{F'_m}$ is not composed with an involution as long as $m > d$. Thus the conclusion will follow similarly by letting $m \to \infty$. □

Lemma 6.3. *Let $\alpha : V \to W$ be a generically finite morphism between two varieties of degree $d > 0$ such that α is not composed with an involution. Let $p > d$ be any prime number. Let $W_p \to W$ be a Galois cover with $G = \mathrm{Gal}(W_p/W)$ a p-group. Let $V_p := V \times_W W_p$, and let $\alpha_p : V_p \to W_p$ be the induced morphism. Then α_p is not composed with an involution.*

Proof. By our assumption, $K(V) \simeq \frac{K(W)(t)}{(f(t))}$, where $f(t)$ is an irreducible polynomial of degree d with coefficients in $K(W)$. Using Galois theory, we can find a variety U and a generically finite map $\beta : U \to V$ such that $K(U)$ is the splitting field of $f(t)$. Thus $K(U)/K(W)$ is a Galois extension. Write

$$H = \mathrm{Gal}(K(U)/K(W)).$$

Then H is a subgroup of S_d. In particular, $|H|$ divides $d!$. Since $p > d$ and G is a p-group, we have $(|G|, |H|) = 1$.

Let $U_p = U \times_V V_p$. We claim that U_p is irreducible. Otherwise, let U_p' be an irreducible component of U_p. Now the morphism $U_p' \to W$ has two factorisations $U_p' \to U \to W$ and $U_p' \to W_p \to W$. Thus both $|H|$ and $|G|$ divide $[K(U_p') : K(W)]$. Since $(|G|, |H|) = 1$, we have

$$[K(U_p') : K(W)] \geq |G||H|.$$

On the other hand, since the degree of the map $U_p' \to V_p$ is strictly less than $\deg \beta$, we have

$$[K(U_p') : K(W)] = [K(U_p') : K(W_p)][K(U) : K(W)] < |G||H|.$$

This is a contradiction. As a result, U_p is irreducible. In particular, the natural morphism $U_p \to U$ is also a Galois cover and

$$G = \mathrm{Gal}(K(U_p)/K(U)).$$

We claim that the extension $K(U_p)/K(W)$ is also Galois. Write

$$G_p = \mathrm{Aut}(K(U_p)/K(W)).$$

It is clear that

$$|G_p| \leq [K(U_p) : K(W)] = [K(U_p) : K(U)][K(U) : K(W)] = |G||H|.$$

On the other hand, since $H = \mathrm{Gal}(K(U_p)/K(W_p))$, we may view both G and H as subgroups of G_p. Since $(|G|, |H|) = 1$, we deduce that

$$|G_p| \geq |G||H|.$$

Therefore, $|G_p| = |G||H|$, and the claim is verified. As a consequence of this claim, G is a normal subgroup in G_p.

Now suppose that α_p is composed with an involution. This means there exists a variety V_p' such that $K(V_p) \supset K(V_p') \supseteq K(W_p)$ and

$$[K(V_p) : K(V_p')] = 2.$$

Write $H_1 = \mathrm{Aut}(K(U_p)/K(V_p))$ and $H_1' = \mathrm{Aut}(K(U_p)/K(V_p'))$. Then the fundamental theorem of Galois theory tells us that $H_1 \subset H_1'$ are both subgroups of G_p and

$$[H_1' : H_1] = 2.$$

Since G is normal, we consider another two subgroups $H_1 G \subset H_1' G$ of G_p. Then we still have

$$[H_1' G : H_1 G] = 2.$$

Note that $K(U_p)^{H_1 G} = K(V)$. Again by the fundamental theorem of Galois theory, $K(U_p)^{H_1' G}$ is a subfield of $K(V)$ and

$$[K(V) : K(U_p)^{H_1' G}] = 2.$$

This implies that α is composed with an involution. However, this is absurd. Thus the proof is completed. □

Remark 6.4. After we finished the first version of the paper, Barja informed us of the result [6, Lemma 2.9], which states that if one further assumes that V is of general type, then $\mathrm{Gal}(\alpha_p) = \mathrm{Gal}(\alpha)$ for any

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prime number p larger than a certain nonexplicit constant, depending on the volume and the dimension of V.

6.4. An example

We provide an example showing that equation (1.1) is sharp.

Let $Y := B \times A$ be a product of a smooth curve B of genus g and an abelian variety A of dimension $n-1$, with two natural projections $p_1 : Y \to B$ and $p_2 : Y \to A$. Take two sufficiently ample divisors L_1 on B and L_2 on A, respectively. Denote $L = p_1^*L_1 + p_2^*L_2$. Choose a smooth divisor $D \in |2L|$ on Y. Let $\pi : X \to Y$ be a double cover branched along D. It is easy to see that

$$f : X \to B$$

is a relatively minimal fibration whose general fibre F is a double cover of A branched along L_2 and thus is of general type. Moreover, f is relatively minimal of maximal Albanese dimension.

Since $K_{X/B} \sim \pi^*L$, we have

$$K_{X/B}^n = (\pi^*L)^n = 2L^n = 2n(\deg L_1)L_2^{n-1}.$$

On the other hand, since

$$\pi_*\omega_X = \mathcal{O}_Y(p_1^*K_B) \oplus \mathcal{O}_Y(L + p_1^*K_B)$$

and

$$\pi_*\omega_F = \mathcal{O}_A \oplus \mathcal{O}_A(L_2),$$

by the Künneth formula, we have

$$\begin{aligned} \chi(X,\omega_X) &= \chi(Y,\mathcal{O}_Y(p_1^*K_B)) + \chi(Y,\mathcal{O}_Y(L+p_1^*K_B)) \\ &= \chi(B,\omega_B)\chi(A,\mathcal{O}_A) + \chi(B,\mathcal{O}_B(L_1+K_B))\chi(A,\mathcal{O}_A(L_2)) \\ &= \chi(B,\mathcal{O}_B(L_1+K_B))\chi(A,\mathcal{O}_A(L_2)), \end{aligned}$$

and

$$\chi(F,\omega_F) = \chi(A,\mathcal{O}_A) + \chi(A,\mathcal{O}_A(L_2)) = \chi(A,\mathcal{O}_A(L_2)).$$

It follows that

$$\chi_f = \chi(A,\mathcal{O}_A(L_2))(\chi(B,\mathcal{O}_B(L_1+K_B)) - \chi(B,\omega_B)) = \frac{L_2^{n-1}}{(n-1)!}\deg L_1.$$

Thus for this fibration f, we have $K_{X/B}^n = 2n!\chi_f > 0$.

6.5. Proof of Theorem 1.3

Since the result is either known or trivial when $n = 2$, in the following, we assume that $n \geq 3$ and

$$K_{X/B}^n = 2n!\chi_f > 0.$$

We first prove Theorem 1.3 (1). Via a base change argument as in Section 6.2.2, we may assume that $g(B) > 0$. Thus we are under the setting of Section 6.1. Moreover, by Theorem 1.4, we know that $a|_F$ is composed with an involution.

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Resume all notation in Section 6.1. Write $\Sigma = a(F)$. Then Σ is a subvariety of an abelian variety A_F, a general fibre of $A \to J(B)$ of dimension $q - g(B)$, and Σ generates A_F. To show that $\Sigma = A_F$, we only need to show that the smooth model of Σ has Kodaira dimension zero.

Let $\sigma : \Sigma' \to \Sigma$ be a resolution of singularities of Σ. Let $\Sigma_m = a_m(F_m)$ and $\Sigma'_m = \Sigma_m \times_\sigma \Sigma'$. Then $\sigma_m : \Sigma'_m \to \Sigma_m$ is also a resolution of singularities of Σ_m. Let $\nu_m : \Sigma'_m \to \Sigma'$ be the induced étale map. Thus we have the following diagram:

$$\begin{tikzcd}
 & \Sigma'_m \arrow[r, "\nu_m"'] \arrow[d, "\sigma_m"] & \Sigma' \arrow[d, "\sigma"'] & \\
F'_m \arrow[r] \arrow[dr, "b_m"] & \Sigma_m \arrow[r] \arrow[d, hook] & \Sigma \arrow[d, hook] & F' \arrow[l] \arrow[dl, "b"'] \\
 & A_F \arrow[r, "\mu_m"] & A_F . &
\end{tikzcd}$$

Denote

$$r' = r'(L|_{F'}, (b^*H)|_{F'}, \Sigma'), \quad r'_m = r'(L_m|_{F'_m}, (b_m^*H)|_{F'_m}, \Sigma'_m).$$

With this notation, by equation (6.1), we have

$$r' := \frac{(L|_{F'})((b^*H)|_{F'})^{n-2}}{K_{\Sigma'}(\sigma^*(H|_\Sigma))^{n-2}} = \frac{(L_m|_{F'_m})\big((b_m^*H)|_{F'_m}\big)^{n-2}}{K_{\Sigma'_m}\big(\sigma_m^*(H|_{\Sigma_m})\big)^{n-2}} =: r'_m.$$

It simply implies that

$$\delta(r') = \delta(r'_m).$$

Now we use the framework of the proof of Theorem 1.2 again and replace equation (6.5) by the one in Proposition 5.3 (3). Together with the above equality, we deduce that

$$K_{X/B}^n \geq \frac{(5\delta(r') - 3)n!}{2\delta(r') - 1} \chi_f .$$

However, if $\kappa(\Sigma') > 0$, we would have $\delta(r') > 1$ and thus $\frac{5\delta(r')-3}{2\delta(r')-1} > 2$. This is a contradiction. As a result, $\kappa(\Sigma') = 0$ and $\Sigma = A_F$.

Now we prove Theorem 1.3 (2). Note that $K_{X/B}^n > 0$ implies that $K_{X/B}$ is also big. In particular, a general fibre F of f is a minimal variety of general type. By [14, Theorem 1-2-5], we have

$$R^i f_* \omega_{X/B}^{[l]} = R^i f_* \omega_X^{[l]} \otimes \omega_B^{\otimes(-l)} = 0$$

for any $i > 0$ and $l \geq 2$. Thus for any $l \geq 2$, we have

$$\chi(B, f_*\omega_{X/B}^{[l]}) = \chi(X, \omega_{X/B}^{[l]}).$$

Let $P_l(F)$ denote the l^{th} plurigenus of F. Then we have

$$\begin{aligned}
\deg f_*\omega_{X/B}^{[l]} &= \chi(B, f_*\omega_{X/B}^{[l]}) - P_l(F)\chi(B, \mathcal{O}_B) \\
&= \chi(X, \omega_{X/B}^{[l]}) - P_l(F)\chi(B, \mathcal{O}_B) \\
&= \frac{l^n K_{X/B}^n}{n!} + o(l^n).
\end{aligned}$$

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In particular, for $l \gg 0$, $\det f_*\omega_{X/B}^{[l]}$ is an ample line bundle on B. By [22, Proposition 4.6], we know that for $l \gg 0$, the vector bundle $f_*\omega_{X/B}^{[l]}$ is ample. Thus, by [19, Theorem 1.4], $mK_{X/B} - F$ is nef for a sufficiently large $m \in \mathbb{Z}$. Replacing B by one of its cyclic covers of degree m, which is either étale (if $g(B) > 0$) or ramified at general points (if $g(B) = 0$), and replacing $f : X \longrightarrow B$ by the fibration induced by this base change, we may assume that $K_{X/B} - F$ is nef. Similar to Section 6.2.2, we know that this induced fibration is also relatively minimal and of maximal Albanese dimension. Moreover, we still have

$$K_{X/B}^n = 2n!\chi_f > 0$$

for this new fibration f.

Using the same strategy as in the proof of Theorem 1.2, but replacing $K_{X/B}$ by $K_{X/B} - F$, we deduce that

$$(K_{X/B} - F)^n \geq 2n! \deg f_*(\omega_{X/B} \otimes \mathcal{P} \otimes \mathcal{O}_X(-F)),$$

where $\mathcal{P} \in \mathrm{Pic}^0(X)$ is a general torsion element. That is,

$$K_{X/B}^n - nK_F^{n-1} \geq 2n!(\chi_f - \chi(F, \omega_F)).$$

By the assumption that $K_{X/B}^n = 2n!\chi_f$, we have

$$K_F^{n-1} \leq 2(n-1)!\chi(F, \omega_F).$$

Since F is minimal of maximal Albanese dimension, together with the absolute Severi inequality for F, we deduce that

$$K_F^{n-1} = 2(n-1)!\chi(F, \omega_F).$$

Thus the proof is completed.

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Conflicts of Interest. None.

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References

[1] S. Ju. Arakelov, *Families of algebraic curves with fixed degeneracies*, *Izv. Akad. Nauk SSSR Ser. Mat.* **35** (1971), 1269–1293. MR 0321933

[2] M. A. Barja, *Generalized Clifford-Severi inequality and the volume of irregular varieties*, *Duke Math. J.* **164** (2015), no. 3, 541–568. MR 3314480

[3] M. A. Barja, R. Pardini, and L. Stoppino, *Higher dimensional Clifford-Severi equalities*, arXiv:1806.03005 (2018).

[4] M. A. Barja and L. Stoppino, *Stability conditions and positivity of invariants of fibrations*, *Algebraic and complex geometry*, Springer Proc. Math. Stat., vol. **71**, Springer, Cham, 2014, pp. 1–40. MR 3278568

[5] Miguel Ángel Barja, Rita Pardini, and Lidia Stoppino, *Surfaces on the Severi line*, *J. Math. Pures Appl. (9)* **105** (2016), no. 5, 734–743. MR 3479190

[6] Miguel Ángel Barja, Rita Pardini, and Lidia Stoppino, *Linear systems on irregular varieties*, *J. Inst. Math. Jussieu* **19** (2020), no. 6, 2087–2125. MR 4167003

[7] C. Birkenhake and H. Lange, *Complex abelian varieties*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **302**, Springer-Verlag, Berlin, 2004. MR 2062673

[8] C. Ciliberto, Th. Dedieu, F. Flamini, R. Pardini, C. Galati, and S. Rollenske, *Open problems*, *Boll. Unione Mat. Ital.* **11** (2018), no. 1, 5–11. MR 3782686

[9] M. Cornalba and J. Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, *Ann. Sci. École Norm. Sup. (4)* **21** (1988), no. 3, 455–475. MR 974412

[10] O. Fujino, *On maximal Albanese dimensional varieties*, *Proc. Japan Acad. Ser. A Math. Sci*. **89** (2013), no. 8, 92–95. MR 3127923

[11] T. Fujita, *On Kähler fiber spaces over curves*, *J. Math. Soc. Japan* **30** (1978), no. 4, 779–794. MR 513085

[12] C. D. Hacon and R. Pardini, *Birational characterization of products of curves of genus 2*, *Math. Res. Lett*. **12** (2005), no. 1, 129–140. MR 2122736

[13] Yong Hu and Tong Zhang, *Fibered varieties over curves with low slope and sharp bounds in dimension three*, *J. Algebraic Geom.* **30** (2021), no. 1, 57–95. MR 4233178

[14] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model problem*, Algebraic Geometry, Adv. Stud. Pure Math., vol. **10**, North-Holland, Amsterdam, 1987, pp. 283–360. MR 946243

[15] K. Konno, *A lower bound of the slope of trigonal fibrations*, *Internat. J. Math*. **7** (1996), no. 1, 19–27. MR 1369903

[16] Xin Lu and Kang Zuo, *On Severi type inequalities for irregular surfaces*, *Int. Math. Res. Not. IMRN* (2019), no. 1, 231–248. MR 3897429

[17] M. Mendes Lopes and R. Pardini, *The geography of irregular surfaces*, *Current developments in algebraic geometry*, Math. Sci. Res. Inst. Publ., vol. **59**, Cambridge Univ. Press, Cambridge, 2012, pp. 349–378. MR 2931875

[18] N. Nakayama, *Invariance of the plurigenera of algebraic varieties under minimal model conjectures*, *Topology* **25** (1986), no. 2, 237–251. MR 837624

[19] K. Ohno, *Some inequalities for minimal fibrations of surfaces of general type over curves*, *J. Math. Soc. Japan* **44** (1992), no. 4, 643–666. MR 1180441

[20] R. Pardini, *The Severi inequality $K^2 \geq 4\chi$ for surfaces of maximal Albanese dimension*, *Invent. Math*. **159** (2005), no. 3, 669–672. MR 2125737

[21] F. Severi, *La serie canonica e la teoria delle serie principali di gruppi di punti sopra una superficie algebrica*, *Comment. Math. Helv*. **4** (1932), no. 1, 268–326 (Italian).

[22] E. Viehweg, *Positivity of direct image sheaves and applications to families of higher dimensional manifolds*, *School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000)*, ICTP Lect. Notes, vol. **6**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, pp. 249–284. MR 1919460

[23] G. Xiao, *Fibered algebraic surfaces with low slope*, *Math. Ann.* **276** (1987), no. 3, 449–466. MR 875340

[24] X. Yuan and T. Zhang, *Effective bounds of linear series on algebraic varieties and arithmetic varieties*, *J. Reine Angew. Math.* **736** (2018), 255–284. MR 3769991

[25] T. Zhang, *Severi inequality for varieties of maximal Albanese dimension*, *Math. Ann.* **359** (2014), no. 3-4, 1097–1114. MR 3231026

[26] T. Zhang, *Slope inequality for families of curves over surfaces*, *Math. Ann.* **371** (2018), no. 3-4, 1095–1136. MR 3831266

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