THE STRUCTURE SHEAF OF AN INCIDENCE ALGEBRA

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Abstract

Let R be the incidence algebra of a finite partially ordered set T over a commutative noetherian ring A. Then the spectrum of R is homeomorphic to the product (Spec A) × T, where Spec A has the usual Zariski topology and T has the order topology. An explicit construction is given for the structure sheaf of R over its spectrum.

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1. Introduction

In Goldston and Mewborn (1975) we define Spec R, the spectrum of an arbitrary left noetherian ring R, to be the set of isomorphism classes of indecomposable injective left R-modules, and define a topology on Spec R which reduces to the usual Zariski topology when R is commutative. In addition, we define a sheaf of rings over Spec R which reduces to the usual structure sheaf when R is commutative. In this paper we show that the spectrum of the incidence algebra R of a finite partially ordered set T over a commutative noetherian ring A is homeomorphic to (Spec A) × T with the product topology, where Spec A has the Zariski topology and T has the order topology. In addition, we explicitly compute the rings of sections over arbitrary open subsets of Spec R.

For a detailed description of the spectrum of a left noetherian ring and the structure sheaf over its spectrum, see Goldston and Mewborn (1977). For standard notions on torsion theories and quotient rings, see Stenström (1971), and for the notions of Krull dimension and critical modules, refer to Gordon and Robson (1973).

The isomorphism class of a module M is denoted by $[M]$, the injective hull of M by $E(M)$. Let R be a noetherian ring, $X = \text{Spec } R$. For each $x \in X$ we fix a representative V_x from the isomorphism class x. Thus, $x = [V_x]$. If $U \subseteq X$, we denote by V_U the direct sum $\coprod \{V_x | x \in U\}$. Since R is noetherian, V_U is injective. $T_U(R)$ denotes the torsion ideal of R in the torsion theory congenerated by V_U; E_U denotes the R-endomorphism ring of V_U and R_U denotes the biendomorphism ring of V_U. If U is a nonempty open subset of X, R_U is the ring of sections over U in the structure sheaf of rings over X.

Throughout, A denotes a commutative noetherian ring with unity, T a finite partially ordered set, R the incidence algebra of T over A. All R-modules are left modules, and all R- and A-modules are unital. Endomorphisms of R-modules are written on the right, and a left R-module is regarded as a right module over its endomorphism ring. If A is a commutative noetherian ring, there is a one-to-one correspondence between prime ideals of A and isomorphism classes of indecomposable injective A-modules given by $P \leftrightarrow [E(A/P)]$. Therefore, to simplify notation in this case, we often refer, as is custom, to the point in Spec A as P rather than $[E(A/P)]$.

2. A special case

We first summarize without proofs the results in the special case where A is a field k. This case should provide some insight into the general situation.

Let T be a finite partially ordered set, k a field, R the incidence algebra of T over k. Then R has k-basis $\{e_{s,t} | s, t \in T, s \leqslant t\}$, with multiplication defined by $e_{st} e_{wu} = \delta_{t,w} e_{s,u}$, where $\delta_{t,w}$ is the Kronecker delta. Denote by V the k-vector space with basis $\{v_t | t \in T\}$. Then V is a left R-module under the module composition defined by $e_{s,t} v_w = \delta_{t,w} v_s$.

Let n be the number of elements in T. If we choose a linear ordering of T compatible with the partial ordering, then we can identify R with a subalgebra of the algebra of $n \times n$ upper triangular matrices in such a way that each $e_{s,t}$ is identified with one of the canonical matrix units. We can also identify V with the set of n-dimensional column vectors in such a way that each v_t is identified with a canonical basis vector and such that the module action of R on V is given by matrix multiplication.

For each $t \in T$, let M_t be the k-subspace of V spanned by $\{v_s | s \not> t\}$, and let N_t be the k-subspace of V spanned by $\{v_s | s \not\geqslant t\}$. Then M_t and N_t are R-submodules of V and M_t/N_t is a one-dimensional simple R-module. In fact, every simple R-module is isomorphic to a module of this form. The module V/N_t is an injective envelope of M_t/N_t. Furthermore, if $s, t \in T$, then $\text{Hom}_R(V/N_s, V/N_t) \neq 0$ if and only if $s \leqslant t$, and in this case $\text{Hom}_R(V/N_s, V/N_t)$ has dimension 1 over the field k. Thus Spec R is homeomorphic to T with the order topology.

Let U be any open subset of T. We wish to describe the ring R_U of sections over U. First we note that $T_U(R)$, which is the kernel of the restriction map $R \to R_U$, is the k-subspace of R spanned by $\{e_{s,u} | s \not\geqslant t, \text{ all } t \in U\}$. Since $R/T_U(R)$ is isomorphic to the incidence algebra of T' over k, where $T' = \{s \in T | s \geqslant t, \text{ some } t \in U\}$, we can reduce to the case where $T_U(R) = (0)$, that is, where U is dense in T. We now make this assumption. Let Y be the set of minimal elements of T and note that the restriction maps $R \to R_U$ and $R_U \to R_Y$ are injective. We first describe R_Y and then characterize R_U as a subring of R_Y.

For the open set Y, $V_Y = \coprod_{y \in Y}(V/N_y)$ and there are no nonzero R-module maps from V/N_y to V/N_z, $y, z \in Y$, $y \neq z$. Thus, $\mathrm{End}_R V_Y = \prod_{y \in Y}(\mathrm{End}_R V/N_y)$, and $\mathrm{Biend}_R V_Y = \prod_{y \in Y}(\mathrm{Biend}_R V/N_y)$. Now $\mathrm{End}_R V/N_y \cong k$, all $y \in Y$ and so $\mathrm{Biend}_R V/N_y$ is the full ring of k-linear transformations of V/N_y. Therefore, R_Y is a product of full linear rings.

We now return to the characterization of R_U. We must characterize those elements $(q_y) \in \prod_{y \in Y}(\mathrm{Biend}_R V/N_y)$ such that (q_y) is the restriction to V_Y of an element of $\mathrm{Biend}_R V_U$. For each $y \in Y$, V/N_y has k-basis $\{v_t + N_y | t \geqslant y\}$. To each linear transformation q_y of V/N_y to itself, we can associate the "matrix" $[\alpha(y)_{s,t}]$ of q_y in this basis: $q_y(v_t + N_y) = \sum_{s \geqslant y} \alpha(y)_{s,t}(v_s + N_y)$, $t \geqslant y$.

With these notations, the ring R_U of sections over U may be identified with the subring of $\prod_{y \in Y}(\mathrm{Biend}_R V/N_y)$ consisting of those elements (q_y) such that the associated matrices $[\alpha(y)_{s,t}]$ satisfy the following two conditions:

(1) For each $y \in Y$ and $s, t \geqslant y$, $\alpha(y)_{s,t} = 0$ if there exists $s' \in U$ such that $y \leqslant s' \leqslant s$, $s' \not\leqslant t$.

(2) If $y, z \in Y$ and $s, t \in T$ such that $s, t \geqslant y$ and $s, t \geqslant z$, then $\alpha(y)_{s,t} = \alpha(z)_{s,t}$ if there exists $s' \in U$ such that $s, t \geqslant s'$ and $s' \geqslant y, z$.

When the open set U has a unique minimal element y, the description of the ring R_U is simpler, since condition (2) above has no content in this case. We define a binary relation σ on T as follows: if $s, t \in T$ then $s\sigma t$ provided that for each $t' \in U$, $t' \leqslant s$ implies $t' \leqslant t$. Then σ is a transitive relation on T which extends the partial order relation. It follows immediately that R_U can be identified with the incidence algebra of T with the relation σ.

For example, let $T = \{1, 2, 3, 4\}$ be the partially ordered set defined by: $1 \leqslant 2 \leqslant 4$, $1 \leqslant 3 \leqslant 4$. R is the ring of 4×4 triangular matrices of the form

$$\begin{bmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Let $U = \{1, 2\}$. Then the relation σ determined by U is obtained from the partial order by the addition of the pairs $(4, 2)$, $(3, 1)$ and $(3, 2)$. Hence, R_U can be identified with the ring of 4×4 matrices of the form

$$\begin{bmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{bmatrix}.$$

3. General case

Let T be a finite partially ordered set and A a commutative noetherian ring. The incidence algebra R of T over A is the free A-module with A-basis $\{e_{s,t} \mid s, t \in T, s \leqslant t\}$, with multiplication defined by $e_{s,t} e_{w,u} = \delta_{t,w} e_{s,u}$, where $\delta_{t,w}$ is the Kronecker delta.

Let n be the number of elements of T and choose a linear ordering of T compatible with partial ordering. Then we can identify R with a subring of the ring of $n \times n$ upper triangular matrices over A in such a way that $e_{s,t}$ is identified with one of the canonical matrix units.

If L is an A-module, define $V(L)$ to be the set of functions from T into L. Let ℓv_t denote the function $v \in V(L)$ such that $v(t) = \ell$ and $v(s) = 0$, $s \neq t$. Let $Lv_t = \{v \in V(L) \mid v(s) = 0, s \neq t\}$. Note that $V(L)$ is a left R-module under the module composition $e_{s,t} \ell v_w = \delta_{t,w} \ell v_s$. If we identify $V(L)$ with the set of n-dimensional column vectors with entries in L, the module action of R on $V(L)$ is given by matrix multiplication. For each $t \in T$, we also define

$$M_t(L) = \{v \in V(L) \mid v(s) = 0 \text{ if } s > t\}$$

and

$$N_t(L) = \{v \in V(L) \mid v(s) = 0 \text{ if } s \geqslant t\}.$$

Note that $M_t(L)$ and $N_t(L)$ are R-submodules of $V(L)$ and $M_t(L)/N_t(L)$ is A-isomorphic to L. To simplify notation, we let $K_t(L)$ denote $M_t(L)/N_t(L)$.

An R-module M is *critical* if the Krull dimension of every proper factor of M is less than the Krull dimension of M (see Gordon and Robson, 1973). If $P \in \operatorname{Spec} A$ then $K_t(A/P)$ is a critical R-module for each $t \in T$. Two critical modules are *equivalent* if they have isomorphic injective envelopes.

PROPOSITION 3.1. *The set $\{K_t(A/P) \mid t \in T, P \in \operatorname{Spec} A\}$ is a complete set of equivalence class representatives of critical R-modules. The R-module $V(E(A/P))/N_t(E(A/P))$ is an injective envelope of $K_t(A/P)$.*

PROOF. First note that if R is critical as a module over itself, then $R \cong A$ and T has only one element. In this case, the result is well known. Now assume C is a nonzero cyclic critical R-module; $C \cong R/I$, where I is a proper left ideal of R. Choose $t \in T$ such that $e_{t,t} \notin I$. Then $Re_{t,t}/(Re_{t,t} \cap I)$ is isomorphic to a submodule of C, and we may assume C is of this form. Write $Re_{t,t} \cap I = \sum_{s \leqslant t}(I_s e_{s,t})$, where I_s is an ideal in A for all $s \leqslant t$ in T and $I_u \subseteq I_s$ if $s \leqslant u$ in T. Choose $u \in T$ minimal such that $u \leqslant t$ and $A/I_u \neq (0)$. Then $K_u(A/I_u) \cong (\sum_{s \leqslant u} A e_{s,t})/(\sum_{s \leqslant u} I_s e_{s,t})$, and this is isomorphic to a submodule of C. Thus we can reduce to the case $C \cong K_u(A/I)$ where I is an ideal of A. Since C is uniform as an R-module, A/I is a uniform A-module. Thus $A/I \subseteq E(A/P)$ for some prime ideal P of A. Furthermore, there is an injection $A/P \to A/I$. This induces an injection of the R-module $K_u(A/P)$ into C. Thus C is equivalent to a critical module of the required form.

Let $V_{P,t} = V(E(A/P)/N_t(E(A/P))$, where $t \in T$, $P \in \operatorname{Spec} A$; and note that every nonzero submodule of $V_{P,t}$ must have nonzero intersection with $K_t(A/P)$. Thus $V_{P,t}$ is indecomposable. We must show that $V_{P,t}$ is injective as an R-module.

If $V_{P,t}$ is not injective, there is a proper left ideal I of R and an R-map $f: I \to V_{P,t}$ which has no extension as an R-module map to a left ideal properly containing I. Choose $s \in T$ such that $e_{s,s} \notin I$. First suppose that $Re_{s,s} \cap I \subseteq \operatorname{Ker} f$. Then define $g: Re_{s,s} + I \to V_{P,t}$ by $g: re_{s,s} + a \to af$, $a \in I$, $r \in R$. If $re_{s,s} + a = 0$,

$$a = -re_{s,s} \in Re_{s,s} \cap I \subseteq \operatorname{Ker} f,$$

and $af = 0$. Hence g is a well-defined R-map properly extending f, a contradiction.

Therefore, $Re_{s,s} \cap I \nsubseteq \operatorname{ker} f$. Since $V_{P,t}$ is uniform, we may choose $r_0 \in R$ such that $r_0 e_{s,s} \in I$ and $0 \neq (r_0 e_{s,s})f \in K_t(A/P)$. Thus $(r_0 e_{s,s})f = \alpha v_t + N_t(E(A/P))$, $\alpha \in A/P$. But then also $(e_{t,t} r_0 e_{s,s})f = \alpha v_t + N_t(E(A/P))$. Write $e_{t,t} r_0 e_{s,s} = a_{t,s} e_{t,s}$, $a_{t,s} \in A$, and let $J = \{a \in A \mid ae_{t,s} \in I\}$. Then J is an ideal of A and the composition $g: J \to E(A/P)$ given by

$$j \mapsto je_{t,s} \mapsto (je_{t,s})f = \beta v_t + N_t(E(A/P)) \mapsto \beta, \quad \text{where } j \in J, \ \beta \in E(A/P),$$

is a nonzero A-map. Since $J \subseteq A$, g extends to a map $g^{\#}: A \to E(A/P)$. Let $c = 1g^{\#}$, and define $h: I + Re_{s,s} \to V_{P,t}$ by $h: a + re_{s,s} \mapsto af + crv_s + N_t(E(A/P))$, $a \in I$, $r \in R$. Suppose that $a = -re_{s,s} \in I \cap Re_{s,s}$, and write $r = \sum_{u \leqslant v} a_{u,v} e_{u,v}$, $a_{u,v} \in A$. Then $re_{s,s} = \sum_{u \leqslant s} a_{u,s} e_{u,s}$, $rv_s + N_t(E(A/P)) = \sum_{u \leqslant s} [a_{u,s} v_u + N_t(E(A/P))]$. Now

$$e_{u,u} re_{s,s} = a_{u,s} e_{u,s}$$

for all $u \leqslant s$ and so $a_{u,s} e_{u,s} \in I$. Further, if $a_{u,s} \neq 0$,

$$(e_{u,u} a_{u,s} e_{u,s})f = (a_{u,s} e_{u,s})f = \beta_u v_u + N_t(E(A/P)),$$

where $\beta_u \in E(A/P)$. If $u \ngeq t$, $(a_{u,s} e_{u,s})f = 0$. If $u \geqslant t$,

$$(e_{t,u} a_{u,s} e_{u,s})f = (a_{u,s} e_{t,s})f = \beta_u v_t + N_t(E(A/P)).$$

But since $a_{u,s} \in J$, we also have $(a_{u,s} e_{t,s})f = a_{u,s} cv_t + N_t(E(A/P))$. Thus $a_{u,s} c = \beta_u$, and

$$
\begin{aligned}
at + crv_s + N_t(E(A/P)) \\
&= (-re_{s,s})f + crv_s + N_t(E(A/P)) \\
&= -\sum_{u \leqslant s} [(a_{u,s} e_{u,s})f] + c \sum_{u \leqslant s} [a_{u,s} v_u + N_t(E(A/P))] \\
&= -\sum_{u \leqslant s} [a_{u,s} cv_u + N_t(E(A/P))] + c \sum_{u \leqslant s} [a_{u,s} v_u + N_t(E(A/P))] \\
&= 0.
\end{aligned}
$$

Therefore, h is well defined and properly extends f. This contradiction implies that $V_{P,t}$ is injective, and the proof is complete.

PROPOSITION 3.2. *If $s, t \in T$, $P, Q \in \operatorname{Spec} A$, then $\operatorname{Hom}_R(V_{P,s}, V_{Q,t}) \neq (0)$ if and only if $s \leqslant t$ and $P \subseteq Q$.*

PROOF. Assume that $s \leqslant t$, $P \subseteq Q$. Then $N_t(E(A/P)) \subseteq N_s(E(A/P))$ and there is a surjection $\prod: V_{P,s} \to V_{P,t}$. Since $P \subseteq Q$, there is a natural surjection $A/P \to A/Q$ which induces a nonzero R-map from $K_t(A/P)$ to $K_t(A/Q)$. This in turn induces a nonzero R-map from $V_{P,t}$ to $V_{Q,t}$. The composition of those two maps is a nonzero map in $\operatorname{Hom}_R(V_{P,s}, V_{Q,t})$.

Conversely, assume that $0 \neq f \in \operatorname{Hom}_R(V_{P,s}, V_{Q,t})$. Then there exists $\alpha \in A/Q$ such that $0 \neq \alpha v_t + N_t(E(A/Q)) = vf$, some $v \in V_{P,s}$. Thus

$$(e_{t,t}v)f = [\beta v_t + N_s(E(A/P))]f = \alpha v_t + N_t(E(A/Q)), \quad \beta \in E(A/P).$$

Since f is well defined, $\beta v_t \notin N_s(E(A/P))$ and hence $s \leqslant t$. Secondly, the map $f: A(\beta v_t) + N_s(E(A/P)) \to A(\alpha v_t) + N_t(E(A/Q))$ induces a nonzero A-map from $E(A/P)$ to $E(A/Q)$. Thus $P \subseteq Q$. This completes the proof.

COROLLARY 3.3. *Spec R with the Zariski topology is homeomorphic to (Spec $A) \times T$ with the product topology where Spec A has the Zariski topology and T has the order topology.*

PROOF. The result follows immediately from Propositions 3.1 and 3.2.

Note that an arbitrary open set U in Spec R may be expressed as a union $U = \cup_{t \in T}(\mathcal{O}(t) \times t)$, where $\mathcal{O}(t)$ is an open subset of Spec A for each t and $\mathcal{O}(s) \subseteq \mathcal{O}(t)$ if $t \leqslant s$ in T. Let T^* denote the set of minimal elements of T and if $t \in T^*$, let n_t denote the number of $s \in T$ such that $t \leqslant s$. If U is an open subset of Spec R, let $\prod_T(U)$ be the set of $t \in T$ such that $\mathcal{O}(t) \neq \varnothing$ in the decomposition above.

PROPOSITION 3.4. *With notations as above, let $U = \bigcup_{t \in T}(\mathcal{O}(t) \times t)$ be an open subset of Spec R, and let*

$$\mathcal{T} = \left\{ \sum_{i \leqslant p} a_{i,p} e_{i,p} \,\middle|\, a_{i,p} \in \bigcap_{s \trianglelefteq t} T_{\mathcal{O}(s)}(A) \right\}.$$

Then $T_U(R) = \mathcal{T}$.

PROOF. Let $a \in R$, $a \notin \mathcal{T}$. Then some "coordinate" $a_{u,s}$ of a is not in $\bigcap_{w \leqslant u} T_{\mathcal{O}(w)}(A)$. Hence, for some $p \leqslant u$, $a_{u,s} \notin T_{\mathcal{O}(p)}(A)$; and there exists $P \in \mathcal{O}(p)$ such that $a_{u,s}E(A/P) \neq (0)$. But then, since $s \geqslant p$, $0 \neq a_{u,s}e_{u,s}\beta_s v_s + N_p(E(A/P)) \in V_{P,p}$ for some $\beta_s \in E(A/P)$. Thus $a \notin T_U(R)$ and $T_U(R) \subseteq \mathcal{T}$.

Now let $a \in R$, $a \notin T_U(R)$. There exists $[V_{P,s}] \in U$, $v \in V_{P,s}$ such that $av \neq 0$. Write $v = \sum_{u \geqslant s}[\beta_u v_u + N_s(E(A/P))]$ and choose $w \in T$ such that $a\beta_w v_w + N_s(E(A/P)) \neq 0$.

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Then

$$0 \neq a\beta_w v_w + N_s(E(A/P)) = \sum_{t \leqslant w} [a_{t,w} e_{t,w} \beta_w v_w + N_s(E(A/P))]$$

$$= \sum_{t \leqslant w} [a_{t,w} \beta_w v_t + N_s(E(A/P))].$$

Choose $t \geqslant s$ such that $a_{t,w}\beta_w \neq 0$. Then $a_{t,w} \notin T_{\mathcal{O}(s)}(A)$. Since $s \leqslant t$, this implies that $a \notin \mathcal{T}$. Thus $\mathcal{T} = T_U(R)$ and the proof is complete.

Let $S^*(U) = T^* \cap \prod_T(U)$ and note that since $s \leqslant t$ implies that $T_{\mathcal{O}(s)}(A) \subseteq T_{\mathcal{O}(t)}(A)$, $T_U(R) = \{\sum_{t \leqslant p} a_{t,p} e_{t,p}\}$, where the coefficients $a_{t,p}$ range over the set

$$\bigcap \{T_{\mathcal{O}(s)}(A) \mid s \in S^*(U), s \leqslant t\}.$$

Thus if $U = \bigcup \{\mathcal{O}(t) \times t \mid t \in T\}$ and $U^* = \bigcup \{\mathcal{O}(t) \times t \mid t \in S^*(U)\}$, where U and U^* are open subsets of Spec R, then $T_U(R) = T_{U^*}(R)$. This implies that the restriction map $\varphi_{U,U^*} \colon R_U \to R_{U^*}$ is an injection. We first describe the rings R_{U^*} where $U^* = \bigcup_{t \in S}(\mathcal{O}(t) \times t)$, $S \subset T^*$; then we describe the rings R_U where U is an arbitrary open set in Spec R.

PROPOSITION 3.5. *If $t \in T^*$, $\mathcal{O}(t)$ is an open subset of Spec A, and $U = \mathcal{O}(t) \times t$, then R_U is isomorphic to the ring of $n_t \times n_t$ matrices over $A_{\mathcal{O}(t)}$. More generally, if $S \subset T^*$ and $U = \bigcup_{t \in S}(\mathcal{O}(t) \times t)$, $\mathcal{O}(t)$ open in Spec A for each $t \in S$, then $R_U = \prod_{t \in S} R_{(\mathcal{O}(t) \times t)}$.*

Note: The proofs of this proposition and of the following Theorem 3.6 are quite technical. We give only outlines of the proofs.

PROOF. Let $U = \mathcal{O}(t) \times t$, as in the statement of the proposition. Let $W_t = \coprod_{P \in \mathcal{O}(t)} E(A/P)$. Then $V_U = \coprod_{P \in \mathcal{O}(t)} V_P \cong V(W_t)/N_t(W_t)$. Let $\varphi \in E_U$. Then for each $s \geqslant t$ and each $u \geqslant t$ we define a map $\varphi_{s,u} \colon W_t \to W_t$ as follows: if $w \in W_t$ then $w\varphi_{s,u}$ is the uth "coordinate" of $\{wv_s + N_t(W_t)\}\varphi$. Hence

$$\{wv_s + N_t(W_t)\}\varphi = \sum_{u \geqslant t} (w\varphi_{s,u}) v_u + N_t(W_t).$$

Then $\varphi_{s,u} \in \mathrm{End}_A(W_t)$. It can be verified that $\varphi_{s,u} = 0$ if $u \neq s$ and that $\varphi_{s,s} = \varphi_{t,t}$ for all $s \geqslant t$. This permits us to identify E_U with $\mathrm{End}_A(W_t)$ via the map $\varphi \to \varphi_{t,t}$.

Let $B = [b_{r,s}]$, $r, s \geqslant t$, be an $n_t \times n_t$ matrix with entries in $A_{\mathcal{O}(t)} = \mathrm{Biend}_A(W_t)$. Define a map $q_B \colon V_U \to V_U$ by

$$q_B \colon wv_s + N_t(W_t) \mapsto \sum_{r \geqslant t} ((b_{r,s} w) v_r + N_t(W_t)), \quad \text{all } s \geqslant t.$$

Then one can check that $q_B \in R_U$. Conversely, if $q \in R_U$, $w \in W_t$, we let $b_{r,s}(w)$ denote the rth "coordinate" of $q\{wv_s + N_t(W_t)\}$. Then it can be shown that $b_{r,s}$ can be chosen independent of w, and $b_{r,s} \in A_{\mathcal{O}(t)}$, for all $r, s \geqslant t$. Thus we obtain an isomorphism between R_U and the ring of $n_t \times n_t$ matrices over $A_{\mathcal{O}(t)}$.

Now suppose that $S \subset T^*$, $U = \bigcup_{t \in S}(\mathcal{O}(t) \times t)$, where $\mathcal{O}(t)$ is an open subset of Spec A for each $t \in S$. Proposition 3.2 implies that if $s, t \in S$, $s \neq t$, then there are no nonzero R-maps between $V_{(\mathcal{O}(t) \times t)}$ and $V_{(\mathcal{O}(s) \times s)}$. Hence

$$\operatorname{End}_R V_U = \prod_{t \in S} \operatorname{End}_R V_{(\mathcal{O}(t) \times t)},$$

which implies that $\operatorname{Biend}_R V_U = \prod_{t \in S} \operatorname{Biend}_R(V_{(\mathcal{O}(t) \times t)}) = \prod_{t \in S} R_{(\mathcal{O}(t) \times t)}$. This concludes the proof.

We now turn to the computation of R_U, where $U = \bigcup_{t \in T}(\theta(t) \times t)$ is an arbitrary open subset of Spec R. If $S^*(U)$ and U^* are defined as above, then we must characterize those elements $(q_p) \in \prod_{p \in S^*(U)} \operatorname{Biend}_R(V_{(\mathcal{O}(p) \times p)})$ such that (q_p) is the restriction to V_{U^*} of an element of $\operatorname{Biend}_R V_U$. As in the proof of Proposition 3.5, we let $W_p = \coprod_{P \in \mathcal{O}(p)} E(A/P)$ and note that $V_{(\mathcal{O}(p) \times p)} = V(W_p)/N_p(W_p)$. By Proposition 3.5, we may associate to each biendomorphism q_p of $V_{\mathcal{O}(p) \times p}$ an $n_p \times n_p$ matrix $[b(p)_{s,t}]$ with entries in $A_{\mathcal{O}(p)}$ such that

$$q_p[wv_t + N_p(W_p)] = \sum_{s \geqslant p} [(b(p)_{s,t} w) v_s + N_p(W_p)], \quad t \geqslant p.$$

THEOREM 3.6. *With notations as above, the ring R_U of sections over U may be identified with the subring of $\prod_{p \in S^*(U)} \operatorname{Biend}_R(V_{(\mathcal{O}(p) \times p)})$ consisting of those elements (q_p) such that the associated matrices $[b(p)_{s,t}]$ satisfy the following two conditions:*

(1) *For each $p \in S^*(U)$ and $s, t \geqslant p$ such that there exists $k \in \prod_T(U)$ with $p \leqslant k \leqslant s$ but $k \not\leqslant t$, it is true that*

$$b(p)_{s,t} \in \bigcap \{T_{\mathcal{O}(r)}(A_{\mathcal{O}(p)}) \mid r \in \prod_T(U), p \leqslant r \leqslant s, r \not\leqslant t\}.$$

(2) *If $p, m \in S^*(U)$ and $s, t \in T$ such that $s, t \geqslant p$, $s, t \geqslant m$, and if there exists $k \in \prod_T(U)$ such that $s, t \geqslant k$ and $k \geqslant p, m$, then*

$$\rho_{\mathcal{O}(p),\mathcal{O}(k)}(b(p)_{s,t}) = \rho_{\mathcal{O}(m),\mathcal{O}(k)}(b(m)_{s,t}),$$

where $\rho_{\mathcal{O}(p),\mathcal{O}(k)}$ and $\rho_{\mathcal{O}(m),\mathcal{O}(k)}$ are the restriction maps from $A_{\mathcal{O}(p)}$ to $A_{\mathcal{O}(k)}$ and from $A_{\mathcal{O}(m)}$ to $A_{\mathcal{O}(k)}$, respectively.

PROOF. For $s, t \in T$, $s \leqslant t$, let $\theta_{s,t}: W_s \to W_t$ be the natural projection; and define $F_{s,t}: V(W_s)/N_s(W_s) \to V(W_t)/N_t(W_t)$ by $F_{s,t}: wv_u + N_s(W_s) \mapsto (w\theta_{s,t}) v_u + N_t(W_t)$ for all $u \geqslant s$. For $q \in R_U$ let q_p be the restriction of q to $V(W_p)/N_p(W_p)$, and let $[b(p)_{s,t}]$ be the matrix associated with q_p, $p \in S^*(U)$. Let $p \in S^*(U)$, $s, t \geqslant p$ and $k \in \prod_T(U)$ such that $p \leqslant k \leqslant s$, $s \not\leqslant t$. Then one can show that $\{q_p(wv_t + N_p(W_p))\} F_{p,k} = 0$. From this one can deduce that $b(p)_{s,t} \in T_{\mathcal{O}(k)}(A_{\mathcal{O}(p)})$.

Now suppose that p, m, s, t, k are as in (2). Then

$$q_k[wv_t + N_k(W_k)] = \sum_{l \geqslant k} [(b(p)_{l,t} w) v_l + N_l(W_l)] = \sum_{l \geqslant k} [b(m)_{l,t} w) v_l + N_l(W_l)].$$

So $\rho_{\mathcal{O}(p),\mathcal{O}(k)}(b(p)_{l,t}) = \rho_{\mathcal{O}(m),\mathcal{O}(k)}(b(p)_{l,t})$, for all $l \geqslant k$. Hence

$$\rho_{\mathcal{O}(p),\mathcal{O}(k)}(b(p)_{s,t}) = \rho_{\mathcal{O}(m)\,\mathcal{O}(k)}(b(m)_{s,t}).$$

Now assume that $(q_p) \in \prod_{p \in S^*(U)} \operatorname{Biend}_R(V(W_p)/N_p(W_p))$ and assume that the matrices $[b(p)_{s,t}]$ associated with the q_p satisfy (1) and (2). For every $t \in \prod_T(U)$ choose $p \in S^*(U)$ such that $p \leqslant t$, and define $q_t \in \operatorname{Biend}_R(V(W_t)/N_t(W_t))$ by

$$q_t[wv_s + N_t(W_t)] = \sum_{k \geqslant t} [(b(p)_{k,s}\, w)\, v_s + N_t(W_t)],$$

where $s \geqslant t$. Then q_t is well defined independent of p by (1) and (2). Now $\{q_t \,|\, t \in \prod_T(U)\} = q$ defines an A-linear map of V_U to itself, and the restriction of q to $V(W_p)/N_p(W_p)$, $p \in S^*(U)$, is q_p. It can be verified that, for $r, w \in \prod_T(U), r \leqslant w$, one has

$$\{q_r(w_t v_t + N_r(W_r))\}\, \varphi_{r,w} = q_w\{(w_t v_t + N_r(W_r))\}\, \varphi_{r,w}.$$

From this one deduces that $q \in \operatorname{Biend}_R(V_U)$. This completes the proof.

4. Examples

(1a) Let T be a linearly ordered set of n elements, $T = \{1, 2, ..., n\}$, R the incidence algebra of T over a field k. Then R is the algebra of $n \times n$ upper triangular matrices over k. An open set U is of the form $U(i) = \{1, 2, ..., i\}$, $1 \leqslant i \leqslant n$. Then $R_{U(n)} = R$, and $R_{u(1)}$ is the full ring of $n \times n$ matrices over k. Let $1 < i < n$. The relation σ on T defined by $U(i)$ (see Section 2) is such that $j\sigma k$ whenever $j \leqslant k$ and also whenever $k \geqslant i$. Thus $R_{U(i)}$ can be identified with the ring of all $n \times n$ matrices $[\alpha_{j,k}]$ such that $\alpha_{j,k} = 0$ if $k < i$ and $k < j$.

(1b) Let T be as above, R the incidence algebra of T over a commutative noetherian ring A. Then R is the algebra of $n \times n$ upper triangular matrices over A. An open set U in $\operatorname{Spec} R$ is of the form $U = \bigcup_{i \leqslant i}(\mathcal{O}(t) \times t)$, where $\varnothing \neq \mathcal{O}(t) \subseteq \mathcal{O}(s)$ if $t \geqslant s$. If $i = 1$, R_U is the full ring of $n \times n$ matrices over $A_{\mathcal{O}(1)}$. If $i > 1$, R_U is the ring of $n \times n$ matrices over $A_{\mathcal{O}(1)}$ of the form $[\alpha_{j,k}]$ where $\alpha_{j,k} \in T_{\mathcal{O}(k+1)}(A_{\mathcal{O}(1)})$ if $k < j$ and $k < i$.

(2) Let T be the linearly ordered set $\{1, 2, 3\}$, $A = k[x, y]/(xy)$, where k is a field, x and y are indeterminates. Let R be the incidence algebra of T over A, and let U be the open set $U = (\operatorname{Spec} A \times \{1\}) \cup (\mathcal{O}(y) \times \{2\})$, where $\mathcal{O}(y) = \{P \in \operatorname{Spec} A \,|\, y \notin P\}$. Then by (1b) above, R_U is the ring of 3×3 matrices over A of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}x & a_{22} & a_{23} \\ a_{31}x & a_{32} & a_{33} \end{bmatrix}.$$

(3) If A is an integral domain and $\varnothing \neq \mathcal{O} \subseteq \mathcal{O}'$ are open subsets of $\operatorname{Spec} A$, then $T_{\mathcal{O}}(A_{\mathcal{O}'}) = (0)$. Thus condition (1) of Theorem 3.6 reduces to: (1*) For each

$y \in S^*(U)$ and $s, t \geq y$ such that there exists $s' \in \prod_T(U)$ with $y \leq s' \leq s$, but $s' \nleq t$, $b(y)_{s,t} = 0$.

Hence if $T = \{1, 2, 3, 4, 5\}$ is the partially ordered set defined by: $1 \leq 3 \leq 5$, $2 \leq 4 \leq 5$, $1 \leq 4$, $2 \leq 3$, $A = \mathbf{Z}$, and R is the incidence algebra of T over $\mathbf{Z}$, then R is the ring of 5×5 matrices over $\mathbf{Z}$ of the form

$$\begin{bmatrix} z_{11} & 0 & z_{13} & z_{14} & z_{15} \\ 0 & z_{22} & z_{23} & z_{24} & z_{25} \\ 0 & 0 & z_{33} & 0 & z_{35} \\ 0 & 0 & 0 & z_{44} & z_{45} \\ 0 & 0 & 0 & 0 & z_{55} \end{bmatrix}.$$

Let $U = (\mathcal{O}(2) \times \{1\}) \cup (\mathcal{O}(3) \times \{2\}) \cup (\mathcal{O}(6) \times \{4\})$, where $\mathcal{O}(i) = \{P \in \mathrm{Spec} \mid i \notin P\}$. Then R_U is the set of ordered pairs of the form

$$\begin{bmatrix} \alpha_{11} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ 0 & \alpha_{33} & \alpha_{34} & \alpha_{35} \\ 0 & 0 & \alpha_{44} & \alpha_{45} \\ 0 & 0 & \alpha_{54} & \alpha_{55} \end{bmatrix}, \quad \begin{bmatrix} \beta_{22} & \beta_{23} & \beta_{24} & \beta_{25} \\ 0 & \beta_{33} & \beta_{34} & \beta_{35} \\ 0 & 0 & \beta_{44} & \beta_{45} \\ 0 & 0 & \beta_{54} & \beta_{55} \end{bmatrix}$$

with each $\alpha_{j,k} \in \mathbf{Z}_{\mathcal{O}(2)}$, each $\beta_{s,i} \in \mathbf{Z}_{\mathcal{O}(3)}$, and the restrictions of $\alpha_{44}, \alpha_{45}, \alpha_{54}, \alpha_{55}$ to $\mathbf{Z}_{\mathcal{O}(6)}$ agree with the restrictions of $\beta_{44}, \beta_{45}, \beta_{54}, \beta_{55}$ to $\mathbf{Z}_{\mathcal{O}(6)}$, respectively.

If A is a field k, T the partially ordered set above, R the incidence algebra of T over k and U the open set $\{1, 2, 4\}$; then R_U is the set of ordered pairs of matrices over k of the form

$$\begin{bmatrix} \alpha_{11} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ 0 & \alpha_{33} & \alpha_{34} & \alpha_{35} \\ 0 & 0 & \alpha_{44} & \alpha_{45} \\ 0 & 0 & \alpha_{54} & \alpha_{55} \end{bmatrix}, \quad \begin{bmatrix} \beta_{22} & \beta_{23} & \beta_{24} & \beta_{25} \\ 0 & \beta_{33} & \beta_{34} & \beta_{35} \\ 0 & 0 & \alpha_{44} & \alpha_{45} \\ 0 & 0 & \alpha_{54} & \alpha_{55} \end{bmatrix}.$$

(4) Let $T = \{1, 2, 3, 4, 5\}$ be the partially ordered set defined by: $1 \leq 3 \leq 5$, $2 \leq 4 \leq 5$, $1 \leq 4$ and $A = k[x, y, z]/(xyz)$, where k is a field and x, y and z are indeterminates. Let R be the incidence algebra of T over A, U the open set $(\mathcal{O}(x) \times \{1\}) \cup (\mathcal{O}(z) \times \{2\}) \cup (\mathcal{O}(xy) \times \{3\}) \cup (\mathcal{O}(xz) \times \{4\})$, where $\mathcal{O}(a) = \{P \in \mathrm{Spec}\, A \mid a \notin P\}$. Then R_U is the set of ordered paris of matrices of the form

$$\begin{bmatrix} \alpha_{11} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ \alpha_{31} z & \alpha_{33} & \alpha_{34} z & \alpha_{35} \\ \alpha_{41} y & \alpha_{43} y & \alpha_{44} & \alpha_{45} \\ 0 & \alpha_{53} y & \alpha_{54} z & \alpha_{55} \end{bmatrix}, \quad \begin{bmatrix} \beta_{22} & \beta_{24} & \beta_{25} \\ \beta_{42} y & \beta_{44} & \beta_{45} \\ \beta_{52} y & \beta_{54} & \beta_{55} \end{bmatrix}$$

such that each $\alpha_{j,k} \in A_x$, each $\beta_{s,t} \in A_z$, and such that the restrictions of α_{44}, α_{45}, α_{54}, α_{55} to A_{xz} agree respectively with the restrictions of β_{44}, β_{45}, β_{54}, β_{55} to A_{xz} (A_a is used to denote $A_{\theta(a)}$.)

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