Canad. Math. Bull. Vol. 14 (2), 1971

AN APPROXIMATELY CONTINUOUS PERRON INTEGRAL

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1. **Introduction**. J. C. Burkill [1] has defined the *AP*-integral whose indefinite integral is approximately continuous. An (approximately continuous) function which is approximately derivable at all points of an interval is necessarily an indefinite AP-integral of its approximate derivative.

The aim of this note is to define an integral of the Perron type such that the above assertion also holds for the *symmetric* approximate derivative. The resulting integral (SAP-integral) is more general than the AP-integral and is not compatible to the SCP-integral [2].

2. **The SAP-integral**. Let $F(x)$ be a measurable function defined on the closed interval $[a, b]$. We shall call upper symmetric approximate derivate of F at a point $c \in (a, b)$, $\overline{\text{SAD}}\, F(c)$, the greatest lower bound of all the number α ($+\infty$ included) for which the set

$$\{t : \{F(c+t)-F(c-t)\}/2t \leq \alpha\}$$

has c as a point of density. At the end points a and b, we mean $\overline{\text{SAD}}\, F(a)=\overline{\text{AD}}\, F(a)$ and $\overline{\text{SAD}}\, F(b)=\overline{\text{AD}}\, F(b)$ where $\overline{\text{AD}}$ [$\underline{\text{AD}}$] is ordinary upper [lower] approximate derivate. Similarly we can define lower symmetric approximate derivate $\underline{\text{SAD}}\, F(c)$. When they are equal, their common value is termed symmetric approximate derivative of F at c and is written SAD $F(c)$. It is easily seen that

$$\underline{\text{AD}}\, F(c) \leq \underline{\text{SAD}}\, F(c) \leq \overline{\text{SAD}}\, F(c) \leq \overline{\text{AD}}\, F(c).$$

THEOREM 1. *If $f(x)$ is approximately continuous on $[a, b]$ and* $\text{SAD}\, f(x) \geq 0$ *everywhere then $f(x)$ is nondecreasing on $[a, b]$.*

To prove the theorem we need a lemma [3, p. 964].

LEMMA. *If $f(x)$ is approximately continuous on $[a, b]$ then the set $E=\{x : f(x) \geq f(a)\}$ contains every point of positive density of E.*

Proof of Theorem 1. We first assume $\underline{\text{SAD}}\, f(x)>0$ on $[a, b]$. Let $E=\{x : f(x) \geq f(a)\}$ and $0<\alpha<1$. Let A be a subset of E such that $x'<x''$ and $x', x'' \in A$ imply $|E(x', x'')|/(x''-x') \geq \alpha$. Since $\underline{\text{SAD}}\, f(a)=\underline{\text{AD}}\, f(a)>0$ such a set A exists. Let F be the family of all such sets A. Then F is partially ordered by set inclusion, and every linearly ordered subset of F has an upper bound in F. It follows from Zorn's lemma that F has a maximal element M. Let $\beta=\sup M$. We shall show that $\beta\in M$. Let x be any point of M. Then there exists an increasing sequence $\{x_n\}$ converging to β, such that $x<x_n<\beta(x_n\in M)$. Since $|E(x,x_n)|/(x_n-x)\geq\alpha$, we have $|E(x,\beta)|/(\beta-x)\geq\alpha$. Hence β is a positive upper density point of E and therefore $\beta\in E$ by lemma, so that $\beta\in M$.

Next we shall show that $\beta=b$. Suppose that $\beta<b$. Since $\underline{\text{SAD}}\,f(\beta)>0$, the set $B=\{t:f(\beta+t)-f(\beta-t)>0,\ t>0\}$ has 0 as a right density point. Also, since $f(x)$ is approximately continuous at β, the set $C=\{t:f(\beta)+\varepsilon>f(\beta-t)>f(\beta)-\varepsilon\}$ has 0 as a point of density, so that the set $B\cap C$ has 0 as a right density point. Therefore the set $D=\{t:f(\beta+t)>f(\beta)-\varepsilon,\ t>0\}$ is also so. It follows from the relation

$$\{t:f(\beta+t)\geq f(\beta),\ t>0\}=\bigcap_{n=1}^{\infty}\{t:f(\beta+t)>f(\beta)-1/n,\ t>0\}$$

and $f(\beta)\geq f(a)$ that the set $\{t:f(\beta+t)\geq f(a),\ t>0\}$ has 0 as a right density point. Hence there exists a point $\gamma\in E$, $\beta<\gamma$ such that $|E(\beta,\gamma)|/(\gamma-\beta)\geq\alpha$. Hence $\gamma\in M$, which is impossible. Thus $f(b)\geq f(a)$.

If $\underline{\text{SAD}}\,f(x)\geq 0$ on $[a,b]$, then for any $\varepsilon>0$, $\underline{\text{SAD}}\,(f(x)+\varepsilon x)>0$. Hence $f(a)\leq f(b)+\varepsilon(b-a)$, and since ε is arbitrary, we have $f(a)\leq f(b)$. This completes the proof.

Let $f(x)$ be a measurable function defined on $[a,b]$. A function $M(x)$ is termed *major* function of $f(x)$ on $[a,b]$ if the following conditions are satisfied;

(i) $M(a)=0$,

(ii) $M(x)$ is approximately continuous on $[a,b]$,

(iii) $\underline{\text{SAD}}\,M(x)>-\infty$ at each point,

(iv) $\underline{\text{SAD}}\,M(x)\geq f(x)$ at each point.

We can define *minor* functions of $f(x)$ in similar way.

If $f(x)$ has major and minor functions in $[a,b]$ and $\inf_M M(b)=\sup_m m(b)$, then $f(x)$ is termed SAP-integrable on $[a,b]$. The common value of the two bounds is called the definite SAP-integral of $f(x)$ on $[a,b]$, and is denoted by (SAP) $\int_a^b f(t)\,dt$.

By Theorem 1 we can prove that for any upper function $M(x)$ and any minor function of $f(x)$, the function $M(x)-m(x)$ is nondecreasing on $[a,b]$. Then we may establish the following properties of the SAP-integral in usual Perron's method.

THEOREM 2. (i) *The indefinite* SAP-*integral* $F(x)=$(SAP) $\int_a^x f(t)\,dt$ *is approximately continuous on* $[a,b]$. (ii) *The set of all* SAP-*integrable functions on* $[a,b]$ *is a linear space, and the* SAP-*integral is a linear functional on it.*

THEOREM 3. (i) *If* $f(x)$ *is* SAP-*integrable on* $[a,b]$ *then its indefinite integral* $F(x)$ *is symmetric approximate derivable almost everywhere and* SAD $F(x)=f(x)$ a.e. (ii) *If* $f(x)$ *is approximately continuous on* $[a,b]$ *and symmetric approximate derivable everywhere then the function* SAD $f(x)$ *is* SAP-*integrable and*

$$\text{(SAP)}\int_a^b \text{SAD}\,f(t)\,dt=f(b)-f(a).$$

https://doi.org/10.4153/CMB-1971-048-0 Published online by Cambridge University Press

THEOREM 4. (i) *If* $f(x)$ *is* AP-*integrable on* $[a, b]$ *then* $f(x)$ *is also* SAP-*integrable, and the converse does not hold.* (ii) *The* SAP-*integral is not compatible to the* SCP-*integral.*

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