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A COLLECTIONWISE HAUSDORFF, NON-NORMAL MOORE SPACE

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A topological space X is said to be *collectionwise Hausdorff* if every discrete collection of points of X can be simultaneously separated by a collection of pairwise disjoint open sets. The question of whether there exists a collectionwise Hausdorff, non-normal Moore space was first asked by R. L. Moore. In 1964, J. M. Worrell announced that such a space did indeed exist (see [**7**]), but his proof has never appeared in print. More recently, works such as [**3**; **4**; **5** and **6**] have shown that the property of collectionwise Hausdorff plays an important role in the study of generalized metric spaces. With interest in Moore's question increased by such works, W. G. Fleissner [**2**] and Alster and Pol [**1**] exhibited *consistent* examples of collectionwise Hausdorff non-normal Moore spaces under assumptions such as Martin's axiom or $2^\omega < 2^{\omega_1}$. The purpose of this note is to construct an example of a collectionwise Hausdorff non-normal Moore space without the use of set theoretic assumptions beyond the axiom of choice. Background material and undefined terminology can be found in [**2**] or [**5**].

In the construction of the example, we will use the following lemma which is independently interesting as a fact about subsets of the real line.

LEMMA. *There exist subsets of the real line A and B such that $B \subset A$ and every countable subset of B is contained in a G_δ that does not meet $A - B$, yet every G_δ containing B does meet $A - B$.*

Proof. For each $\alpha < \omega_1$, inductively choose Y_α and Z_α such that $Y_\alpha \subset Z_\alpha \subset \mathbf{R}$, Y_α is a countable subset of $\mathbf{R} - \bigcup \{Z_\beta : \beta < \alpha\}$, and Z_α is a G_δ of (Lebesgue) measure zero that contains $\bigcup \{Z_\beta : \beta < \alpha\} \cup Y_\alpha$. The fact that Lebesgue measure is regular and countably additive guarantees that the above sets can actually be chosen. Let $Y = \bigcup \{Y_\alpha : \alpha < \omega_1\}$ and $Z = \bigcup \{Z_\alpha : \alpha < \omega_1\}$.

It is well known and easy to prove that Y is a λ-set (i.e. that each *countable* subset of Y is a relative G_δ). If Y is not a Q-set (i.e. it is not true that each subset of Y is a relative G_δ) then there exists $Y' \subset Y$ such that every G_δ containing Y' meets $Y - Y'$. In this case we can simply set $A = Y$ and $B = Y'$ to complete the proof. Also, every countable subset of Z is contained in some Z_α, so that if Z happens not to be a G_δ, we are done by letting $A = \mathbf{R}$ and $B = Z$. Similarly, we are done unless $Z - Y$ is a G_δ (for if not, let $A = \mathbf{R}$ and $B = Z - Y$).

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The only remaining case is when Y is a Q-set and both $Z - Y$ and Z are G_δ. This implies that the uncountable Q-set Y is a Borel set. But every uncountable Borel set has cardinality 2^ω whereas no Q-set can have cardinality 2^ω. Hence this last case cannot occur, and the proof is complete.

We will now construct an example of a collectionwise Hausdorff non-normal space that is not a Moore space and then show how to modify it to get an example that is a Moore space.

Let A and B be as in the lemma. Topologize A by letting the points of B have the usual (interval) neighborhoods and by letting each point of $A - B$ be open. Let $X = A \times (\omega + 1) - B \times \{\omega\}$ have the product topology. Then X is a first countable, T_3 space.

Every discrete subset of B, and hence every discrete subset of $B \times \omega$, is countable. Since the points of $(A - B) \times \omega$ are open, X will be shown to be collectionwise Hausdorff once we show that every countable subset of $B \times \omega$ is contained in an open set whose closure misses $(A - B) \times \{\omega\}$. Fix $C \subset B \times \omega$ with C countable. Every countable subset of B is contained in a G_δ that misses $A - B$, thus there exist $U_n (n \in \omega)$ open subsets of A such that $U_n \supset U_{n+1}$, $\bigcap \{U_n : n \in \omega\} \cap (A - B) = \emptyset$, and $C \subset \bigcup \{U_n \times \{n\} : n \in \omega\}$. Since the closure of $\bigcup \{U_n \times \{n\} : n \in \omega\}$ contains no points of $(A - B) \times \{\omega\}$, X is collectionwise Hausdorff.

X is not normal since the closed sets $B \times \omega$ and $(A - B) \times \{\omega\}$ cannot be separated by disjoint open sets. To see this, suppose U is an open set containing $B \times \omega$ whose closure misses $(A - B) \times \{\omega\}$. Then (by reversing the argument used in the above paragraph) U gives rise to a G_δ containing B that does not meet $A - B$. This contradiction proves that X is not normal.

The space described above is not a Moore space since $B \times \{n\}$ is a closed set that is not a G_δ. We can construct a Moore space example by letting $Y = X' \times \{\omega\} \cup X - X' \times \omega$ where X' is the set of all non-isolated points of X. Y is given the product topology by considering it as a subspace of $X \times (\omega + 1)$. Now Y is a Moore space, and the proof used above shows that Y is collectionwise Hausdorff but not normal. Note that Y is also pseudo-normal (i.e. any two disjoint closed sets, one of which is countable, can be separated by disjoint open sets) and hence is another example of a pseudo-normal space that is not normal.

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