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SPECTRAL SYNTHESIS AND APPLICATIONS TO C_0-GROUPS

M. ZARRABI

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Abstract

Let $k \geq 0$ be an integer, $T = (T(t))_{t\in\mathbb{R}}$ a C_0-group of bounded operators and A the infinitesimal generator of T. We prove that if,

$$\|T(t)\| = o(t^{k+1}) \qquad \text{and} \qquad \log^+ \|T(-t)\| = o(t^{1/2}) \qquad (t \to +\infty),$$

and if the spectrum of A is equal to $\{\lambda\}$, then A is bounded and $(A - \lambda)^{k+1} = 0$. Examples are given to show that these conditions are, essentially, the best possible.

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1. Introduction

Let X be a Banach space and let $T : \mathbb{R} \to L(X)$ be a C_0-group, that is, T satisfies the following conditions:

(i) $T(0) = I$, where I is the identity operator;
(ii) $T(t+s) = T(t)T(s) \quad (t, s \in \mathbb{R})$;
(iii) For each $x \in \mathbb{R}$, the map $t \to T(t)x$ is norm-continuous from $\mathbb{R}$ into X.

Let $\mathcal{D}$ be the set of all $x \in X$ such that the limit

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t}$$

exists. Then $\mathcal{D}$ is a linear space dense in X and A is a closed linear operator with domain $\mathcal{D}$ [11, p. 5]; A is called the *infinitesimal generator* of T.

In this paper we consider the following problem: Under what conditions on T do we have $(A - \lambda)^{k+1} = 0$, when the spectrum of A, $\sigma(A)$, is equal to $\{\lambda\}$ and where $k \geq 0$ is a given integer?

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We prove that if T satisfies the conditions

(a) $\|T(t)\| = o(t^{k+1}) \qquad (t \to +\infty)$,

(b) $\log^+ \|T(-t)\| = o(t^{1/2}) \qquad (t \to +\infty)$,

and if $\sigma(A) = \{\lambda\}$, then A is bounded and $(A - \lambda)^{k+1} = 0$.

We obtain this result by using spectral sythesis arguments in weighted algebras. For $k = 0$ we give the equivalence between the above result and the spectral synthesis property for points. We prove that if ω is a weight on $\mathbb{R}$ then points satisfy the ω-spectral synthesis if and only if for all C_0-groups T dominated by ω such that $\sigma(A) = \{\lambda\}$, we have $A = \lambda I$.

For the analogous problem in the discrete case, some results are known. Let R be a bounded and invertible operator on a Banach space with spectrum equal to $\{\lambda\}$. Gelfand proved that if the sequence $(R^n)_{n\in\mathbb{Z}}$ is bounded then $R = \lambda I$ (see [1, 12]). This result is extended in [2] and [14] to contractions R such that

(b′) $\log^+ \|R^{-n}\| = o(n^{\frac{1}{2}}) \qquad (n \to +\infty)$.

More generally, it is proved in [2] that if (b') and

(c) $\|R^n\| = O(n^k) \qquad (n \to +\infty)$

hold, then $(R-\lambda)^{k+1} = 0$. We improve this result by replacing (c) by the condition

(a′) $\|R^n\| = o(n^{k+1}) \qquad (n \to +\infty)$.

Finally, we construct examples to show that (a), (b) and (a′), (b′) are, essentially, the best growth conditions for the two results.

2. Entire functions of exponential type zero

Let f be an entire function. Then f is said to be of *exponential type* if there exists a constant $c > 0$ such that $|f(z)| = O(e^{c|z|})$ $(z \in \mathbb{C})$.

The infimum τ of all constants c for which the above inequality holds is called the *exponential type* of f. It is easy to see that

$$\tau = \limsup_{|z|\to+\infty} \frac{\log|f(z)|}{|z|}.$$

THEOREM 2.1. *Let f be an entire function of exponential type zero. If f is bounded on $[0, +\infty)$ and satisfies $\log^+ |f(-t)| = o(t^{1/2})$ $(t \to +\infty)$, then f is constant.*

PROOF. Set, for $z \in \mathbb{C} \setminus (-\infty, 0]$, $\log z = \log|z| + i \operatorname{Arg}(z)$, where $\operatorname{Arg}(z)$ is the determination of the argument of z which belongs to $(-\pi, +\pi]$. The function $z \to z^{1/2} = e^{(\log z)/2}$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$ and continuous on $\mathbb{C} \setminus (-\infty, 0)$. Denote by $\mathbb{C}_+$ the right-hand half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Let $\epsilon > 0$ and set

$$g_\epsilon(z) = e^{-\epsilon z^{1/2}} f(iz) \quad (z \in \bar{\mathbb{C}}_+).$$

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Then g_ϵ is analytic on $\mathbb{C}_+$ and continuous on $\bar{\mathbb{C}}_+$. We have

$$|g_\epsilon(z)| = e^{-\epsilon \operatorname{Re}(z^{1/2})}|f(iz)| = e^{-\epsilon|z|^{1/2}\cos(\frac{1}{2}\operatorname{Arg} z)}|f(iz)|.$$

For $z \in \mathbb{C}_+$, $\cos(\frac{1}{2}\operatorname{Arg} z) \geq 0$ and so $|g_\epsilon(z)| \leq |f(iz)|$. Since f is of exponential type zero, we have for an arbitrary $a > 0$,

$$|g_\epsilon(z)| = O\left(e^{a|z|}\right) \qquad (|z| \to +\infty,\ \operatorname{Re} z \geq 0).$$

Moreover, we obtain for $t \in \mathbb{R}$, $|g_\epsilon(it)| = e^{-\epsilon(|t|/2)^{1/2}}|f(-t)|$. The hypothesis on f implies that there exists a constant $m_\epsilon > 0$ such that $|f(t)| \leq m_\epsilon e^{\epsilon(|t|/2)^{1/2}}$ $(t \in \mathbb{R})$. Thus g_ϵ is bounded by m_ϵ on the imaginary axis. It follows from [4, Theorem 1.4.3] that g_ϵ is bounded by m_ϵ on $\bar{\mathbb{C}}_+$. We have

$$|f(iz)| = |e^{\epsilon z^{1/2}} g_\epsilon(z)| = e^{\epsilon|z|^{1/2}\cos(\frac{1}{2}\operatorname{Arg} z)}|g_\epsilon(z)| \leq m_\epsilon e^{\epsilon|z|^{1/2}}.$$

So

$$\limsup_{|z|\to+\infty,\ \operatorname{Re} z\geq 0} \frac{\log|f(iz)|}{|iz|^{1/2}} = 0.$$

Applying the same method to the function $f(-z)$, we obtain

$$\limsup_{|z|\to+\infty,\ \operatorname{Re} z\leq 0} \frac{\log|f(iz)|}{|iz|^{1/2}} = 0.$$

Finally, the function f is of growth $(1/2, 0)$ in the sense of [4, Chapter 2] and bounded on $[0, +\infty)$. Hence f is constant by [4, Theorem 3.1.5].

COROLLARY 2.2. *Let $k \geq 0$ be an integer and let f be an entire function of exponential type zero. If f satisfies the two following conditions,*

(i) $f(t) = o(t^{k+1}) \quad (t \to +\infty)$,
(ii) $\log^+|f(t)| = o(t^{\frac{1}{2}}) \quad (t \to +\infty)$,

then f is a polynomial of degree $\leq k$.

PROOF. Consider the function

$$g(z) = \frac{1}{z^{k+1}}\left(f(z) - \sum_{i=0}^{k} \frac{f^{(i)}(0)}{i!} z^i\right).$$

It is easily verified that g satisfies the hypothesis of Theorem 1.1 and thus g is constant. It follows from the definition of g that f is a polynomial of degree $\leq k+1$, and from condition (i) that f is polynomial of degree $\leq k$.

3. Spectral synthesis in weighted algebras

Let ω be a continuous function on $\mathbb{R}$ such that $\omega(t) \geq 1$ and $\omega(t+s) \leq \omega(t)\omega(s)$ $(t, s \in \mathbb{R})$. The function ω is called a *weight*.

Denote by $M_\omega(\mathbb{R})$ the space of all complex-valued measures μ on $\mathbb{R}$ such that

$$\|\mu\|_\omega = \int_{-\infty}^{+\infty} \omega(t)\, d|\mu|(t) < +\infty,$$

where $|\mu|$ is the total variation of μ. Let $\mu, \nu \in M_\omega(\mathbb{R})$. By the Riesz representation theorem there exists a unique measure, denoted $\mu \star \nu$ and called the *convolution product* of μ and ν, such that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(s+t)\, d\mu(s) d\nu(t) = \int_{-\infty}^{+\infty} f(s)\, d(\mu \star \nu)(s)$$

for all continuous functions f on $\mathbb{R}$ which vanish at infinity. We have $\mu \star \nu \in M_\omega(\mathbb{R})$ and $\|\mu \star \nu\|_\omega \leq \|\mu\|_\omega \|\nu\|_\omega$.

The space $M_\omega(\mathbb{R})$ with convolution product and norm $\|.\|_\omega$ is a unital Banach algebra (see [9, Sec. 4.16]). If we denote by δ_t the Dirac measure concentrated at $\{t\}$, then δ_0 is the unit of $M_\omega(\mathbb{R})$ and $\|\delta_t\|_\omega = \omega(t)$ $(t \in \mathbb{R})$.

Set $L_\omega(\mathbb{R})$ to be the space of all measurable functions f such that

$$\|f\|_\omega = \int_{-\infty}^{+\infty} |f(t)|\omega(t) dt < +\infty.$$

The space $L_\omega(\mathbb{R})$ is naturally identifiable with a closed ideal of $M_\omega(\mathbb{R})$. The convolution product of $f, g \in L_\omega(\mathbb{R})$ is defined almost everywhere in the formula

$$(f \star g)(s) = \int_{-\infty}^{+\infty} f(t) g(s-t)\, dt.$$

The dual of $L_\omega(\mathbb{R})$, denoted by $L^\infty_{\omega^{-1}}(\mathbb{R})$, is the set of all measurable functions g such that $\operatorname{esssup}_{t\in\mathbb{R}} |g(t)|/\omega(-t) < +\infty$, and the duality is implemented by the formula

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(t) g(-t)\, dt \qquad (f \in L_\omega(\mathbb{R}),\ g \in L^\infty_{\omega^{-1}}(\mathbb{R})).$$

We say the weight ω is *regular* if

$$\int_{-\infty}^{+\infty} \frac{\log \omega(t)}{1+t^2}\, dt < +\infty.$$

It is well-known that the algebra $L_\omega(\mathbb{R})$ is regular in the sense of [10, pp. 221, 226] if and only if the weight ω is regular [10, p. 118].

For $\mu \in M_\omega(\mathbb{R})$ we denote by $\hat{\mu}$ the Fourier transformation of μ, so that

$$\hat{\mu}(x) = \int_{-\infty}^{+\infty} e^{-ixt}\, d\mu(t) \qquad (x \in \mathbb{R}).$$

If I is a closed ideal of $L_\omega(\mathbb{R})$, the *hull* of I is the set $h(I) = \{x \in \mathbb{R} : \hat{f}(x) = 0\ (f \in I)\}$.

Let E be a closed subset of $\mathbb{R}$. We set $I_\omega(E) = \{f \in L_\omega(\mathbb{R}) : \hat{f}_{|E} = 0\}$, and we denote by $J_\omega(E)$ the closure, with respect to the norm $\|.\|_\omega$, of the set $\{f \in L_\omega(\mathbb{R}) : \hat{f} = 0$ on some neighbourhood of $E\}$. It is easily seen that $I_\omega(E)$ and $J_\omega(E)$ are closed ideals of $L_\omega(\mathbb{R})$. If the weight ω is regular then $h(I_\omega(E)) = h(J_\omega(E)) = E$ and $J_\omega(E) \subseteq I \subseteq I_\omega(E)$ for all closed ideals I of $L_\omega(\mathbb{R})$ such that $h(I) = E$ [10, p. 224].

DEFINITION 3.1. Let ω be a regular weight and let E be a closed subset of $\mathbb{R}$. Then

(i) *E satisfies ω-synthesis* if $J_\omega(E) = I_\omega(E)$;
(ii) *f satisfies ω-synthesis for E* if $f \in J_\omega(E)$.

Thus E satisfies the ω-synthesis when there exists exactly one closed ideal in $L_\omega(\mathbb{R})$ with hull E.

In [13] it is proved that closed countable subsets of $\mathbb{R}$ satisfy the ω-synthesis for all weights ω such that

$$(1) \qquad \begin{cases} \omega(t) = 1 & (t \geq 0), \\ \log\omega(-t) = o(t^{\frac{1}{2}}) & (t \to +\infty). \end{cases}$$

THEOREM 3.2. *Let $x \in \mathbb{R}$, let $k \geq 0$ be an integer and let ω be a weight such that*:

(i) $\omega(t) = o(t^{k+1}) \qquad (t \to +\infty)$;
(ii) $\log\omega(-t) = o(t^{\frac{1}{2}}) \qquad (t \to +\infty)$;
(iii) $\liminf_{|t|\to+\infty} \omega(t)/(1+|t|^k) > 0$.

Then a function $f \in L_\omega(\mathbb{R})$ satisfies ω-synthesis for $\{x\}$ if and only if $\hat{f}^{(i)}(x) = 0$ for $i = 0, 1, \dots, k$.

PROOF. By using the transformation $f(t) \to e^{ixt} f(t)$ we can suppose that $x = 0$. Assume that f satisfies the ω-synthesis for $\{0\}$. Let $(f_n)_{n\geq 0}$ be a sequence in $L_\omega(\mathbb{R})$ such that for each integer n, $\hat{f}_n$ vanishes on a neighborhood of $\{0\}$ and $\|f_n - f\|_\omega \to 0$ as $n \to +\infty$. Condition (iii) ensures the existence of the first k derivatives for all

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Fourier transforms of elements of $L_\omega(\mathbb{R})$, and we have for $i = 0, \ldots, k$ $\hat{f}_n^{(i)}(0) = 0$, $\hat{f}_n^{(i)}(0) \to \hat{f}^{(i)}(0)$ as $n \to +\infty$, and so $\hat{f}^{(i)}(0) = 0$.

Conversely, let π be the canonical surjection from $L_\omega(\mathbb{R})$ onto $L_\omega(\mathbb{R})/J_\omega(\{0\})$. There exists $p \in L_\omega(\mathbb{R})$ such that $\pi(p)$ is the unit of the quotient algebra $L_\omega(\mathbb{R})/J_\omega\{(0)\}$ (see [13, Proposition 1.2]).

For all $t \in \mathbb{R}$ set $\pi(\delta_t) = \pi(p \star \delta_t)$; $(\pi(\delta_t))_{t\in\mathbb{R}}$ is a norm-continuous C_0-group, and so there exists $u = \lim_{t\to+\infty}(\pi(\delta_t) - \pi(\delta_0))/t$ such that $\pi(\delta_t) = e^{tu}$ $(t \in \mathbb{R})$ [9, Theorem 9.4.2].

The weight ω is regular and so $h(J_\omega(\{0\})) = \{0\}$. Thus the set of characters of the algebra $L_\omega(\mathbb{R})/J_\omega(\{0\})$ is equal to $\{\chi_0\}$, where $\chi_0(\pi(f)) = \hat{f}(0)$, $(f \in L_\omega(\mathbb{R}))$. Hence the spectrum of u equals $\{\chi_0(u)\} = \{0\}$ and so $\lim_{n\to+\infty} \|u^n\|^{1/n} = 0$. Hence, for each $\epsilon > 0$, there exists a constant $m_\epsilon > 0$ such that $\|u^n\| \leq m_\epsilon \epsilon^n$ $(n \geq 0)$.

Set $\varphi(z) = e^{zu}$ $(z \in \mathbb{C})$. Then,

$$\|\varphi(z)\| \leq \sum_{n=0}^{+\infty} \frac{\|u^n\||z|^n}{n!} \leq m_\epsilon \sum_{n=0}^{+\infty} \frac{\epsilon^n |z|^n}{n!} = m_\epsilon e^{\epsilon|z|},$$

and so $\varphi(z)$ is a vector-valued entire function of exponential type zero. Moreover $\varphi(t) = \pi(\delta_t)$ $(t \in \mathbb{R})$, and so

$$\|\varphi(t)\| = \|\pi(\delta_t)\| = \|\pi(p \star \delta_t)\| \leq \|p\|_\omega \omega(t).$$

It follows from the Hahn-Banach theorem and from Corollary 2.2 that $\varphi(z)$ is a polynomial of degree $\leq k$. Let $a_0, \ldots, a_k \in L_\omega(\mathbb{R})/J_\omega(\{0\})$ be such that

$$\varphi(z) = a_0 + a_1 z + \cdots + a_k z^k.$$

Recall that the dual of $L_\omega(\mathbb{R})/J_\omega(\{0\})$ can be identified with $J_\omega(\{0\})^\perp$ by using the isomorphism

$$\theta : J_\omega(\{0\})^\perp \to (L_\omega(\mathbb{R})/J_\omega(\{0\}))^\star$$

defined by $\theta(g) = \tilde{g}$ where $g = \tilde{g} \circ \pi$ $(g \in J_\omega(\{0\})^\perp)$. We obtain, for $g \in J_\omega(\{0\})^\perp$, $t \in \mathbb{R}$,

$$\langle p \star \delta_t, g\rangle = \langle \pi(\delta_t), \tilde{g}\rangle = \sum_{i=0}^{k} \langle a_i, \tilde{g}\rangle t^i.$$

Let $f \in L_\omega(\mathbb{R})$ be such that $\hat{f}_n^{(i)}(0) = 0$ for $i = 0, \ldots, k$ and let $g \in J_\omega(\{0\})^\perp$. We

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have

$$
\begin{aligned}
\langle f, g\rangle &= \langle f \star p, g\rangle \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s)p(t-s)g(-t)\,dt\,ds \\
&= \int_{-\infty}^{+\infty} f(s)\int_{-\infty}^{+\infty} p(t-s)g(-t)\,dt\,ds \\
&= \int_{-\infty}^{+\infty} f(s)\langle p\star\delta_s, g\rangle\,ds = \int_{-\infty}^{+\infty} f(s)\left(\sum_{j=0}^{k}\langle a_j,\tilde{g}\rangle s^j\right)ds \\
&= \sum_{j=0}^{k}\langle a_j,\tilde{g}\rangle\int_{-\infty}^{+\infty} f(s)s^j\,ds = \sum_{j=0}^{k}\langle a_j,\tilde{g}\rangle(-i)^j\hat{f}^{(j)}(0) = 0.
\end{aligned}
$$

So $f \in J_\omega(\{0\})$.

REMARK. For $k = 0$, Theorem 3.2 shows that points satisfy ω-synthesis for all weights ω such that

$$
\begin{cases} \omega(t) = o(t) & (t \to +\infty), \\ \log\omega(-t) = o(t^{\frac{1}{2}}) & (t \to +\infty), \end{cases} \tag{2}
$$

and this improves [8, Theorem 8.1].

Suppose that ω satisfies $\liminf_{|t|\to+\infty}\omega(t)/|t| > 0$. Set $I = \{f \in L_\omega(\mathbb{R}) : \hat{f}(0) = \hat{f}'(0) = 0\}$; I is a closed ideal of $L_\omega(\mathbb{R})$ different from $I_\omega(\{0\})$ and with $h(I) = \{0\}$. So $\{0\}$ does not satisfy the ω-synthesis.

It follows from this observation that (2) gives, essentially, the best growth conditions on ω for which points satisfy ω-synthesis.

4. C_0-groups satisfying some growth conditions

Let T be a C_0-group with a generator A and let ω be a regular weight such that $\|T(t)\| = O(\omega(t))$ ($|t| \to +\infty$). For example, ω can be the weight $\omega_T = (\|T(t)\|)_{t\in\mathbb{R}}$.

For $\mu \in M_\omega(\mathbb{R})$, we define

$$
\mu(T) = \int_{-\infty}^{+\infty} T(t)\,d\mu(t).
$$

(Here the integral is the Bochner integral with respect to the strong operator topology [9, Theorem 3.8.2].) The map $\psi : M_\omega(\mathbb{R}) \to \mathscr{L}(X)$, $\mu \to \mu(T)$ is a continuous algebra homomorphism satisfying

$$\|\mu(T)\| \leq \sup_{t\in\mathbb{R}} \frac{\|T(t)\|}{\omega(t)} \|\mu\|_\omega.$$

Denote by $\sigma(A)$ the spectrum of A [9, Definition 2.16.1]. Let $\mathscr{R}(\lambda) = (\lambda - A)^{-1}$ be the resolvent of A, so that $\mathscr{R}$ is defined and analytic on $\mathbb{C} \setminus \sigma(A)$.

Since ω is a weight, the limits $\lim_{t\to\pm\infty} \log\omega(t)/t$ exist; and since ω is regular, $\lim_{|t|\to+\infty} \log\omega(t)/|t| = 0$. In particular, $\|T(n)\|^{1/n} \to 1$ as $|n| \to +\infty$, and so $\sigma(T(1)) \subset \Gamma$, where Γ denotes the unit circle. By [9, p. 457] we have $e^{\sigma(A)} \subset \sigma(T(1))$. Hence $\operatorname{Re}\lambda = 0$ for all $\lambda \in \sigma(A)$, and we have [9, p. 344]

$$\mathscr{R}(z) = \begin{cases} \int_0^{+\infty} T(t)e^{-zt}\,dt & (\operatorname{Re} z > 0), \\ -\int_{-\infty}^0 T(t)e^{-zt}\,dt & (\operatorname{Re} z < 0). \end{cases}$$

LEMMA 4.1. *Let* $\mu \in M_\omega(\mathbb{R})$. *Then*

$$\mu(T)x = \lim_{\epsilon\to0+} \int_{-\infty}^{+\infty} \hat{\mu}(t)\,[\mathscr{R}(\epsilon - it)x - \mathscr{R}(-\epsilon - it)x]\,dt,$$

for all x *in the domain of* A^2.

PROOF. Let $x \in X$. For $\epsilon > 0$ and $t \in \mathbb{R}$, we set $S_\epsilon(t) = e^{-\epsilon|t|}T(-t)$ and

$$g_\epsilon(t) = \int_{-\infty}^{+\infty} S_\epsilon(t-s)x\,d\mu(s),$$

so that $g_\epsilon : \mathbb{R} \to X$ is continuous. Since $\|T(t)\| = O(\omega(t))$ ($|t| \to +\infty$), we have $\int_{-\infty}^{+\infty} \|S_\epsilon(t)\|\,dt < +\infty$. Hence by Fubini's theorem

$$\begin{aligned}\int_{-\infty}^{+\infty} \|g_\epsilon(t)\|\,dt &\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \|S_\epsilon(t-s)\|\,\|x\|\,d|\mu|(s)dt \\ &\leq \|x\|\,\|\mu\|_\omega \int_{-\infty}^{+\infty} \|S_\epsilon(t)\|\,dt < +\infty.\end{aligned}$$

The Fourier transform of g_ϵ is given by the formula

$$\begin{aligned}\hat{g}_\epsilon(u) &= \int_{-\infty}^{+\infty} e^{-iut} g_\epsilon(t)\,dt \\ &= \int_{-\infty}^{+\infty} e^{-iut}\left[\int_{-\infty}^{+\infty} S_\epsilon(t-s)x\,d\mu(s)\right]dt \\ &= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} e^{-iut}e^{-\epsilon|t-s|}T(s-t)x\,dt\right]d\mu(s).\end{aligned}$$

We have

$$\int_{-\infty}^{+\infty} e^{-iut}e^{-\epsilon|s-t|}T(s-t)x\,dt = \int_{-\infty}^{+\infty} e^{iu(v-s)}e^{-\epsilon|v|}T(v)x\,dv$$
$$= e^{-ius}\left[\int_{-\infty}^{0} e^{(iu+\epsilon)v}T(v)x\,dv + \int_{0}^{+\infty} e^{(iu-\epsilon)v}T(v)x\,dv\right]$$
$$= e^{-ius}\left[\mathscr{R}(\epsilon - iu)x - \mathscr{R}(-\epsilon - iu)x\right].$$

So

$$\hat{g}_\epsilon(u) = \hat{\mu}(u)\left(\mathscr{R}(\epsilon - iu)x - \mathscr{R}(-\epsilon - iu)x\right)$$
$$= -2\epsilon\hat{\mu}(u)\mathscr{R}(\epsilon - iu)\mathscr{R}(-\epsilon - iu)x.$$

Suppose that x is in the domain of A. We have for all $\lambda \notin \sigma(A)$, $\mathscr{R}(\lambda)(\lambda - A)x = x$. So $\mathscr{R}(\lambda)x = (x + \mathscr{R}(\lambda)Ax)/\lambda$. It follows from the fact that $\lim_{|t|\to+\infty} \log\omega(t)/|t| = 0$ and [11, Remark 5.4], that there exists a constant M such that $\|\mathscr{R}(\lambda)\| \le M$ for all λ, $|\operatorname{Re}\lambda| \ge \epsilon$. Then we obtain

$$\|\mathscr{R}(\lambda)x\| \le (\|x\| + M\|Ax\|)/|\lambda|, \qquad |\operatorname{Re}\lambda| \ge \epsilon.$$

Suppose now that x is in the domain of A^2. We deduce from the above inequality that

$$\|\mathscr{R}(\epsilon - iu)\mathscr{R}(-\epsilon - iu)x\| \le \frac{1}{(\epsilon^2 + u^2)^{1/2}}\left(\|\mathscr{R}(-\epsilon - iu)x\| + M\|\mathscr{R}(-\epsilon - iu)Ax\|\right)$$
$$\le \frac{1}{(\epsilon^2 + u^2)}\left(\|x\| + 2M\|Ax\| + M^2\|A^2x\|\right).$$

This estimate and the fact that $\hat{\mu}$ is a bounded function show that $\hat{g}_\epsilon$ is integrable. Since g_ϵ is continuous, we obtain by the inverse Fourier transform, for all $t \in \mathbb{R}$,

$$g_\epsilon(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{itu}\hat{g}_\epsilon(u)\,du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\mu}(s)e^{ist}\left(\mathscr{R}(\epsilon - is)x - \mathscr{R}(-\epsilon - is)x\right)\,ds.$$

We have

$$g_\epsilon(0) = \int_{-\infty}^{+\infty} e^{-\epsilon|s|}T(s)x\,d\mu(s),$$

which implies that $\mu(T)x = \lim_{\epsilon\to 0} g_\epsilon(0)$, and this proves the lemma.

The following result is certainly known but we have not been able to find a precise reference.

THEOREM 4.2. *Let T be a C_0-group with a generator A and let ω be a regular weight such that $\|T(t)\| = O(\omega(t))$ $(|t| \to +\infty)$. Let*

$$\varphi : L_\omega(\mathbb{R}) \to L(X), \qquad f \to f(T) = \int_{-\infty}^{+\infty} f(t)T(t)\,dt.$$

Then $\ker\varphi$ *is a closed ideal of* $L_\omega(\mathbb{R})$ *and* $h(\ker\varphi) = i\sigma(A)$.

PROOF. Note that φ is the restriction of ψ to $L_\omega(\mathbb{R})$ and so φ is a continuous algebra homomorphism. So $\ker\varphi$ is a closed ideal of $L_\omega(\mathbb{R})$.

Set $\mathscr{A} = L_\omega(\mathbb{R}) + \mathbb{C}\delta_0$; $\ker\varphi$ is also a closed ideal of $\mathscr{A}$. The set of characters of $\mathscr{A}$ can be identified with the set $\{\chi_x : x \in \mathbb{R}\} \cup \{\chi_\infty\}$ where $\chi_x(\mu) = \hat{\mu}(x)$ and $\chi_\infty(\mu) = \lim_{|x|\to\infty} \hat{\mu}(x)$ $(\mu \in \mathscr{A})$. So, the set of characters of $\mathscr{A}/\ker\varphi$ can be indentified with the set $\{\chi_x : x \in h(\ker\varphi)\} \cup \{\chi_\infty\}$. Since $\ker\varphi \subset \ker\psi \cap \mathscr{A}$ there exists a homomorphism $\tilde{\psi} : \mathscr{A}/\ker\varphi \to L(X)$ such that $\tilde{\psi} \circ \pi = \psi$, where π is the canonical surjection from $\mathscr{A}$ onto $\mathscr{A}/\ker\varphi$.

Let $v(x) = e^{-x}$ $(x \geq 0)$ and $v(x) = 0$ $(x < 0)$, so that $v \in L_\omega(\mathbb{R})$. The spectrum $\sigma(\pi(v))$ of $\pi(v)$ in $\mathscr{A}/\ker\varphi$ is given by

$$\sigma(\pi(v)) = \{(1+ix)^{-1} : x \in h(\ker\varphi)\} \cup \{0\}.$$

We have

$$v(T) = \int_0^{+\infty} T(t)e^{-t}\,dt,$$

and the relation between the Carlemann transform of T and the resolvent of A, recalled in the beginning of this paragraph, implies that $v(T) = (I - A)^{-1}$. Since $v(T) = \varphi(v) = \tilde{\psi}(\pi(v))$, we have $\sigma(v(T)) \subset \sigma(\pi(v))$. So if $x \in \sigma(A)$ then $x \neq 1$, $1/(1-x) \in \sigma(v(T)) \subset \sigma(\pi(v))$, and $1/(1-x) = 1/(1+iy)$ with $y \in h(\ker\varphi)$, and so $ix = y \in h(\ker\varphi)$. Thus $i\sigma(A) \subset h(\ker\varphi)$.

For $n \geq 1$, set $e_n = n1_{[0,1/n]}$, where $1_{[0,1/n]}$ is the characteristic function of the interval $[0, 1/n]$. We have for all $f \in L_\omega(\mathbb{R})$, $\|f \star e_n - f\|_\omega \to 0$ as $n \to +\infty$ [13, Proposition 1.1].

Let I be the closure of the set of all functions $f \in L_\omega(\mathbb{R})$ such that $\hat{f}$ has compact support; I is a closed ideal of $L_\omega(\mathbb{R})$ such that $h(I) = \phi$. Wiener's Tauberian theorem for regular Beurling algebras shows that $I = L_\omega(\mathbb{R})$ (see [2] and [5]). It follows that for $n \geq 1$, there exists $k_n \in L_\omega(\mathbb{R})$ such that the support of $\hat{k}_n$ is compact and $\|k_n - e_n\|_\omega < 1/n$. Thus $\|f \star k_n - f\|_\omega \to 0$ as $n \to +\infty$ $(f \in L_\omega(\mathbb{R}))$.

Fix now $f \in L_\omega(\mathbb{R})$ and let x be in the domain of A^2, which we denote by $\mathscr{D}(A^2)$. It follows from Lemma 4.1 that

$$(f \star k_n)(T)x = \frac{1}{2\pi} \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} \hat{f}(t)\hat{k}_n(t)\,[\mathscr{R}(\epsilon - it)x - \mathscr{R}(-\epsilon - it)x]\,dt.$$

Suppose that f vanishes on a neighbourhood of $i\sigma(A)$. Then it is clear that $(f \star k_n)(T)x = 0$. Since $\mathscr{D}(A^2)$ is dense in X [11, Theorem 2.7] and since $(f \star k_n)(T)$ is a bounded operator, we have $(f \star k_n)(T) = 0$. Using the inequality

$$\|f(T) - (f \star k_n)(T)\| \leq \sup_{t\in\mathbb{R}} \frac{\|T(t)\|}{\omega(t)} \|f \star k_n - f\|_\omega$$

we see that $f(T) = 0$.

The map φ is continuous and so $J_\omega(i\sigma(A)) \subseteq \ker\varphi$. Thus $h(\ker\varphi) \subseteq h(J_\omega(i\sigma(A))$. Since the algebra $L_\omega(\mathbb{R})$ is regular, $h(J_\omega(i\sigma(A))) = i\sigma(A)$, which concludes the proof of the theorem.

THEOREM 4.3. *Let $k \geq 0$ be an integer and let T be a C_0-group with a generator A. Assume that the following conditions hold:*

(i) $\|T(t)\| = o(t^{k+1}) \qquad (t \to +\infty)$;

(ii) $\log^+ \|T(-t)\| = o(t^{1/2}) \qquad (t \to +\infty)$.

If $\sigma(A) = \{\lambda\}$, then A is bounded and $(A - \lambda)^{k+1} = 0$.

PROOF. We have $\operatorname{Re}\lambda = 0$ and multiplying $T(t)$ by $e^{-\lambda t}$ we can assume without loss of generality that $\lambda = 0$.

Set $\omega(t) = \max(\|T(t)\|, (1 + |t|)^k)$, $(t \in \mathbb{R})$; ω is a weight satisfying the three conditions of Theorem 3.2 and we have $\|T(t)\| = O(\omega(t))$ $(t \to +\infty)$.

Let

$$u(t) = \begin{cases} 0 & \text{if } |t| \geq 1, \\ ce^{-1/(1-|t|^2)} & \text{if } |t| < 1. \end{cases}$$

the constant c is chosen such that $\int_{-1}^{1} u(t)\,dt = 1$. The function u is infinitely differentiable and $\operatorname{Supp}(u) = [-1, 1]$. Set, for $n \geq 1$, and for $t \in \mathbb{R}$, $u_n(t) = nu(nt)$ and $e_n(t) = u_n^{(k+1)}(t)$. Clearly, $\operatorname{Supp}(e_n) = [-1/n, 1/n]$ and $e_n \in L_\omega(\mathbb{R})$. We have $\hat{e}_n(t) = (it)^{k+1}\hat{u}_n(t)$. Thus $\hat{e}_n^{(i)}(0) = 0$ for $i = 0, 1, \ldots, k$ and Theorem 3.2 shows that e_n satisfies ω-synthesis for $\{0\}$. It follows from Theorem 4.2 that $h(\ker\varphi) = \{0\}$, so that $e_n \in \ker\varphi$. So $e_n(T) = 0$.

Let x be in the domain of A^{k+1}. The map $h_x : \mathbb{R} \to X$, $t \mapsto T(t)x$ is $(k+1)$-times continuously differentiable and $h_x^{(k+1)}(t) = T(t)A^{k+1}x$ [9, Theorem 11.5.3].

We have

$$e_n(T)x = \int_{-\infty}^{+\infty} e_n(t)h_x(t)\,dt;$$

applying integration by parts $(k + 1)$-times we obtain

$$e_n(T) = (-1)^{k+1} \int_{-\infty}^{+\infty} u_n(t)h_x^{(k+1)}(t)\,dt.$$

So

$$\begin{aligned}\|e_n(T)x-(-1)^{k+1}A^{k+1}x\| &= \left\|\int_{-\infty}^{+\infty} u_n(t)(T(t)A^{k+1}x - A^{k+1}x)\,dt\right\| \\ &= \left\|\int_{-\infty}^{+\infty} \frac{1}{n}u_n\left(\frac{t}{n}\right)\left(T\left(\frac{t}{n}\right)A^{k+1}x - A^{k+1}x\right)dt\right\| \\ &= \left\|\int_{-1}^{1} u(t)\left(T\left(\frac{t}{n}\right)A^{k+1}x - A^{k+1}x\right)dt\right\| \\ &\le \sup_{-1\le t\le 1}\left\|T\left(\frac{t}{n}\right)A^{k+1}x - A^{k+1}x\right)\right\|.\end{aligned}$$

Since the map $t \mapsto T(t)A^{k+1}x$ is continuous, the latter quantity converges to 0 as $n \to +\infty$. Thus $e_n(T)x \to (-1)^{k+1}A^{k+1}x$ as $n \to +\infty$. Since $e_n(T) = 0$, we obtain $A^{k+1}x = 0$. Hence $A^{k+1} = 0$. It follows from [9, p. 56] that A is bounded.

We will make precise the relation between Theorem 3.2 and Theorem 4.3 in the case $k = 0$. We will say that a regular weight ω *satisfies property* (P) if :

(P) *For all C_0-groups T with $\sigma(A) = \{0\}$ and $\|T(t)\| = O(\omega(t))$ $(|t| \to +\infty)$, we have $A = 0$.*

THEOREM 4.4. *Let ω be a regular weight. Then ω satisfies property (P) if and only if points satisfy ω-synthesis.*

PROOF. We will use the notations introduced in the proof of Theorem 3.2. Suppose that ω satisfies property (P). The C_0-group $(\pi(\delta_t))_{t\in\mathbb{R}} \subseteq L_\omega(\mathbb{R})/J_\omega(\{0\})$ is such that $\sigma(u) = 0$ and $\|\pi(\delta_t)\| \le \|p\|_\omega \omega(t)$ $(t \in \mathbb{R})$. Hence $u = 0$ and $\pi(\delta_t) = \pi(\delta_0)$ $(t \in \mathbb{R})$.

Let $f \in I_\omega(\{0\})$ and let $g \in J_\omega(\{0\})^\perp$. Then

$$\begin{aligned}\langle f, g\rangle = \langle f \star p, g\rangle &= \int_{-\infty}^{+\infty} f(s)\langle p \star \delta_s, g\rangle\, ds \\ &= \int_{-\infty}^{+\infty} f(s)\langle \pi(\delta_s), \tilde{g}\rangle\, ds = \langle \pi(p), \tilde{g}\rangle \hat{f}(0) = 0.\end{aligned}$$

Thus $g \in I_\omega(\{0\})^\perp$. It follows that $I_\omega(\{0\}) = J_\omega(\{0\})$.

Conversely, suppose that $\{0\}$ satisfies ω-synthesis and let T be such that $\sigma(A) = \{0\}$ and $\|T(t)\| = O(\omega(t))$ $(|t| \to +\infty)$.

Consider the function $\varphi : L_\omega(\mathbb{R}) \to \mathscr{L}(X)$, $f \mapsto f(T)$. By Theorem 4.2, $h(\ker\varphi) = \{0\}$. Since $\{0\}$ satisfies ω-synthesis, $\ker\varphi = I_\omega(\{0\})$. For each integer $n \ge 1$ let e_n be the function defined in the proof of Theorem 4.3 for $k = 0$. We have $e_n(T)x \to -Ax$ for all x in the domain of A. Since $\hat{e}_n(0) = 0$, $e_n(T) = 0$. Hence $A = 0$.

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5. The discrete case

We now give a result, analogous to Theorem 4.3 as announced in the introduction.

THEOREM 5.1. *Let $k \geq 0$ be an integer and let R be a bounded operator on a Banach space with spectrum equal to $\{\lambda\}$ and such that*

(i) $\|R^n\| = o(n^{k+1})$ $(n \to +\infty)$;

(ii) $\log^+ \|R^{-n}\| = o(n^{1/2})$ $(n \to +\infty)$.

Then $(R-\lambda)^{k+1} = 0$.

PROOF. We have $|\lambda| = 1$ and we can assume without loss of generality that $\lambda = 1$. Let

$$A = \log R = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(R-I)^n.$$

(This series is convergent since $\lim_{n\to+\infty} \|(R-I)^n\|^{1/n} = 0$.) Set $T(t) = e^{tA}$ $(t \in \mathbb{R})$. It is well-known and easy to check that $\sigma(A) = \{0\}$ and $T(n) = R^n$ $(n \in \mathbb{Z})$.

For all $t \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $|t-n| < 1$ and $|n| \leq |t|$. We have

$$\|T(t)\| \leq \|T(t-n)\|\|T(n)\| \leq \sup_{-1\leq s\leq 1} \|T(s)\|\|T(n)\| \leq e^{\|A\|}\|T(n)\|.$$

Since $|n| \leq |t|$, we see that the group T satisfies conditions (i) and (ii) of Theorem 4.3. Hence $A^{k+1} = 0$. We have

$$A^{k+1} = (R-I)^{k+1}\left(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(R-I)^{n-1}\right)^{k+1}.$$

Since the operator $\sum_{n\geq 1}((-1)^{n-1}/n)(R-I)^{n-1}$ is invertible, we obtain $(R-I)^{k+1} = 0$.

REMARK. Let ω be a weight on $\mathbb{Z}$ and let

$$A_\omega(\Gamma) = \left\{ f \in \mathscr{C}(\Gamma) : \sum_{n\in\mathbb{Z}} |\hat{f}(n)|\omega(n) < +\infty \right\},$$

where Γ is the unit circle. As in Definition 3.1 we can define the ω-spectral synthesis property for a closed subset of Γ [14]. By the same methods we can prove results analogous to Theorems 3.2 and 4.4 in the discrete case. The result analogous to Theorem 4.2 is given in [14].

Finally we give examples to show that conditions (i) and (ii) in Theorem 4.3 and Theorem 5.1 are essentially the best possible.

EXAMPLE 1. Let $k \geq 0$ be an integer and let $A : \mathbb{C}^{k+2} \to \mathbb{C}^{k+2}$ be the operator defined by the matrix

$$\begin{pmatrix} 0 & 1 & & & \\ & \cdot & \cdot & 0 & \\ & & \cdot & \cdot & \\ & 0 & & \cdot & 1 \\ & & & & 0 \end{pmatrix}$$

with respect to the canonical basis of $\mathbb{C}^{k+2}$. Define $T(t) = e^{tA}$ $(t \in \mathbb{R})$. A direct calculation gives $\|T(t)\| = O(|t|^{k+1})$ $(|t| \to +\infty)$. But is is clear that $A^{k+1} \neq 0$.

EXAMPLE 2. Let H^2 be the Hardy space, let $\theta(z) = e^{(z+1)/(z-1)}$, $(|z| < 1)$ and let $\mathscr{H} = H^2 \ominus \theta H^2$. Denote by $P_{\mathscr{H}}$ the orthogonal projection on $\mathscr{H}$ and set $R(f) = P_{\mathscr{H}}(zf)$ $(f \in \mathscr{H})$. Then R is a contraction with spectrum $\{1\}$ such that $\log \|R^{-n}\| = O(n^{1/2})$ $(n \to +\infty)$ [11, Remark 2.c]. But $(R - I)^k \neq 0$ for all integers $k \geq 0$.

If we set

$$A = \sum_{n \geq 1} \frac{(-1)^n}{n}(R - I)^n,$$

then $\sigma(A) = \{0\}$ and A is the infinitesimal generator of a norm-continuous group T which satisfies $\|T(t)\| = O(1)$ $(t \to +\infty)$, $\log^+ \|T(-t)\| = O(t^{1/2})$ $(t \to +\infty)$ and $A^k \neq 0$ for all integers $k \geq 0$ (see the proof of Theorem 5.1).

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References

[1] G. R. Allan and T. J. Ransford, 'Power-dominated elements in a Banach algebra', *Studia Math.* **94** (1989), 63–79.

[2] A. Atzmon, 'Operators which are annihilated by analytic functions and invariant subspaces', *Acta Math.* **144** (1980), 27–63.

[3] A. Beurling, 'Sur les integrales de Fourier absolument convergentes et leur application à une transformation fonctionnelle', in: *Neuvième Cong. Math. Scandinaves, Helsinki 1938* (Tryekeri, Helsinki, 1939) pp. 199–210.

[4] R. P. Boas, *Entire functions* (Academic Press, New York, 1954).

[5] H. G. Dales and W. K. Hayman, 'Esterle's proof of the Tauberian theorem for Beurling algebras', *Ann. Inst. Fourier (Grenoble)* **31** (1981), 141–150.

[6] J. Esterle, 'Distributions on Kronecker sets, strong form of uniqueness and closed ideals of A^+', *J. Reine Angew. Math.* **450** (1994), 43–82.
[7] J. Esterle, E. Strouse and F. Zouakia, 'Stabilité asymptotique de certains semi-groupes d'opérateurs', *J. Operator Theory* **28** (1992), 203–227.
[8] V. P. Gurarii, 'Harmonic analysis in spaces with weights', *Trans. Moscow Math. Soc.* **35** (1979), 21–75.
[9] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. 31 (Amer. Math. Soc., Providence, 1957).
[10] Y. Katznelson, *An introduction to harmonic analysis* (Wiley, New York, 1968).
[11] A. Pazy, *Semigroups of linear operators and applications to partial differential equations* (Springer, New York, 1983).
[12] H. H. Schaefer, M. Wolf and W. Arendt, 'On lattice isomorphisms with positive real spectrum and group operators', *Math. Z.* **164** (1978), 115–123.
[13] M. Zarrabi, 'Ensembles de synthèse pour certaines algèbres de Beurling', *Rev. Roumaine Math. Pures Appl.* **35** (1990), 385–396.
[14] ———, 'Contractions à spectre dénombrable et propriétés d'unicité des fermés dénombrables du cercle', *Ann. Inst. Fourier (Grenoble)* **43** (1993), 251–263.

U.F.R de Mathématiques et Informatique
Université Bordeaux I
351, cours de la Liberation
33405 Talence
France
e-mail: zarrabi@math.u-bordeaux.fr