PSEUDO-RINGS OF INFINITE MATRICES

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1. Introduction

Patterson (4) introduced the concept of a pseudo-ring and considered the pseudo-ring of infinite matrices over a ring. In this paper we shall generalize and improve the work of Patterson, using certain additions to the general theory of pseudo-rings which have recently been introduced (1). We shall follow the conventions and notations used in (1) and (4).

We shall consider a more general type of pseudo-ring of infinite matrices over a pseudo-ring; we define such a pseudo-ring as follows.

Let $\mathfrak{S}$ be an infinite set of cardinality $\mathfrak{c}$, let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring and let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers such that $\mathfrak{b} \geqq \mathfrak{a} \geqq \aleph_0$. For any subset $\mathfrak{S}'$ of $\mathfrak{S}$, we denote by $\kappa(\mathfrak{S}')$ the cardinality of $\mathfrak{S}'$. Let $M(A)$ be the set of infinite matrices of type $\mathfrak{S}$ over A; formally, $M(A)$ is the set of mappings of $\mathfrak{S} \times \mathfrak{S}$ into A. For each $\Gamma \in M(A)$ and each $s \in \mathfrak{S}$, define

$$\mathfrak{S}(\Gamma, s) = \{t \in \mathfrak{S}\colon\ (s, t)\Gamma \neq 0\};$$

let $\mathfrak{S}(\Gamma) = \bigcup_{s \in \mathfrak{S}} \mathfrak{S}(\Gamma, s)$. Let $M^*(A^*)$ be the set of row-finite matrices of type $\mathfrak{S}$ over A^*; formally,

$$M^*(A^*) = \{\Gamma^* \in M(A^*)\colon\ \kappa(\mathfrak{S}(\Gamma^*, s)) < \aleph_0 \text{ for all } s \in \mathfrak{S}\}.$$

Define $M(A, \mathfrak{b}) = \{\Gamma \in M(A)\colon\ \kappa(\mathfrak{S}(\Gamma)) < \mathfrak{b}\}$ and similarly define

$$M^*(A^*, \mathfrak{a}) = \{\Gamma^* \in M^*(A^*)\colon\ \kappa(\mathfrak{S}(\Gamma^*)) < \mathfrak{a}\}.$$

We note that if $\mathfrak{a} > \mathfrak{c}$, then $M^*(A^*, \mathfrak{a}) = M^*(A^*)$ and $M(A, \mathfrak{b}) = M(A)$; however, if $\mathfrak{a} = \aleph_0$, then $M^*(A^*, \mathfrak{a})$ is the set of row-bounded infinite matrices of type $\mathfrak{S}$ over A^*.

Under pointwise addition, $M(A, \mathfrak{b})$ is a group and $M^*(A^*, \mathfrak{a})$ is a subgroup of $M(A, \mathfrak{b})$. For each $\Gamma^* \in M^*(A^*, \mathfrak{a})$ and each $\Gamma \in M(A, \mathfrak{b})$ we define $\Gamma^*\Gamma \in M(A, \mathfrak{b})$ by

$$(s, t)(\Gamma^*\Gamma) = \sum_{u \in \mathfrak{S}} ((s, u)\Gamma^*(u, t)\Gamma) \quad \text{for all } (s, t) \in \mathfrak{S} \times \mathfrak{S}.$$

Under this multiplication, $(M^*(A^*, \mathfrak{a}), M(A, \mathfrak{b}))$ is a pseudo-ring, which we denote by $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$.

We note that, if $\mathfrak{a} > \mathfrak{c}$ and $A^* = A$, then $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ is just the pseudo-ring $\mathfrak{M}(A^*)$ as studied by Patterson (4); it was shown that, if J^* is the Jacobson radical of the ring A^*, then the Jacobson radical of $\mathfrak{M}(A^*)$ is contained in $\mathfrak{M}(J^*)$.

We shall extend this result of Patterson to the more general pseudo-rings of the form $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$; indeed we shall improve slightly the result of Patterson. We shall show that the Jacobson radical of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ is contained in a normal right ideal $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$. If $\mathfrak{R} = (R^*, R)$ is the Jacobson radical of $\mathfrak{A}$, $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ is of the form $(M^*(R^*, \mathfrak{a}), G)$ where $G \subseteq M(R, \mathfrak{b})$. We shall show by an example that the latter containment may be strict; this example also shows that the Jacobson radical of $\mathfrak{M}(A^*)$ may be strictly contained in $\mathfrak{M}(J^*)$.

Finally, we shall discuss the existence of analogues for pseudo-rings of certain results of Patterson (**2, 3**). We shall show that the Jacobson radical of $\mathfrak{M}(\mathfrak{A}, \aleph_0, \mathfrak{b})$ is exactly $\mathfrak{G}(\mathfrak{A}, \aleph_0, \mathfrak{b})$; this is, of course, the analogue of (**2**), Theorem 2. However, we shall show that there exist rings A^* with non-right-vanishing Jacobson radical J^*, such that the Jacobson radical of $\mathfrak{M}(A^*)$ is $\mathfrak{M}(J^*)$; thus Theorem 5 of (**2**) has no strict analogue.

2. Preliminary Results

In this section we prove some results concerning the general theory of pseudo-rings; these results will be used in the proofs of our main theorems. The first of these is a result of ring theory, stated explicitly as a lemma.

Lemma 2.1. *Let A^* be a ring, B^* an ideal of A^*, and M^* a right ideal of B^*, modular with respect to $e^* \in B^*$. Let $N^* = \{a^* \in A^* : a^*B^* \subseteq M^*\}$; then N^* is a right ideal of A^*, modular with respect to e^*, and $M^* \subseteq N^* \cap B^*$. If, in addition, M^* is maximal in B^*, then $M^* = N^* \cap B^*$ and N^* is maximal in A^*.*

Proof. Clearly N^* is an additive subgroup of A^*; also,

$$(N^*A^*)B^* \subseteq N^*B^* \subseteq M^*$$

so that $N^*A^* \subseteq N^*$. Thus N^* is a right ideal of A^*. Now,

$$((1-e^*)A^*)B^* \subseteq (1-e^*)B^* \subseteq M^*;$$

hence $(1-e^*)A^* \subseteq N^*$. Thus, N^* is modular in A^* with respect to e^*. Also, since $M^*B^* \subseteq M^*$, $M^* \subseteq N^* \cap B^*$.

We now suppose that M^* is maximal in B^*. Then, since $N^* \cap B^*$ is a right ideal of B^*, either $M^* = N^* \cap B^*$ or $B^* = N^* \cap B^*$. If $B^* = N^* \cap B^*$, $e^* \in N^*$ so that $e^*B^* \subseteq M^*$; since $(1-e^*)B^* \subseteq M^*$ it follows that $M^* = B^*$, which contradicts the maximality of M^*. Therefore $M^* = N^* \cap B^*$. Finally we show that N^* is maximal in A^*. Clearly, since $M^* = N^* \cap B^*$, $N^* \neq A^*$. Suppose that K^* is a right ideal of A^* such that $N^* \subseteq K^*$; then $K^* \cap B^*$ is a right ideal of B^* such that $K^* \cap B^* \supseteq M^*$. Thus $K^* \cap B^* = M^*$ or $K^* \cap B^* = B^*$. If $K^* \cap B^* = M^*$, $K^*B^* \subseteq K^* \cap B^* = M^*$ so that $K^* = N^*$. If $K^* \cap B^* = B^*$, $e^* \in K^*$; but $(1-e^*)A^* \subseteq N^* \subseteq K^*$, so that $K^* = A^*$. Thus $K^* = N^*$ or $K^* = A^*$. The proof is now complete.

Lemma 2.2. *Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring, $\mathfrak{N} = (N^*, N)$ a maximal modular normal right ideal of $\mathfrak{A}$, and B^* a right ideal of the ring A^* such that $N^* \not\supseteq B^*$. Then for some $e^* \in B^*$, $\mathfrak{N}$ is modular with respect to e^*.*

Proof. Suppose that $\mathfrak{N}$ is modular with respect to $f^* \in A^*$. N^* is a maximal right ideal of A^* and N^*+B^* is a right ideal of A^* such that $N^* \subset N^*+B^*$. Thus $N^*+B^* = A^*$, so that there exist $e^* \in B^*$ and $n^* \in N^*$ satisfying $n^*+e^* = f^*$. It follows that $(1-e^*)A \subseteq (1-f^*)A+n^*A \subseteq N$, as required.

Lemma 2.3. *Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring, $\mathfrak{B} = (B^*, B)$ a normal ideal of $\mathfrak{A}$, and $\mathfrak{N} = (N^*, N)$ a maximal modular normal right ideal of $\mathfrak{A}$ such that $N^* \not\supseteq B^*$. Then $\mathfrak{M} = \mathfrak{N} \cap \mathfrak{B}$ is a maximal modular normal right ideal of $\mathfrak{B}$.*

Proof. By Lemma 2.2, $\mathfrak{N}$ is modular with respect to some $e^* \in B^*$. Then clearly $\mathfrak{M}$ is a normal right ideal of $\mathfrak{B}$, modular with respect to $e^* \in B^*$. Finally we show that $\mathfrak{M}$ is maximal in $\mathfrak{B}$; by (1), Theorem 2.5, it is sufficient to show that M^* is maximal in B^*. Since $N^* \not\supseteq B^*$, $M^* \neq B^*$. Suppose that K^* is a right ideal of B^* such that $K^* \supseteq M^*$. Then $K^* \supseteq (1-e^*)B^*$. Consider $L^* = \{a^* \in A^*: a^*B^* \subseteq K^*\}$; then, by Lemma 2.1, L^* is a right ideal of A^* and $K^* \subseteq L^* \cap B^*$. Now, since $N^*B^* \subseteq N^* \cap B^* = M^* \subseteq K^*$, $N^* \subseteq L^*$ so that $L^* = N^*$ or $L^* = A^*$. If $L^* = N^*$, $M^* = N^* \cap B^* = L^* \cap B^* \supseteq K^*$ and hence $K^* = M^*$. If $L^* = A^*$, $e^* \in L^*$ so that $e^*B^* \subseteq K^*$; since $K^* \supseteq (1-e^*)B^*$, $K^* = B^*$. Thus $K^* = M^*$ or $K^* = B^*$. The proof is now complete.

Lemma 2.4. *Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring, $\mathfrak{B} = (B^*, B)$ a normal ideal of $\mathfrak{A}$ such that $A = A^*+B$, and $\mathfrak{M} = (M^*, M)$ a maximal quasi-accessible normal right ideal of $\mathfrak{B}$. Let $N^* = \{a^* \in A^*;\ a^*B^* \subseteq M^*\}$ and $N = N^*+M$. Then $\mathfrak{N} = (N^*, N)$ is a maximal quasi-accessible normal right ideal of $\mathfrak{A}$ such that $\mathfrak{M} = \mathfrak{N} \cap \mathfrak{B}$.*

Proof. By (1), Theorem 2.3(i), $\mathfrak{M}$ is modular in $\mathfrak{B}$ with respect to some $e^* \in B^*$. Then Lemma 2.1 shows that N^* is a maximal right ideal of A^*, modular with respect to e^*, and $M^* = N^* \cap B^*$. Now $(1-e^*)N^* \subseteq N^*$ so that $e^*N^* \subseteq N^* \cap B^* = M^*$; thus $e^*N^*B \subseteq M^*B \subseteq M$. Since

$$(1-e^*)N^*B \subseteq (1-e^*)B \subseteq M,$$

it follows that $N^*B \subseteq M$. Then $N^*A \subseteq N^*A^*+N^*B \subseteq N^*+M = N$, so that $\mathfrak{N}$ is a right ideal of $\mathfrak{A}$.

Since $N^* \subseteq A^*$, $N \cap A^* = N^*+(M \cap A^*) = N^*+(M \cap B^*) = N^*+M^* = N^*$. Since $M \subseteq B$, $N \cap B = (N^* \cap B)+M = (N^* \cap B^*)+M = M^*+M = M$. Thus $\mathfrak{N}$ is normal in $\mathfrak{A}$, and $\mathfrak{N} \cap \mathfrak{B} = \mathfrak{M}$. Further,

$$(1-e^*)A = (1-e^*)A^*+(1-e^*)B \subseteq N^*+M = N,$$

so that $\mathfrak{N}$ is modular in $\mathfrak{A}$; thus, using Theorem 2.5 of (1), $\mathfrak{N}$ is maximal in $\mathfrak{A}$.

Finally, $\mathfrak{N}$ is quasi-accessible. Clearly $N \supseteq N^*+N^*A+(1-e^*)A$. Now

$$N = N^*+M = N^*+M^*+M^*B+(1-e^*)B \subseteq N^*+N^*A+(1-e^*)A.$$

Therefore $N = N^*+N^*A+(1-e^*)A$ so that $\mathfrak{N}$ is quasi-accessible. Thus $\mathfrak{N}$ is a maximal quasi-accessible normal right ideal of $\mathfrak{A}$ such that $\mathfrak{M} = \mathfrak{N} \cap \mathfrak{B}$, as required.

3. The Main Theorems

We are now ready to prove our main results concerning the Jacobson radical of a pseudo-ring of infinite matrices as defined in §1. We shall adopt the following notation.

Let $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ be a pseudo-ring of infinite matrices. Let $\mathfrak{A}_1$ be the extension of $\mathfrak{A}$ as in (1), Lemma 2.8; we recall that, as additive groups, $A_1^* = A^* \oplus Z^*$ and $A_1 = A \oplus Z^*$, where Z^* is the group of integers. Let

$$M'(A_1, \mathfrak{a}, \mathfrak{b}) = M(A, \mathfrak{b}) + M^*(A_1^*, \mathfrak{a});$$

then, under matrix multiplication, $(M^*(A_1^*, \mathfrak{a}), M'(A_1, \mathfrak{a}, \mathfrak{b}))$ is a pseudo-ring, which we denote by $\mathfrak{M}'(\mathfrak{A}_1, \mathfrak{a}, \mathfrak{b})$. We note that $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ is a normal ideal of $\mathfrak{M}'(A_1, \mathfrak{a}, \mathfrak{b})$ which satisfies the condition of Lemma 2.4.

Let $a_1 \in A_1$ and let $(s, t) \in \mathfrak{S} \times \mathfrak{S}$; then we denote by $[a_1, s, t]$ the element of $M'(A_1, \mathfrak{a}, \mathfrak{b})$ such that $(s, t)[a_1, s, t] = a_1$ and $(u, v)[a_1, s, t] = 0$ for all $(u, v) \neq (s, t)$. We note that if $a \in A$, $[a, s, t] \in M(A, \mathfrak{b})$; if $a_1^* \in A_1^*$,

$$[a_1^*, s, t] \in M^*(A_1^*, \mathfrak{a}); \text{ and if } a^* \in A^*, [a^*, s, t] \in M^*(A^*, \mathfrak{a}).$$

Let $\mathfrak{B} = (B^*, B)$ be any maximal quasi-accessible normal right ideal of $\mathfrak{A}$; then $\mathfrak{B}$ is modular with respect to some $e^* \in A^*$. Let $s \in \mathfrak{S}$. Then we define $H^*(B^*, \mathfrak{a}, s) = \{\Gamma^* \in M^*(A^*, \mathfrak{a})\colon (s, t)\Gamma^* \in B^* \text{ for all } t \in \mathfrak{S}\}$, and

$$H(B, \mathfrak{b}, s) = H^*(B^*, \mathfrak{a}, s) + H^*(B^*, \mathfrak{a}, s)M(A, \mathfrak{b}) + (1-[e^*, s, s])M(A, \mathfrak{b}).$$

We note that $H(B, \mathfrak{b}, s)$ is independent of the choice of e^*; for, if f^* is another such element of A^*, then Lemma 2.2 of (1) shows that $e^*-f^* \in B^*$ and hence $[e^*, s, s]-[f^*, s, s] \in H^*(B^*, \mathfrak{a}, s)$. Then $(H^*(B^*, \mathfrak{a}, s), H(B, \mathfrak{b}, s))$ is a pseudo-ring, which we denote by $\mathfrak{H}(\mathfrak{B}, \mathfrak{a}, \mathfrak{b}, s)$.

Theorem 3.1. *Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring and $\mathfrak{B} = (B^*, B)$ a maximal quasi-accessible normal right ideal of $\mathfrak{A}$; let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinals such that $\mathfrak{b} \geqq \mathfrak{a} \geqq \aleph_0$ and let $s \in \mathfrak{S}$. Then $\mathfrak{H}(\mathfrak{B}, \mathfrak{a}, \mathfrak{b}, s)$ is a maximal quasi-accessible normal right ideal of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$.*

Proof. Clearly $\mathfrak{H}(\mathfrak{B}, \mathfrak{a}, \mathfrak{b}, s)$ is a right ideal of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$. Suppose $\Gamma^* \in H(B, \mathfrak{b}, s) \cap M^*(A^*, \mathfrak{a})$; then, for all $t \in \mathfrak{S}$,

$$(s, t)\Gamma^* \in A^* \cap (B^*+B^*A+(1-e^*)A) = A^* \cap B = B^*,$$

so that $\Gamma^* \in H^*(B^*, \mathfrak{a}, s)$. Thus $\mathfrak{H}(\mathfrak{B}, \mathfrak{a}, \mathfrak{b}, s)$ is a quasi-accessible normal right ideal of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$. It remains to show that $\mathfrak{H}(\mathfrak{B}, \mathfrak{a}, \mathfrak{b}\, s)$ is maximal in $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$; by Theorem 2.5 of (1), it is sufficient to show that $H^*(B^*, \mathfrak{a}, s)$ is maximal in $M^*(A^*, \mathfrak{a})$. Suppose that N^* is a right ideal of $M^*(A^*, \mathfrak{a})$ such that $H^*(B^*, \mathfrak{a}, s) \subset N^*$. For all $t \in \mathfrak{S}$, define $C^*(s, t) = \{c^* \in A^*: [c^*, s, t] \in N^*\}$. For any $a^* \in A^*$, and any $c^* \in C^*(s, t)$, $[c^*a^*, s, t] = [c^*, s, t][a^*, t, t] \in N^*$, so that $C^*(s, t)$ is a right ideal of A^*; clearly $B^* \subseteq C^*(s, t)$ for all $t \in \mathfrak{S}$. Now, there exists $\hat{\Gamma}^* \in N^*$ such that $\hat{\Gamma}^* \notin H^*(B^*, \mathfrak{a}, s)$; then, for some $t_1 \in \mathfrak{S}$, $(s, t_1)\hat{\Gamma}^* = d^* \notin B^*$. Since $(1-[e^*, s, s])\hat{\Gamma}^* \in H^*(B^*, \mathfrak{a}, s) \subset N^*$, $[e^*, s, s]\hat{\Gamma}^* \in N^*$; also, since $(1-e^*)d^* \in B^*$, $e^*d^* \notin B^*$. Now, $B^* = \{a^* \in A^*: a^*A^* \subseteq B^*\}$ by Lemma 2.1; thus there exists $a^* \in A^*$ such that $e^*d^*a^* \notin B^*$. Then

$$[e^*d^*a^*, s, t_1] = [e^*, s, s]\hat{\Gamma}^*[a^*, t_1, t_1] \in N^*M^*(A^*, \mathfrak{a}) \subseteq N^*.$$

Thus $e^*d^*a^* \in C^*(s, t_1)$ but $e^*d^*a^* \notin B^*$; since B^* is maximal, $C^*(s, t_1) = A^*$.

Now
$$[e^*, s, s] = [(1-e^*)e^*, s, s] + [e^{*2}, s, s]$$
$$= (1-[e^*, s, s])[e^*, s, s] + [e^*, s, t_1][e^*, t_1, s].$$

Since $(1-[e^*, s, s])[e^*, s, s] \in H^*(B^*, \mathfrak{a}, s) \subset N^*$ and $[e^*, s, t_1] \in N^*$, it follows that $[e^*, s, s] \in N^*$. Then, for all $\Gamma^* \in M^*(A^*, \mathfrak{a})$,

$$\Gamma^* = (1-[e^*, s, s])\Gamma^* + [e^*, s, s]\Gamma^*$$

so that $N^* = M^*(A^*, \mathfrak{a})$. Therefore $H^*(B^*, \mathfrak{a}, s)$ is maximal in $M^*(A^*, \mathfrak{a})$.

Theorem 3.2. *Let* $\mathfrak{A} = (A^*, A)$ *be a pseudo-ring and let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be cardinals such that* $\mathfrak{b} \geqq \mathfrak{a} \geqq \aleph_0$; *let* $\mathfrak{K} = (K^*, K)$ *be a maximal quasi-accessible normal right ideal of* $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ *such that* $K^* \not\supseteq M^*(A^*, \aleph_0)$. *Then*

$$\mathfrak{K} \supseteq \bigcap_{s \in \mathfrak{S}_0} \mathfrak{H}(\mathfrak{B}_s, \mathfrak{a}, \mathfrak{b}, s),$$

where $\mathfrak{S}_0$ *is a finite non-empty subset of* $\mathfrak{S}$, *and where, for all* $s \in \mathfrak{S}_0$, $\mathfrak{B}_s$ *is a maximal quasi-accessible normal right ideal of* $\mathfrak{A}$.

Proof. Let $L^* = \{\Gamma^* \in M^*(A_1^*, \mathfrak{a}): \Gamma^*M^*(A^*, \mathfrak{a}) \subseteq K^*\}$ and let $L = L^* + K$. Then, by Lemma 2.4, $\mathfrak{L} = (L^*, L)$ is a maximal quasi-accessible normal right ideal of $\mathfrak{M}'(\mathfrak{A}_1, \mathfrak{a}, \mathfrak{b})$ such that $\mathfrak{K} = \mathfrak{L} \cap \mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$. Let

$$C(s, t) = \{c \in A_1: [c, s, t] \in L\} \text{ and let } C^*(s, t) = A_1^* \cap C(s, t).$$

Let $(u, v) \in \mathfrak{S} \times \mathfrak{S}$ and let 1^* be the multiplicative identity in A_1^*; then, under matrix multiplication, $[1^*, u, v]$ is a right multiplier in $\mathfrak{M}'(\mathfrak{A}_1, \mathfrak{a}, \mathfrak{b})$ in the sense of (**1**), §4. Thus, for all $c \in C(s, t)$ and all $v \in \mathfrak{S}$, we have

$$[c, s, v] = [c, s, t][1^*, t, v] \in L[1^*, t, v] \subseteq L$$

by (**1**), Theorem 4.2. It follows that $C(s, t)$ and $C^*(s, t)$ are independent of t. Also, for all $c^* \in C^*(s, s)$ and all $a_1 \in A_1$, $[c^*, s, s] \in L \cap M^*(A_1^*, \mathfrak{a}) = L^*$ so that

$$[c^*a_1, s, s] = [c^*, s, s][a_1, s, s] \in L^*M'(A_1, \mathfrak{a}, \mathfrak{b}) \subseteq L;$$

therefore $c^*a_1 \in C(s, s.)$ It follows that, for all $s \in \mathfrak{S}$, $(C^*(s, t), C(s, t))$ is a normal right ideal of $\mathfrak{A}_1$, modular with respect to 1^*, and independent of t.

Now $M^*(A^*, \aleph_0)$ is a right ideal of $M^*(A_1^*, \mathfrak{a})$; since $K^* \not\supseteq M^*(A^*, \aleph_0)$, clearly $L^* \not\supseteq M^*(A^*, \aleph_0)$. Thus, by Lemma 2.2, there exists $\Gamma_0^* \in M^*(A^*, \aleph_0)$

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such that $\mathfrak{L}$ is modular with respect to Γ_0^*. Then $\kappa(\mathfrak{S}(\Gamma_0^*))<\aleph_0$. Let Γ be any element of $M'(A_1, \mathfrak{a}, \mathfrak{b})$ such that $(u, v)\Gamma = 0$ for all $u \in \mathfrak{S}(\Gamma_0^*)$; then

$$\Gamma = (1-\Gamma_0^*)\Gamma \in L.$$

Let $\mathfrak{S}_0 = \{u \in \mathfrak{S}: C^*(u, u) \neq A_1^*\}$; clearly $\mathfrak{S}_0 \subseteq \mathfrak{S}(\Gamma_0^*)$ so that $\kappa(\mathfrak{S}_0)<\aleph_0$. Also $\mathfrak{S}_0 \neq \varnothing$. Suppose, to the contrary, $\mathfrak{S}_0 = \varnothing$; then $C^*(u, u) = A_1^*$ for all $u \in \mathfrak{S}$. Now any element Γ^* of $M^*(A_1^*, \mathfrak{a})$ may be written $\Gamma^* = \Gamma_1^* + \Gamma_2^*$, where $(s, t)\Gamma_1^* = 0$ if $s \in \mathfrak{S}(\Gamma_0^*)$ and $(s, t)\Gamma_2^* = 0$ if $s \notin \mathfrak{S}(\Gamma_0^*)$. As above, $\Gamma_1^* \in L$, while $\Gamma_2^* = \sum_{s \in \mathfrak{S}(\Gamma_0^*)} \sum_{t \in \mathfrak{S}(\Gamma^*, s)} [(s, t)\Gamma^*, s, t] \in L$. Then

$$M^*(A_1^*, \mathfrak{a}) = L \cap M^*(A_1^*, \mathfrak{a}) = L^*!$$

Next, if $s \in \mathfrak{S}_0$, $C^*(s, s)$ is maximal in A_1^*. For, suppose there exists a right ideal N^* of A_1^* such that $C^*(s, s) \subset N^*$. There exists $n^* \in N^*$ such that $n^* \notin C^*(s, s)$; then $[n^*, s, s] \notin L^*$. Now, L^* is maximal in $M^*(A_1^*, \mathfrak{a})$; thus $M^*(A_1^*, \mathfrak{a}) = L^* + [n^*, s, s]M^*(A_1^*, \mathfrak{a})$. In particular,

$$[1^*, s, s] = \hat{\Gamma}_1^* + [n^*, s, s]\hat{\Gamma}_2^*$$

where $\hat{\Gamma}_1^* \in L^*$ and $\hat{\Gamma}_2^* \in M^*(A_1^*, \mathfrak{a})$. Now, since $\hat{\Gamma}_1^* = [1^*, s, s] - [n^*, s, s]\hat{\Gamma}_2^*$, clearly $(u, s)\hat{\Gamma}_1^* = 0$ for all $u \neq s$. Now, $\hat{\Gamma}_1^*[1^*, s, s] \in L^*$ so that

$$(s, s)\hat{\Gamma}_1^* \in C^*(s, s).$$

It follows that $1^* = (s, s)\hat{\Gamma}_1^* + n^*(s, s)\hat{\Gamma}_2^* \in C^*(s, s) + N^*A_1^* \subseteq N^*$. Since $1^* \in N^*$, $N^* = A_1^*$; therefore $C^*(s, s)$ is maximal in A_1^*.

Then, by (1), Theorem 2.5, $(C^*(s, s), C(s, s))$ is a maximal modular normal right ideal of $\mathfrak{A}_1$ for every $s \in \mathfrak{S}_0$. Also, if $s \in \mathfrak{S}_0$, $A^* \nsubseteq C^*(s, s)$. Suppose, to the contrary, $A^* \subseteq C^*(s, s)$; then it follows that

$$[1^*, s, s]M^*(A^*, \mathfrak{a}) \subseteq L^* \cap M^*(A^*, \mathfrak{a}) = K^*.$$

Then $[1^*, s, s] \in L^*$ since $L^* = \{\Gamma^* \in M^*(A_1^*, \mathfrak{a}): \Gamma^* M^*(A^*, \mathfrak{a}) \subseteq K^*\}$; thus $1^* \in C^*(s, s)$ so that $C^*(s, s) = A_1^*$. Since $s \in \mathfrak{S}_0$, this is a contradiction.

By Lemma 2.2, there exists $e_s^* \in A^*$ such that $(C^*(s, s), C(s, s))$ is modular with respect to e_s^*; by (1), Lemma 2.2, $1^* - e_s^* \in C^*(s, s)$. Let $B_s^* = A^* \cap C^*(s, s)$ and let $B_s = B_s^* + B_s^* A + (1-e_s^*)A \subseteq A \cap C(s, s)$. By Lemma 2.3, $(B_s^*, A \cap C(s, s))$ is a maximal modular normal right ideal of $\mathfrak{A}$. Then Theorem 2.6 of (1) shows that $\mathfrak{B}_s = (B_s^*, B_s)$ is a maximal quasi-accessible normal right ideal of $\mathfrak{A}$.

We are now ready to show that $\mathfrak{K} \supseteq \bigcap_{s \in \mathfrak{S}_0} \mathfrak{H}(\mathfrak{B}_s, \mathfrak{a}, \mathfrak{b}, s)$. Let

$$\Gamma \in \bigcap_{s \in \mathfrak{S}_0} H(B_s, \mathfrak{b}, s).$$

We decompose Γ as follows. Define $\Gamma_1 \in M(A, \mathfrak{b})$ by $(u, v)\Gamma_1 = (u, v)\Gamma$ if $u \notin \mathfrak{S}(\Gamma_0^*)$; $(u, v)\Gamma_1 = 0$ if $u \in \mathfrak{S}(\Gamma_0^*)$. Then

$$\Gamma_1 = (1-\Gamma_0^*)\Gamma_1 \in L \cap M(A, \mathfrak{b}) = K.$$

Define $\Gamma_2 \in M(A, \mathfrak{b})$ by $(u, v)\Gamma_2 = (u, v)\Gamma$ if $u \in \mathfrak{S}(\Gamma_0^*)$ and $u \notin \mathfrak{S}_0$, and $(u, v)\Gamma_2 = 0$ if $u \notin \mathfrak{S}(\Gamma_0^*)$ or if $u \in \mathfrak{S}_0$.

Now, if $u \in \mathfrak{S}(\Gamma_0^*)$ and $u \notin \mathfrak{S}_0$, $C^*(u, u) = A_1^*$ so that $[1^*, u, u] \in L^*$. Since $\mathfrak{S}(\Gamma_0^*)$ is finite,

$$\hat{\Gamma}^* = \sum_{u \in \mathfrak{S}(\Gamma_0^*),\, u \notin \mathfrak{S}_0,} [1^*, u, u] \in L^*,$$

and thus $\Gamma_2 = \hat{\Gamma}^*\Gamma_2 \in L^*M(A, \mathfrak{b}) \subseteq L \cap M(A, \mathfrak{b}) = K$.

For all $s \in \mathfrak{S}_0$, define $\Gamma_s \in M(A, \mathfrak{b})$ by $(s, t)\Gamma_s = (s, t)\Gamma$ for all $t \in \mathfrak{S}$, and $(u, t)\Gamma_s = 0$ if $u \neq s$. Now $(s, t)(\Gamma - \Gamma_s) = 0$ for all $t \in \mathfrak{S}$ so that

$$\Gamma - \Gamma_s = (1 - [e_s^*, s, s])(\Gamma - \Gamma_s) \in H(B_s, \mathfrak{b}, s);$$

therefore

$$\Gamma_s \in H(B_s, \mathfrak{b}, s) = H^*(B_s^*, \mathfrak{a}, s) + H^*(B_s^*, \mathfrak{a}, s)M(A, \mathfrak{b}) + (1 - [e_s^*, s, s])M(A, \mathfrak{b}).$$

Thus

$$\Gamma_s = \Gamma_{(0,s)}^* + \sum_{r=1}^{k_s} \Gamma_{(r,s)}^* \Gamma_{(r,s)} + (1 - [e_s^*, s, s])\Gamma_{(0,s)},$$

where, for $r = 0, 1, 2, \ldots, k_s$, $\Gamma_{(r,s)}^* \in H^*(B_s^*, \mathfrak{a}, s)$ and $\Gamma_{(r,s)} \in M(A, \mathfrak{b})$.

For $r = 0, 1, 2, \ldots, k_s$, define $\hat{\Gamma}_{(r,s)}^* \in H^*(B_s^*, \mathfrak{a}, s)$ by $(s, t)\hat{\Gamma}_{(r,s)}^* = (s, t)\Gamma_{(r,s)}^*$ for all $t \in \mathfrak{S}$, and $(u, t)\hat{\Gamma}_{(r,s)}^* = 0$ if $u \neq s$. Similarly, define $\hat{\Gamma}_{(0,s)} \in M(A, \mathfrak{b})$ by $(s, t)\hat{\Gamma}_{(0,s)} = (s, t)\Gamma_{(0,s)}$ for all $t \in \mathfrak{S}$, and $(u, t)\hat{\Gamma}_{(0,s)} = 0$ if $u \neq s$. Then

$$\Gamma_s = \hat{\Gamma}_{(0,s)}^* + \sum_{r=1}^{k_s} \hat{\Gamma}_{(r,s)}^* \Gamma_{(r,s)} + (1 - [e_s^*, s, s])\hat{\Gamma}_{(0,s)}.$$

Now, for $r = 0, 1, 2, \ldots, k_s$, $(s, t)\hat{\Gamma}_{(r,s)}^* \in B_s^*$ for all $t \in \mathfrak{S}$; thus

$$\hat{\Gamma}_{(r,s)}^* = \sum_{t \in \mathfrak{S}(\Gamma_{(r,s)}^*)} [(s, t)\hat{\Gamma}_{(r,s)}^*, s, t] \in L \cap M^*(A^*, \mathfrak{a}) = K^*,$$

$$(1 - [e_s^*, s, s])\hat{\Gamma}_{(0,s)} = [1^* - e_s^*, s, s]\hat{\Gamma}_{(0,s)} \in L^*M(A, \mathfrak{b}) \subseteq K.$$

Therefore $\Gamma_s \in K^* + K^*M(A, \mathfrak{b}) + K = K$ for all $s \in \mathfrak{S}$.

Now, $\Gamma = \Gamma_1 + \Gamma_2 + \sum_{s \in \mathfrak{S}_0} \Gamma_s \in K$. Therefore $K \supseteq \bigcap_{s \in \mathfrak{S}_0} H(B_s, \mathfrak{b}, s)$ so that $\mathfrak{K} \supseteq \bigcap_{s \in \mathfrak{S}_0} \mathfrak{H}(\mathfrak{B}_s, \mathfrak{a}, \mathfrak{b}, s)$, as required.

Let $\mathfrak{E}$ be the set of maximal quasi-accessible normal right ideals of $\mathfrak{A}$. Define $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) = \bigcap_{\mathfrak{B} \in \mathfrak{E}} \bigcap_{s \in \mathfrak{S}} \mathfrak{H}(\mathfrak{B}, \mathfrak{a}, \mathfrak{b}, s)$.

Let $\mathfrak{I}$ be the set of maximal quasi-accessible normal right ideals $\mathfrak{K} = (K^*, K)$ of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ such that $K^* \supseteq M^*(A^*, \aleph_0)$. Define $\mathfrak{F}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) = \bigcap_{\mathfrak{K} \in \mathfrak{I}} \mathfrak{K}$.

Then $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ and $\mathfrak{F}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ are clearly normal right ideals of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$. The next theorem gives a characterisation of the Jacobson radical of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ in terms of these right ideals.

Theorem 3.3. *Let $\mathfrak{A}$ be a pseudo-ring and let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinals such that $\mathfrak{b} \geqq \mathfrak{a} \geqq \aleph_0$. Let $\mathfrak{J}$ be the Jacobson radical of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$. Then*

$$\mathfrak{J} = \mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) \cap \mathfrak{F}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}).$$

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Proof. By Theorem 3.1, $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ is an intersection of maximal quasi-accessible normal right ideals of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$; by definition, $\mathfrak{F}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$, also, is such an intersection. It follows that $\mathfrak{J} \subseteq \mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) \cap \mathfrak{F}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$.

Conversely, let $\mathfrak{K} = (K^*, K)$ be a maximal quasi-accessible normal right ideal of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$. Either $\mathfrak{K} \in \mathfrak{J}$ or $\mathfrak{K} \notin \mathfrak{J}$. If $\mathfrak{K} \in \mathfrak{J}$ then clearly

$$\mathfrak{K} \supseteq \mathfrak{F}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) \supseteq \mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) \cap \mathfrak{F}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}).$$

If $\mathfrak{K} \notin \mathfrak{J}$ then, by Theorem 3.2, $\mathfrak{K} \supseteq \bigcap_{s \in \mathfrak{S}_0} \mathfrak{H}(\mathfrak{B}_s, \mathfrak{a}, \mathfrak{b}, s)$, where $\mathfrak{S}_0$ is a finite non-empty subset of $\mathfrak{S}$ and where, for all $s \in \mathfrak{S}_0$, $\mathfrak{B}_s \in \mathfrak{E}$. Therefore, if $\mathfrak{K} \notin \mathfrak{J}$,

$$\mathfrak{K} \supseteq \mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) \supseteq \mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) \cap \mathfrak{F}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}).$$

Then by (1), Theorem 2.7, $\mathfrak{J} \supseteq \mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) \cap \mathfrak{F}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$. This completes the proof.

Corollary 3.4. *Let* $\mathfrak{A}$ *be a pseudo-ring and let* $\mathfrak{b}$ *be a cardinal such that* $\mathfrak{b} \geqq \aleph_0$. *Then the Jacobson radical of* $\mathfrak{M}(\mathfrak{A}, \aleph_0, \mathfrak{b})$ *is* $\mathfrak{G}(\mathfrak{A}, \aleph_0, \mathfrak{b})$.

Proof. If $\mathfrak{K}$ is a maximal quasi-accessible normal right ideal of $\mathfrak{M}(\mathfrak{A}, \aleph_0, \mathfrak{b})$, then $K^* \not\supseteq M^*(A^*, \aleph_0)$. Thus $\mathfrak{J} = \varnothing$, so that $\mathfrak{F}(\mathfrak{A}, \aleph_0, \mathfrak{b}) = \mathfrak{W}(\mathfrak{A}, \aleph_0, \mathfrak{b})$.

Thus, in general, the Jacobson radical of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ is contained in $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ and, in particular, the Jacobson radical of $\mathfrak{M}(\mathfrak{A}, \aleph_0, \mathfrak{b})$ is exactly $\mathfrak{G}(\mathfrak{A}, \aleph_0, \mathfrak{b})$. It is an open question whether the Jacobson radical of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ is $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ for cardinal numbers $\mathfrak{a} > \aleph_0$.

In our next theorem we obtain a more useful characterisation of $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$.

Let $\mathfrak{B} \in \mathfrak{E}$ and let e^* be any element of A^* such that $(1-e^*)A \subseteq B$. Define $\Gamma^*(\mathfrak{B}) \in M^*(A^*)$ by $(s, s)(\Gamma^*(\mathfrak{B})) = e^*$ for all $s \in \mathfrak{S}$ and $(s, t)(\Gamma^*(\mathfrak{B})) = 0$ if $s \neq t$. By (1), Lemma 2.2, $M^*(B^*, \mathfrak{b}) + M^*(B^*)M(A, \mathfrak{b}) + (1-\Gamma^*(\mathfrak{B}))M(A, \mathfrak{b})$ is independent of the choice of e^* used to define $\Gamma^*(\mathfrak{B})$.

Theorem 3.5. *Let* $\mathfrak{A} = (A^*, A)$ *be a pseudo-ring with Jacobson radical* $\mathfrak{R} = (R^*, R)$; *let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be cardinals such that* $\mathfrak{b} \geqq \mathfrak{a} \geqq \aleph_0$, *and let*

$$\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) = (G^*, G).$$

Then $G^* = M^*(R^*, \mathfrak{a})$ *and*

$$G = \bigcap_{\mathfrak{B} \in \mathfrak{E}} (M^*(B^*, \mathfrak{b}) + M^*(B^*)M(A, \mathfrak{b}) + (1-\Gamma^*(\mathfrak{B}))M(A, \mathfrak{b})).$$

Proof. Clearly, since $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) = \bigcap_{\mathfrak{B} \in \mathfrak{E}} \bigcap_{s \in \mathfrak{S}} \mathfrak{H}(\mathfrak{B}, \mathfrak{a}, \mathfrak{b}, s)$, it follows that

$$G^* = \bigcap_{\mathfrak{B} \in \mathfrak{E}} \bigcap_{s \in \mathfrak{S}} H^*(B^*, \mathfrak{a}, s) = \bigcap_{\mathfrak{B} \in \mathfrak{E}} M^*(B^*, \mathfrak{a}) = M^*(R^*, \mathfrak{a}).$$

Let $G' = \bigcap_{\mathfrak{B} \in \mathfrak{E}} (M^*(B^*, \mathfrak{b}) + M^*(B^*)M(A, \mathfrak{b}) + (1-\Gamma^*(\mathfrak{B}))M(A, \mathfrak{b}))$. Suppose $\Gamma \in G'$; let $\mathfrak{B} \in \mathfrak{E}$ and let $s \in \mathfrak{S}$. Then

$$\Gamma = \Gamma_0^* + \sum_{r=1}^{k} \Gamma_r^* \Gamma_r + (1-\Gamma^*(\mathfrak{B}))\Gamma_0$$

where $\Gamma_r^* \in M^*(B^*)$ for $r = 1, 2, \ldots, k$, $\Gamma_0^* \in M^*(B^*, \mathfrak{b})$ and $\Gamma_r \in M(A, \mathfrak{b})$ for $r = 0, 1, 2, \ldots, k$. For $r = 0, 1, 2, \ldots, k$, define $\Gamma_{(r,s)}^* \in H^*(B^*, \mathfrak{a}, s)$ by $(s, t)\Gamma_{(r,s)}^* = (s, t)\Gamma_r^*$ for all $t \in \mathfrak{S}$ and $(u, t)\Gamma_{(r,s)}^* = 0$ if $u \neq s$.

Let e^* be the element of A^* used to define $\Gamma^*(\mathfrak{B})$. Now,

$$(1-[e^*, s, s])(\Gamma_r^* - \Gamma_{(r,s)}^*) = \Gamma_r^* - \Gamma_{(r,s)}^* \text{ for } r = 0, 1, 2, \ldots, k;$$

also,

$$(1-[e^*, s, s])([e^*, s, s] - \Gamma^*(\mathfrak{B})) = [e^*, s, s] - \Gamma^*(\mathfrak{B}).$$

Therefore

$$\Gamma = \Gamma_{(0,s)}^* + \sum_{r=1}^{k} \Gamma_{(r,s)}^* \Gamma_r + (1-[e^*, s, s])(\Gamma_0 + ([e^*, s, s] - \Gamma^*(\mathfrak{B}))\Gamma_0)$$
$$+ (1-[e^*, s, s])\left((\Gamma_0^* - \Gamma_{(0,s)}^*) + \sum_{r=1}^{k} (\Gamma_r^* - \Gamma_{(r,s)}^*)\Gamma_r\right).$$

Hence

$$\Gamma \in H^*(B^*, \mathfrak{a}, s) + H^*(B^*, \mathfrak{a}, s)M(A, \mathfrak{b}) + (1-[e^*, s, s])M(A, \mathfrak{b}) = H(B, \mathfrak{b}, s)$$

for all $\mathfrak{B} \in \mathfrak{E}$ and all $s \in \mathfrak{S}$; therefore $\Gamma \in G$.

Conversely, suppose $\Gamma \in G$; let $\mathfrak{B} = (B^*, B)$ be any element of $\mathfrak{E}$. For all $s \in \mathfrak{S}$, define $\Gamma_s \in M(A, \mathfrak{b})$ by $(s, t)\Gamma_s = (s, t)\Gamma$ for all $t \in \mathfrak{S}$, and $(u, t)\Gamma_s = 0$ if $u \neq s$. Now $\Gamma \in H(B, \mathfrak{b}, s)$; also, if e^* is the element of A^* used to define $\Gamma^*(\mathfrak{B})$,

$$\Gamma - \Gamma_s = (1-[e^*, s, s])(\Gamma - \Gamma_s) \in H(B, \mathfrak{b}, s).$$

Thus

$$\Gamma_s \in H(B, \mathfrak{b}, s) = H^*(B^*, \mathfrak{a}, s) + H^*(B^*, \mathfrak{a}, s)M(A, \mathfrak{b}) + (1-[e^*, s, s])M(A, \mathfrak{b}).$$

Then

$$\Gamma_s = \Gamma_{(0,s)}^* + \sum_{r=1}^{k_s} \Gamma_{(r,s)}^* \Gamma_{(r,s)} + (1-[e^*, s, s])\Gamma_{(0,s)},$$

where for $r = 0, 1, 2, \ldots, k_s$, $\Gamma_{(r,s)}^* \in H^*(B^*, \mathfrak{a}, s)$ and $\Gamma_{(r,s)} \in M(A, \mathfrak{b})$. For $r = 1, 2, \ldots, k_s$, we define $\hat{\Gamma}_{(r,s)}^* \in H^*(B^*, \mathfrak{a}, s)$ and $\hat{\Gamma}_{(r,s)} \in M(A, \mathfrak{b})$ by $(s, t)\hat{\Gamma}_{(r,s)}^* = (s, t)\Gamma_{(r,s)}^*$ for all $t \in \mathfrak{S}$, $(u, t)\hat{\Gamma}_{(r,s)}^* = 0$ if $u \neq s$,
$(u, t)\hat{\Gamma}_{(r,s)} = (u, t)\Gamma_{(r,s)}$ if $t \in \mathfrak{S}(\Gamma)$, and $(u, t)\hat{\Gamma}_{(r,s)} = 0$ if $t \notin \mathfrak{S}(\Gamma)$.

Define $\hat{\Gamma}_{(0,s)}^* \in H^*(B^*, \mathfrak{a}, s)$ and $\hat{\Gamma}_{(0,s)} \in M(A, \mathfrak{b})$ by

$$(s, t)\hat{\Gamma}_{(0,s)}^* = (s, t)\Gamma_{(0,s)}^* \text{ if } t \in \mathfrak{S}(\Gamma),\ (u, t)\hat{\Gamma}_{(0,s)}^* = 0 \text{ if } u \neq s \text{ or if } t \notin \mathfrak{S}(\Gamma),$$

$$(s, t)\hat{\Gamma}_{(0,s)} = (s, t)\Gamma_{(0,s)} \text{ if } t \in \mathfrak{S}(\Gamma),\ (u, t)\hat{\Gamma}_{(0,s)} = 0 \text{ if } u \neq s \text{ or if } t \notin \mathfrak{S}(\Gamma).$$

By definition of Γ_s, $\mathfrak{S}(\Gamma_s) \subseteq \mathfrak{S}(\Gamma)$ so that

$$\Gamma_s = \hat{\Gamma}_{(0,s)}^* + \sum_{r=1}^{k_s} \hat{\Gamma}_{(r,s)}^* \hat{\Gamma}_{(r,s)} + (1-[e^*, s, s])\hat{\Gamma}_{(0,s)}.$$

Define $\hat{\Gamma}_0^* \in M^*(B^*)$ by $(s, t)\hat{\Gamma}_0^* = (s, t)\hat{\Gamma}_{(0,s)}^*$ for all $(s, t) \in \mathfrak{S} \times \mathfrak{S}$. Then,

$$\mathfrak{S}(\hat{\Gamma}_0^*) = \bigcup_{s \in \mathfrak{S}} \mathfrak{S}(\hat{\Gamma}_{(0,s)}^*) \subseteq \mathfrak{S}(\Gamma)$$

so that $\hat{\Gamma}_0^* \in M^*(B^*, \mathfrak{b})$.

Define $\hat{\Gamma}_0 \in M(A)$ by $(s,t)\hat{\Gamma}_0 = (s,t)\hat{\Gamma}_{(0,s)}$ for all $(s,t) \in \mathfrak{S}\times\mathfrak{S}$. Then similarly, $\mathfrak{S}(\hat{\Gamma}_0) \subseteq \mathfrak{S}(\Gamma)$ so that $\hat{\Gamma}_0 \in M(A, \mathfrak{b})$. Also

$$(s,t)((1-\Gamma^*(\mathfrak{B}))\hat{\Gamma}_0) = (s,t)((1-[e^*, s, s])\hat{\Gamma}_{(0,s)})$$

for all $(s,t) \in \mathfrak{S}\times\mathfrak{S}$.

Next, we consider the set $\mathfrak{S}'$ of triples of the form (s, t, n), where $s \in \mathfrak{S}$, n is a natural number such that $1 \leqq n \leqq k_s$, and $t \in \mathfrak{S}(\hat{\Gamma}^*_{(n,s)})$. Since k_s is finite for a given s, and since $\mathfrak{S}(\hat{\Gamma}^*_{(n,s)})$ is a finite subset of $\mathfrak{S}$ for all s and all n such that $1 \leqq n \leqq k_s$, it follows that the cardinality of $\mathfrak{S}'$ is exactly $\mathfrak{c}$. Then there exists a one to one mapping η from $\mathfrak{S}'$ into $\mathfrak{S}$. We now define elements $\hat{\Gamma}^*$ and $\hat{\Gamma}$ of $M(A)$ by

$$(s,u)\hat{\Gamma}^* = (s,t)\hat{\Gamma}^*_{(n,s)}$$

if there exist $t \in \mathfrak{S}$ and $n \in N$ such that $(s,t,n) \in \mathfrak{S}'$ and $u = \eta(s,t,n)$; $(s,u)\hat{\Gamma}^* = 0$ otherwise;

$$(u,v)\hat{\Gamma} = (t,v)\hat{\Gamma}_{(n,s)}$$

if there exist $s \in \mathfrak{S}$, $t \in \mathfrak{S}$ and $n \in N$ such that $(s,t,n) \in \mathfrak{S}'$ and $u = \eta(s,t,n)$; $(u,v)\hat{\Gamma} = 0$ otherwise.

We first remark that, because η is one to one, $\hat{\Gamma}^*$ and $\hat{\Gamma}$ are well defined. Now, for all $(s,u) \in \mathfrak{S}\times\mathfrak{S}$, $(s,u)\hat{\Gamma}^* \in B^*$; also, for a fixed $s \in \mathfrak{S}$, there are only a finite number of elements of $\mathfrak{S}'$ of the form (s,t,n) where $t \in \mathfrak{S}$ and $n \in N$, so that $\kappa(\mathfrak{S}(\hat{\Gamma}^*, s)) < \aleph_0$ for all $s \in \mathfrak{S}$. Therefore $\hat{\Gamma}^* \in M^*(B^*)$. If $v \notin \mathfrak{S}(\Gamma_s)$, $(t,v)\hat{\Gamma}_{(n,s)} = 0$ for all $t \in \mathfrak{S}$, all $s \in \mathfrak{S}$ and all $n \in N$ such that $1 \leqq n \leqq k_s$; thus $(u,v)\hat{\Gamma} = 0$ for all $u \in \mathfrak{S}$ so that $\mathfrak{S}(\hat{\Gamma}) \subseteq \mathfrak{S}(\Gamma)$. Thus $\hat{\Gamma} \in M(A, \mathfrak{b})$. Then, for all $(s,v) \in \mathfrak{S}\times\mathfrak{S}$, $(s,v)(\hat{\Gamma}^*\hat{\Gamma}) = \Sigma((s,u)\hat{\Gamma}^*(u,v)\hat{\Gamma})$, the summation being taken over all u of the form $u = \eta(s,t,n)$ where $(s,t,n) \in \mathfrak{S}'$. Therefore

$$\begin{aligned}(s,v)(\hat{\Gamma}^*\hat{\Gamma}) &= \sum_{n=1}^{k_s} \Big(\sum_t ((s,t)\hat{\Gamma}^*_{(n,s)}(t,v)\hat{\Gamma}_{(n,s)})\Big) \\ &= (s,v)\left(\sum_{n=1}^{k_s} \hat{\Gamma}^*_{(n,s)}\hat{\Gamma}_{(n,s)}\right),\end{aligned}$$

where $\sum_t$ denotes summation over $t \in \mathfrak{S}(\hat{\Gamma}^*_{(n,s)})$. It follows that, for all

$$(s,v) \in \mathfrak{S}\times\mathfrak{S},$$

$$\begin{aligned}(s,v)\Gamma = (s,v)\Gamma_s &= (s,v)\left(\hat{\Gamma}^*_{(0,s)} + \sum_{n=1}^{k_s} \hat{\Gamma}^*_{(n,s)}\hat{\Gamma}_{(n,s)} + (1-[e^*, s, s])\hat{\Gamma}_{(0,s)}\right) \\ &= (s,v)(\hat{\Gamma}^*_0 + \hat{\Gamma}^*\hat{\Gamma} + (1-\Gamma^*(\mathfrak{B}))\hat{\Gamma}_0),\end{aligned}$$

and so

$$\Gamma = \hat{\Gamma}^*_0 + \hat{\Gamma}^*\hat{\Gamma} + (1-\Gamma^*(\mathfrak{B}))\hat{\Gamma}_0 \in M^*(B^*, \mathfrak{b}) + M^*(B^*)M(A, \mathfrak{b}) + (1-\Gamma^*(\mathfrak{B}))M(A, \mathfrak{b}).$$

But $\mathfrak{B}$ was chosen at random from $\mathfrak{E}$; thus $\Gamma \in G'$. Then

$$G = G' = \bigcap_{\mathfrak{B} \in \mathfrak{E}} (M^*(B^*, \mathfrak{b}) + M^*(B^*)M(A, \mathfrak{b}) + (1-\Gamma^*(\mathfrak{B}))M(A, \mathfrak{b})).$$

Thus if the maximal quasi-accessible normal right ideals of a pseudo-ring $\mathfrak{A}$ are known, then by using Theorem 3.5 we may determine the normal right ideal $\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$ of $\mathfrak{M}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b})$.

It is clear from the result of Theorem 3.5 that $G \subseteq \bigcap_{\mathfrak{B} \in \mathfrak{C}} M(B, \mathfrak{b}) = M(R, \mathfrak{b})$. We now give an example of a pseudo-ring such that this containment is strict; this example is particularly interesting because the pseudo-ring is equivalent to a ring, so that the conclusion applies equally well to the pseudo-rings of infinite matrices over a ring defined by Patterson (**4**).

Example 1. Consider the pseudo-ring $\mathfrak{A} = (A^*, A^*)$, where A^* is the ring defined as follows. Let E^* and R^* be additive groups of order 2, generated by e^* and r^* respectively, Let $A^* = E^* \oplus R^*$, with multiplication defined by $e^*e^* = e^*$, $e^*r^* = r^*$ and $r^*e^* = r^*r^* = 0$.

It is not difficult to show that R^* is the only maximal right ideal of A^*; R^* is modular with respect to e^*. Then $\mathfrak{R} = (R^*, R^*)$ is the only maximal quasi-accessible normal right ideal of $\mathfrak{A}$. It follows that the Jacobson radical of the ring A^* is R^*, and the Jacobson radical of $\mathfrak{A}$ is $\mathfrak{R}$.

Now let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinals such that $\mathfrak{b} \geqq \mathfrak{a} \geqq \aleph_0$. We note that, since $R^*A^* = 0$, $M^*(R^*)M(A^*, \mathfrak{b}) = 0$; also, we may use e^* to define $\Gamma^*(\mathfrak{R})$, so that $(1-\Gamma^*(\mathfrak{R}))M(A^*, \mathfrak{b}) = 0$. Then Theorem 3.5 shows that

$$\mathfrak{G}(\mathfrak{A}, \mathfrak{a}, \mathfrak{b}) = (M^*(R^*, \mathfrak{a}), M^*(R^*, \mathfrak{b})).$$

Clearly if $\mathfrak{b} > \aleph_0$, $M^*(R^*, \mathfrak{b}) \subset M(R^*, \mathfrak{b})$. Two cases are of special interest. Choosing $\mathfrak{a} = \aleph_0$, we see from Corollary 3.4 that the Jacobson radical of $\mathfrak{M}(\mathfrak{A}, \aleph_0, \mathfrak{b})$ is exactly $(M^*(R^*, \aleph_0), M^*(R^*, \mathfrak{b}))$. If, however, we choose $\mathfrak{a} > \mathfrak{c}$, then the Jacobson radical of $\mathfrak{M}(A^*)$, as defined by Patterson (**4**), is contained in $(M^*(R^*), M^*(R^*))$; further, since R^* is right-vanishing, the results of Patterson (**2, 3**) show that every element of $M^*(R^*)$ is right quasi-regular. Thus, by Theorem 5 of (**4**), the Jacobson radical of $\mathfrak{M}(A^*)$ is exactly $(M^*(R^*), M^*(R^*))$.

Finally, suppose A^* is a ring with Jacobson radical J^*, and consider the pseudo-ring $\mathfrak{M}(A^*)$. Then Theorem 7 of Patterson (**4**) shows that the Jacobson radical of $\mathfrak{M}(A^*)$ is contained in $\mathfrak{M}(J^*)$.

Now Theorem 5 of Patterson (**2**) states that, if the Jacobson radical of $M^*(A^*)$ is exactly $M^*(J^*)$, then J^* is right-vanishing. The following example shows that there exist rings A^* such that J^* is not right-vanishing and such that the Jacobson radical of $\mathfrak{M}(A^*)$ is exactly $\mathfrak{M}(J^*)$. Thus Theorem 5 of (**2**) has no strict analogue for pseudo-rings.

Example 2. Let p be a prime integer, and let P be the p-adic completion of the ring of integers; then P is a ring with Jacobson radical pP. Also, P is complete with respect to the topology $\{x+p^nP\colon x \in P, n \in N\}$. Then $M(P)$ is complete with respect to the topology $\{\Gamma+M(p^nP)\colon \Gamma \in M(P), n \in N\}$. We first note that every element Γ^* of $M^*(pP)$ is right quasi-regular in $\mathfrak{M}(P)$. For all $n \in N$, define $\Gamma_n = -\sum_{k=1}^{n} (\Gamma^*)^k$; then $\{\Gamma_n\}_{n \in N}$ is a Cauchy sequence with respect to the topology on $M(P)$, so that $\{\Gamma_n\}_{n \in N}$ has a limit $\Gamma \in M(P)$. Thus there exists an increasing sequence $\{k(n)\}_{n \in N}$ such that, for each $n \in N$, $\Gamma - \Gamma_k \in M(p^n P)$ for all $k \geqq k_n$. Consider $\Gamma^* + \Gamma - \Gamma^*\Gamma$; for each $n \in N$, let $m(n) = \max(n, k(n))$. Then $\Gamma^* + \Gamma_{m(n)} - \Gamma^*\Gamma_{m(n)} = (\Gamma^*)^{m(n)+1} \in M(p^n P)$ and $\Gamma - \Gamma_{m(n)} \in M(p^n P)$ so that

$$\Gamma^* + \Gamma - \Gamma^*\Gamma = (\Gamma^* + \Gamma_{m(n)} - \Gamma^*\Gamma_{m(n)}) + (\Gamma - \Gamma_{m(n)}) - \Gamma^*(\Gamma - \Gamma_{m(n)}) \in M(p^n P).$$

Thus $\Gamma^* + \Gamma - \Gamma^*\Gamma \in \bigcap_{n \in N} M(p^n P) = 0$, so that Γ^* is right quasi-regular. Applying Theorem 5 of Patterson (4) and using the fact that the Jacobson radical of $\mathfrak{M}(P)$ is a right ideal of $\mathfrak{M}(P)$, we see that the Jacobson radical of $\mathfrak{M}(P)$ is exactly $\mathfrak{M}(pP)$; however pP is not right-vanishing.

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