Glasgow Math. J. **51** (2009) 385–404. © 2009 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089509005060. Printed in the United Kingdom

A CHARACTERIZATION OF SELF-ADJOINT OPERATORS DETERMINED BY THE WEAK FORMULATION OF SECOND-ORDER SINGULAR DIFFERENTIAL EXPRESSIONS

MOHAMED EL-GEBEILY

Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
e-mail: mgebeily@kfupm.edu.sa

and DONAL O'REGAN

Department of Mathematics, National University of Ireland Galway, Ireland
e-mail: donal.oregan@nuigalway.ie

(Received 31 March 2008; revised 11 September 2008; accepted 20 November 2008)

Abstract. In this paper we describe a special class of self-adjoint operators associated with the singular self-adjoint second-order differential expression ℓ. This class is defined by the requirement that the sesquilinear form $q(u, v)$ obtained from ℓ by integration by parts once agrees with the inner product $\langle \ell u, v\rangle$. We call this class Type I operators. The Friedrichs Extension is a special case of these operators. A complete characterization of these operators is given, for the various values of the deficiency index, in terms of their domains and the boundary conditions they satisfy (separated or coupled).

AMS Subject Classification. 34B15, 34A34.

1. Introduction. In this paper we give a complete characterization of certain self-adjoint operators associated with the differential expression

$$\ell u(x) = \frac{1}{w(x)}(-(p(x)u'(x))' + g(x)u(x)) \tag{1}$$

which is assumed to be defined for almost all $x \in I = (a, b)$, with $-\infty \le a < b \le \infty$. The expression ℓ gives rise to the formal sesquilinear form

$$q(u, v) = \int_I pu'\overline{v'} + gu\overline{v}$$

in addition to the form

$$\langle \ell u, v\rangle = \int_I (-(pu')' + gu)\overline{v}.$$

The equality

$$q(u, v) = \langle \ell u, v\rangle \tag{2}$$

requires the vanishing of the boundary term

$$-pu'\overline{v}]_a^b \tag{3}$$

which is the most general condition for (2) to hold. As we shall see in the following sections equality (2) gives rise to a class of self-adjoint operators, which we termed Type I operators in [**8**]. The study of these operators was necessary to handle certain nonlinear equations and devise numerical methods associated with the formal expression ℓ. It is to be expected of course that boundary term (3) will vanish for all functions in the domain of definition of any Type I operator. The vanishing of this boundary term could occur because $pu'\overline{v}\,(a) = 0 = pu'\overline{v}\,(b)$ or simply because $pu'\overline{v}\,(a) = pu'\overline{v}\,(b)$. The former case is referred to as separated boundary conditions and the latter case is referred to as coupled boundary conditions. In this paper we give a full characterization of Type I operators in terms of both kinds of boundary conditions. As we shall see, Type I operators with the separated boundary conditions always exist while those with coupled boundary conditions exist only under further restrictions on the data of the problem. We should also point out that the Friedrichs Extension [**3**], which is similarly defined, satisfies Dirichlet (i.e. separated) boundary conditions [**5**, **7**] in the regular case (see the next section). Since the Dirichlet boundary conditions are a special form of the more general separated boundary conditions mentioned above, the Friedrichs Extension is a special case of Type I operators. Our work in this paper will establish that other separated boundary conditions such as $u\,(a) = u'\,(b) = 0$ give rise to self-adjoint operators which are essentially different from the Friedrichs Extension.

All self-adjoint operators associated with the expression ℓ are realized through the requirement

$$\langle \ell u, v\rangle = \langle u, \ell v\rangle$$

which, in turn, requires the vanishing of the more general boundary term

$$-pu'\overline{v} + pu\overline{v'}]_a^b. \tag{4}$$

Type I operators are a special class of these operators in that

$$\langle \ell u, v\rangle = q\,(u, v) = \langle u, \ell v\rangle$$

and (consequently)

$$-pu'\overline{v}]_a^b = -pu\overline{v'}]_a^b = 0.$$

Of course not all self-adjoint extensions of L_0 are Type I operators. For example, the expression $\ell u = -u'' + u$ defined on $(0, 1)$ and the boundary conditions $u\,(0) + u'\,(0) = u\,(1) + u'\,(1) = 0$ give rise to a self-adjoint operator in $L^2\,(I)$. The function $u\,(x) = -3x^3 + 4x^2$ is in the domain of this operator but $\{u, u\}_0^1 \neq 0$.

The study of self-adjoint operators associated with ℓ is not new (see [**4**, **6**, **9**, **10**] and the references therein), while the study of boundary conditions associated with them can be found in refs. [**1**, **2**, **4**, **10**]. The study of Type I operators appears to be new and to the best of our knowledge, this is the first time that the study of boundary conditions associated with Type I operators is carried out.

This paper consists of three sections in addition to the introduction. In Section 2 we present some preliminary material that includes definitions, theorems and discussions needed for the rest of the paper. It is designed to be, more or less, self-contained and should help the reader to better follow the terminology used in connection with singular operators. In Section 3 we show that Type I operators with separated boundary conditions always exist while those with coupled boundary conditions exist only when the deficiency index (see next section) is 2. We also give a full characterization of the domains of these operators.

2. Preliminaries. In this section we introduce notation, definitions and discussions that are necessary for this work. The main definitions and theorems can be found in refs. [**4**, **6**, **9**, **10**]. We work with the formally self-adjoint differential expression

$$\ell u = \frac{1}{w}[(-pu')' + gu]$$

defined on the interval $I = (a, b)$, $-\infty \leq a < b \leq \infty$. We assume that

$$1/p, g, w \in L_{\text{loc}}(I)$$

and that $w > 0$ almost everywhere in I.

Let $H = L^2_w(I)$, be the Hilbert space of square integrable functions with respect to the weight w. The inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ in this space are given by

$$\langle f, h \rangle = \int_I f(t)\,\overline{h}(t)\,w(t)\,dt$$

and

$$\|f\|^2 = \int_I |f(t)|^2\,w(t)\,dt,$$

respectively. Also let $u^{[1]} := pu'$. $u^{[1]}$ is called the first pseudo-derivative of u with respect to the function p. The maximal operator L generated by the expression ℓ in H is defined by

$$D(L) = D = \left\{u \in H : u, u^{[1]} \in AC(I) \text{ and } \ell u \in H\right\},$$
$$Lu = \ell u, \quad u \in D.$$

Since D is dense in H, L has a uniquely defined adjoint. Let $L_0 = L^*$ (the adjoint of L) and $D_0 = D(L_0)$. The operator L_0 is called the minimal operator generated by ℓ and it is known [**6**] that $D_0 \subseteq D$, D_0 is dense in $\mathcal{H}$ and $L_0^* = L$. In other words, $L_0 \subset L = L_0^*$. Therefore, L_0 is a symmetric closed operator. Moreover, any self-adjoint extension S of L_0 is a self-adjoint restriction of L and vice versa, i.e. $L_0 \subset S = S^* \subset L_0^* = L$.

For $y, z \in D$ and $x \in I$ define the Lagrange bracket

$$[y, z](x) = -y^{[1]}(x)\overline{z}(x) + \overline{z}^{[1]}(x)\,y(x). \tag{5}$$

Note that the limits of the terms in (5) as $x \to a^+, b^-$ both exist and are finite. Thus, the notation

$$[y, z](a) = \lim_{x \to a^+}[y, z](x), \quad [y, z](b) = \lim_{x \to b^-}[y, z](x)$$

is justified. We use $[y, z]_\alpha^\beta$ to denote $[y, z](\beta) - [y, z](\alpha)$.

The endpoint a is regular if $1/p, g, w \in L(a, c)$ for some (and hence all) $c \in I$; is limit circle (LC) if all solutions of

$$\ell u = 0 \tag{6}$$

are in $L_w^2(a, c)$ for some $c \in I$; is limit point (LP) if it is not LC. Similar definitions hold at b. An endpoint is singular if it is not regular. The *deficiency index* of the operator L_0 is defined to be the number of linearly independent solutions of (6) which belong to H (see [**4**, **10**] for more details).

Proposition 1.
(1) $d = 0 \iff$ *a and b are LP.*
(2) $d = 1 \iff$ *one end point is LP and the other is LC.*
(3) $d = 2 \iff$ *a and b are LC.*

Proof. See [**6**, p. 72]. □

Let $c \in I$ and let θ, ϕ be the unique real solutions of the initial value problems

$$\ell u = 0, \tag{7}$$
$$\theta(c) = -\phi^{[1]}(c) = 1, \tag{8}$$
$$\theta^{[1]}(c) = \phi(c) = 0. \tag{9}$$

Observe that $[\theta, \phi](x) = -1 = -[\theta, \phi](x)$ for all $x \in I$. If a (b) is LC then θ, ϕ belong to $L_w^2(a, c)$ $\left(L_w^2(c, b)\right)$.

If X, Y are vector spaces and $Y \subset X$, the notation $x_1, x_2, \ldots, x_m \in X \bmod Y$ means that these elements are in X and are linearly independent modulo Y (see [**6**]). If

$$X = Y \dotplus \operatorname{span}\{x_1, x_2, \ldots, x_m\} \tag{10}$$

then it will be sufficient for our purposes to consider only the elements in $X \bmod Y$ that are linear combinations of $x_1, x_2, \ldots, x_m$. We also use the notation $\dim(X \bmod Y)$ for the number of elements that can be found in $X \bmod Y$ such that (10) is valid.

The proofs of the following two lemmas can be found in ref. [**10**].

Lemma 2. *Suppose a (b) is LC, then there are real functions ψ_1, ψ_2 $(\psi_3, \psi_4) \in D \bmod D_0$ such that*
(1) $[\psi_1, \psi_2](x) = -1$ *near a* $([\psi_3, \psi_4](x) = -1$ *near b).*
(2) ψ_1 *and* $\psi_2 = 0$ *near b (ψ_3 and $\psi_4 = 0$ near a).*

The functions ψ_1, ψ_2 (ψ_3, ψ_4) may be constructed by taking them equal to θ, ϕ, respectively, near a (b) and equal to 0 near b (a). We remark that in the above lemma, if both a and b are LC, then $\psi_1, \psi_2, \psi_3, \psi_4 \in D \bmod D_0$.

Lemma 3. *If a and b both are LP, then*

$$D = D_0.$$

If a is LC and b is LP, then

$$D = D_0 \dotplus \operatorname{span}\{\psi_1, \psi_2\}.$$

If b is LC and a is LP, then

$$D = D_0 \dotplus \operatorname{span}\{\psi_3, \psi_4\}.$$

If a and b both are LC then

$$D = D_0 \dotplus \operatorname{span}\{\psi_1, \psi_2, \psi_3, \psi_4\}, \tag{11}$$

where $\psi_1, \psi_2, \psi_3, \psi_4$ are as in Lemma 2.

In the case $d = 1$, to avoid a lot of repetitive statements, let us agree to take a as LC and b as LP.

The domain of definition D_1 of any self-adjoint extension of L_0 is characterized by the existence of d functions $\varphi_1, \dots, \varphi_d \in D \bmod D_0$ such that $[\varphi_i, \varphi_j]_a^b = 0$ and

$$D_1 = D_0 \dotplus \operatorname{span}\{\varphi_1, \dots, \varphi_d\}. \tag{12}$$

The proof of the following lemma is an easy consequence of the results in [**10**].

LEMMA 4. *Assume $d = 2$ and $\psi_1, \psi_2, \psi_3, \psi_4$ are as in Lemma 3. Let $\varphi_1, \varphi_2 \in D \bmod D_0$ and write*

$$\varphi_1 = \eta_0 + \alpha_1\psi_1 + \alpha_2\psi_2 + \alpha_3\psi_3 + \alpha_4\psi_4,$$
$$\varphi_2 = \xi_0 + \beta_1\psi_1 + \beta_2\psi_2 + \beta_3\psi_3 + \beta_4\psi_4,$$

where $\eta_0, \xi_0 \in D_0, \alpha_i, \beta_i \in \mathbb{C},\ i = 1, 2, 3, 4$. Then

(1) *φ_1, φ_2 characterize the domain of definition D_1 of a self-adjoint extension L_1 of L_0 if and only if*

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = e^{i\theta} \begin{vmatrix} \alpha_3 & \alpha_4 \\ \beta_3 & \beta_4 \end{vmatrix} \tag{13}$$

for some $\theta \in [0, 2\pi)$.

(2) *If D_1 is the domain of definition of a self-adjoint extension L_1 of L_0 then the separated boundary condition*

$$[u, v](a) = [u, v](b) = 0$$

is satisfied for all $u, v \in D_1$ if and only if the determinants in (13) vanish. Then we can write

$$\varphi_1 = \alpha_1\psi_1 + \alpha_2\psi_2,$$
$$\varphi_2 = \beta_3\psi_3 + \beta_4\psi_4.$$

(3) *If D_1 is the domain of definition of a self-adjoint extension L_1 of L_0 then the coupled boundary condition*

$$[u, v](a) = [u, v](b)$$

is satisfied for all $u, v \in D_1$ if and only if the determinants in (13) do not vanish. Then we can write

$$\varphi_1 = \psi_1 + \alpha_3\psi_3 + \alpha_4\psi_4,$$
$$\varphi_2 = \psi_2 + \beta_3\psi_3 + \beta_4\psi_4.$$

Next we introduce the formal symmetric sesquilinear form

$$q(u, v) = \int_a^b pu'\overline{v}' + gu\overline{v}$$

and the associated boundary terms

$$\{u, v\}(x) = -u^{[1]}\overline{v}(x), \ \ x \in I,$$
$$\{u, v\}_a^b = \{u, v\}(b^+) - \{u, v\}(a^-)$$

whenever the implied limits exist. Note that

$$[u, v](x) = \{u, v\}(x) - \{\overline{v}, \overline{u}\}(x).$$

Also, for $u, v \in D$, $q(u, v)$ exists and is finite if and only if $\{u, v\}_a^b$ exists and is finite. Then

$$q(u, v) = \langle Lu, v\rangle + \{u, v\}_a^b.$$

Our main assumption on q is the following:

(A) We assume that q is bounded below: $q(u) := q(u, u) \geq M\,\|u\|^2$ for some $M \in \mathbb{R}$.

REMARK 5. Without loss of generality, we may assume that $M > 0$ for, otherwise, we may consider the form $q + \lambda$ for some $\lambda > M$ instead. Let V be the subspace of functions $u \in H$ for which $q(u) < \infty$. Note that V is dense in H since it contains the dense subspace of functions in D with compact support in I. It can easily be checked that V is a Hilbert space if equipped with the inner product induced by q.

REMARK 6. Assumption (A) excludes cases where the differential expression is in limit point case at one end-point but not in strong limit point case (cf. e.g. [**2**]).

PROPOSITION 7.

(1) *$D_0 \subseteq V$ and, for all $u, v \in D_0$, $q(u, v) = \langle L_0 u, v\rangle$.*

(2) *$\{u, v\}_a^b = 0$ for all $u, v \in D_0$.*

Proof. See [**8**]. □

The equation $q(u, v) = \langle L_0 u, v\rangle$ for all $u \in D_0$, $v \in V$ means that, for a fixed $u \in D_0$, $q(u, \cdot)$ is continuous on D_0 with respect to the norm in H. The maximal subspace of V with this property will play a central role in this paper. Therefore, we define the space $\tilde{D}$ by

$$\tilde{D} = \{u \in V : q(u, \cdot) \text{ is continuous on } D_0 \text{ with respect to the norm in } H\}. \qquad (14)$$

The next proposition gives some properties of $\tilde{D}$ and, in particular, the fact that $\tilde{D}$ is an essential extension of D_0.

PROPOSITION 8. *Let $\tilde{D}$ be defined by (14). Then*

(1) *For $u \in \tilde{D}$ and $v \in V$ both $\{u, v\}(a)$ and $\{u, v\}(b)$ exist and are finite.*
(2) *$\tilde{D} \subset D$ and, for all $u \in \tilde{D}$ and $v \in D_0$, $\{u, v\}_a^b = \{v, u\}_a^b = 0$.*
(3) *$\tilde{D} = \left\{u \in D : \{u, v\}_a^b = 0 \forall v \in D_0\right\}$.*
(4) *For $d \geq 1$, there are d real functions $u_1, \dots, u_d \in \tilde{D} \bmod D_0$ and*
(5) *$\{u_i, u_j\}_a^b = 0$, $i, j = 1, \dots, d$.*

Proof. See **[8]**. □

Let ψ_1, ψ_2 be as in Lemma 3 (for $d = 1$ or $d = 2$) and define the matrix C_a by

$$C_a = \begin{bmatrix} \{\psi_1, \psi_1\}(a) & \frac{1}{2}\left(\{\psi_1, \psi_2\}(a) + \{\psi_2, \psi_1\}(a)\right) \\ \frac{1}{2}\left(\{\psi_1, \psi_2\}(a) + \{\psi_2, \psi_1\}(a)\right) & \{\psi_2, \psi_2\}(a) \end{bmatrix},$$

if the implied limits exist. This matrix has eigenvalues

$$\frac{d \pm \sqrt{d^2 + c^2}}{2},$$

where

$$\begin{aligned} d &= \{\psi_1, \psi_1\}(a) + \{\psi_2, \psi_2\}(a), \\ c &= \{\psi_1, \psi_2\}(a) - \{\psi_2, \psi_1\}(a) \\ &= [\psi_1, \psi_2](a) = -1, \end{aligned}$$

where, in order to arrive at the expression for c, we used the observation that

$$\{\psi_1, \psi_1\}\{\psi_2, \psi_2\}(a) = \{\psi_1, \psi_2\}\{\psi_2, \psi_1\}(a).$$

It can be directly verified that the last equation is true for any $x \in I$. If the limits as $x \to a^+$ exist we can then pass to the limit. Denote the positive and negative eigenvalues of C_a by $\lambda_a, -\sigma_a$, respectively. Since C_a is symmetric, there exists an orthogonal matrix B_a such that $B_a^t C_a B_a := \Lambda_a := \operatorname{diag}[\lambda_a, -\sigma_a]$. Introduce the change of base transformation

$$\begin{pmatrix} \widetilde{\psi}_1(x) \\ \widetilde{\psi}_2(x) \end{pmatrix} = B_a \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}.$$

Then $\widetilde{\psi}_1, \widetilde{\psi}_2$ are still linearly independent modulo D_0 and still equal to zero near b. The corresponding $\widetilde{C}_a$ matrix is

$$\widetilde{C}_a = \begin{bmatrix} B_{1,a}^t C_a B_{1,a} & B_{1,a}^t C_a B_{2,a} \\ B_{2,a}^t C_a B_{1,a} & B_{2,a}^t C_a B_{2,a} \end{bmatrix} = B_a^t C_a B_a = \Lambda_a,$$

where $B_{1,a}, B_{2,a}$ are the columns of B_a^t. From this we get

$$\begin{aligned} \left\{\widetilde{\psi}_1, \widetilde{\psi}_1\right\}(a) &= \lambda_a, \\ \left\{\widetilde{\psi}_1, \widetilde{\psi}_2\right\}(a) + \left\{\widetilde{\psi}_2, \widetilde{\psi}_1\right\}(a) &= 0, \\ \left\{\widetilde{\psi}_2, \widetilde{\psi}_2\right\}(a) &= -\sigma_a. \end{aligned}$$

https://doi.org/10.1017/S0017089509005060 Published online by Cambridge University Press

Furthermore,

$$\begin{aligned}\begin{bmatrix}\begin{bmatrix}\widetilde{\psi}_1, \widetilde{\psi}_1\\ \widetilde{\psi}_2, \widetilde{\psi}_1\end{bmatrix} & \begin{bmatrix}\widetilde{\psi}_1, \widetilde{\psi}_2\\ \widetilde{\psi}_2, \widetilde{\psi}_2\end{bmatrix}\end{bmatrix}(x) &= \left[B_a\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} B_a\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\right](x)\\ &= B_a^t\begin{bmatrix}[\psi_1, \psi_1] & [\psi_1, \psi_2]\\ [\psi_2, \psi_1] & [\psi_2, \psi_2]\end{bmatrix}(x)\, B_a\\ &= B_a^t\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} B_a = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\end{aligned}$$

where the last equality holds because B_a and the matrix $R := \left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$ are rotations. Hence, we still have $[\widetilde{\psi}_1, \widetilde{\psi}_2](x) = -1 = -[\widetilde{\psi}_2, \widetilde{\psi}_1](x)$. For the case $d = 2$, similar remarks hold for the end point b and the functions ψ_3, ψ_4. It is now in order to clarify our subsequent use of the ψ-functions appearing in Lemma 3. If the limits at the end point(s) do not exist or have not yet been established, then the ψ-functions are exactly the same as in Lemma 3. If the aforementioned limits exist or have been established then we assume that all expressions involving the ψ-functions have been rewritten, if necessary, in terms of the $\widetilde{\psi}$-functions given above. We do this without distinguishing between the two sets of functions. Therefore, we will always assume that the ψ-functions produce the diagonal matrix Λ_a (Λ_b) whenever it is possible to form the corresponding matrix C_a (C_b). One further observation to be made here is that

$$\lambda_a\sigma_a = \lambda_b\sigma_b = \frac{1}{4}, \tag{15}$$

as

$$\begin{aligned}\lambda_a\sigma_a &= -\left(\frac{d+\sqrt{d^2+c^2}}{2}\right)\left(\frac{d-\sqrt{d^2+c^2}}{2}\right)\\ &= \frac{c^2}{4} = \frac{1}{4}.\end{aligned}$$

Definition 9. A self-adjoint extension L_1 of L_0 with domain D_1 such that $\{u, v\}_a^b = 0$ for all $u, v \in D_1$ will be called a Type I operator and its domain D_1 will be called a Type I domain.

Definition 10. Define the δ-deficiency by

$$\delta = \dim\left(D \bmod \tilde{D}\right).$$

In the case $d = 1$, since $\dim(\tilde{D} \bmod D_0) \geq 1$ and $\dim(D \bmod D_0) = 2$, $\dim(D \bmod \tilde{D}) \geq 0$. Therefore, $\delta \in \{0, 1\}$. Similarly in the case $d = 2$, $\delta \in \{0, 1, 2\}$. In either case, if $\delta > 0$ then $\tilde{D}$ is a proper subspace of D and if $\delta = 0$ then $\tilde{D} = D$.

Observe also that if D_1 is a Type I domain then $D_1 \subset \tilde{D}$. Various Type I operators obviously correspond to the various ways of choosing the functions $u_1, \ldots, u_d \in \tilde{D} \bmod D_0$ in accordance with Proposition 8. This paper is, in a sense, about the possible choices of such functions.

3. Description of Type I domains. We emphasize here that the discussion in this section is valid under our basic assumption that the sesquilinear form q is bounded below. For any value of d, the domain D always contains a Type I domain. For $d = 0$ the only self-adjoint extension of L_0 is L_0 itself. In this case L_0 is a Type I operator as asserted by Part 2 of Proposition 7. For $d \geq 1$ the statement follows from Parts 4 and 5 of Proposition 8. The general requirement for a domain $D_1 \subset \tilde{D}$ to be a Type I domain is that $\{u, v\}_a^b = 0$ for all $u, v \in D_1$. In this section we are going to describe all Type I domains D_1 for which the separated boundary condition

$$\{u, v\}(a) = \{u, v\}(b) = 0 \tag{16}$$

for all $u, v \in D_1$ is satisfied and those for which the coupled boundary condition

$$\begin{gathered} \{u, v\}(a) = \{u, v\}(b) \ \forall u, v \in D_1 \\ \text{and } \{u, v\}(a) \neq 0 \text{ for at least one pair } u, v. \end{gathered} \tag{17}$$

As we shall see, Type I domains with the separated boundary condition (16) always exist. However, Type I domains with coupled boundary condition (17) exist only in the case $d = 2$ and under the condition $\tilde{D} = D$.

Again, for $d = 0$ the situation is very simple since, by Part 2 of Proposition 7 D_0 is a Type I domain with separated boundary condition (16).

3.1. The case $d = 1$. In this subsection we assume that the deficiency index $d = 1$ and that ψ_1, ψ_2 are as in Lemmas 2 and 3.

PROPOSITION 11. *There exists a Type I domain $D_1$with separated boundary condition (16).*

Proof. For any $u \in D$,

$$u = u_0 + \alpha\psi_1 + \beta\psi_2,$$

where $\alpha, \beta \in \mathbb{C}$ and $u_0 \in D_0$. Since ψ_1, ψ_2 vanish near b, it follows from this and Proposition 7 that $\{u, v\}(b) = 0$ for any pair $u, v \in D$. Consequently, the same is true for $\tilde{D}$. Since by Proposition 8 $\tilde{D}$ contains a Type I domain D_1 and $\{u, v\}_a^b = 0$ for all $u, v \in D_1$, separated boundary condition (16) is satisfied. □

COROLLARY 12. *All Type I domains have separated boundary condition (16).*

LEMMA 13. *If $\delta = 1$ then there is a real linear combination η_1 of ψ_1, ψ_2 such that $\eta_1 \in \tilde{D} \bmod D_0$. Furthermore, if η_1, ψ_1 are linearly independent then η_1 can be chosen such that $[\eta_1, \psi_1](x) = 1$ for x near a and if η_1, ψ_2 are linearly independent then η_1 can be chosen such that $[\eta_1, \psi_2](x) = -1$ for x near a.*

Proof. Since $\tilde{D}$ contains a Type I domain, a real linear combination η_1 of ψ_1, ψ_2 belongs to $\tilde{D} \bmod D_0$. Suppose $\eta_1 = \alpha_1\psi_1 + \alpha_2\psi_2$ and η_1, ψ_1 are linearly independent. Then $\alpha_2 \neq 0$, and we can assume that $\eta_1 = \alpha_1\psi_1 + \psi_2$. For x near a,

$$\begin{aligned} [\eta_1, \psi_1](x) &= \begin{vmatrix} \eta_1 & \psi_1 \\ \eta_1^{[1]} & \psi_1^{[1]} \end{vmatrix}(x) = \begin{vmatrix} \alpha_1\psi_1 + \psi_2 & \psi_1 \\ \alpha_1\psi_1^{[1]} + \psi_2^{[1]} & \psi_1^{[1]} \end{vmatrix}(x) \\ &= \begin{vmatrix} \psi_2 & \psi_1 \\ \psi_2^{[1]} & \psi_1^{[1]} \end{vmatrix}(x) = [\psi_2, \psi_1](x) = 1. \end{aligned}$$

The other case is proven similarly. □

https://doi.org/10.1017/S0017089509005060 Published online by Cambridge University Press

Lemma 13 means that the functions ψ_1, ψ_2 in Lemmas 2 and 3 can be chosen such that $\psi_1 \in \tilde{D} \bmod D_0$. This choice will always be assumed in the sequel.

THEOREM 14. *All Type I domains are characterized as follows:*
(1) *If* $\delta = 1$ *then there is only one Type I domain, namely,*

$$D_1 = D_0 \dotplus \operatorname{span}\{\psi_1\}.$$

(2) *If* $\delta = 0$ *then there are two Type I domains given by*

$$D_1 = D_0 \dotplus \operatorname{span}\left\{\psi_1 + \sqrt{\frac{\lambda_a}{\sigma_a}}\psi_2\right\},$$
$$D_2 = D_0 \dotplus \operatorname{span}\left\{\psi_1 - \sqrt{\frac{\lambda_a}{\sigma_a}}\psi_2\right\}.$$

Proof. To prove Part 1 assume $\delta = 1$. Then

$$\tilde{D} = D_0 \dotplus \operatorname{span}\{\psi_1\}$$

and $\tilde{D}$ is the only Type I domain.

To prove Part 2, let D_1 be a Type I domain and select a function $\eta \in D_1 \bmod D_0$. We can write

$$\eta = \alpha_1\psi_1 + \alpha_2\psi_2,$$

where we may take α_1, α_2 to be real since all self-adjoint extensions of L_0 are real (see [**1**]). The boundary condition $\{\eta, \eta\}(a) = 0$ then yields

$$\alpha^t \Lambda_a \alpha = 0.$$

where $\alpha^t = [\alpha_1, \alpha_2]$. This equation gives

$$\alpha_2 = \pm\sqrt{\frac{\lambda_a}{\sigma_a}}\alpha_1.$$

Thus,

$$\eta = \alpha_1\left(\psi_1 \pm \sqrt{\frac{\lambda_a}{\sigma_a}}\psi_2\right).$$

This shows that there are two Type I domains, one defined by the function $\psi_1 + \sqrt{\lambda_a/\sigma_a}\psi_2$ and the other defined by the function $\psi_1 - \sqrt{\lambda_a/\sigma_a}\psi_2$. □

3.2. The Case $d = 2$. In this subsection we assume that the deficiency index $d = 2$ and that $\psi_1, \psi_2, \psi_3, \psi_4$ be as in Lemmas 2 and 3.

PROPOSITION 15. *There exists a Type I domain with separated boundary condition (16).*

Proof. The result will be established if we can show that we can select two functions $\varphi_1, \varphi_2 \in \tilde{D} \bmod D_0$ and

$$\{\varphi_i, \varphi_j\}(a) = \{\varphi_i, \varphi_j\}(b) = 0, \; i, j = 1, 2.$$

Let φ_1, φ_2 be as in Parts 4 and 5 of Proposition 8. Since $\{\varphi_i, \varphi_j\}_a^b = 0, \; i, j = 1, 2$, the domain $\widehat{D}$ defined by

$$\widehat{D} = D_0 \dotplus \operatorname{span}\{\varphi_1, \varphi_2\}$$

is a Type I domain. We can write

$$\begin{aligned} \varphi_1 &= \alpha_1\psi_1 + \alpha_2\psi_2 + \alpha_3\psi_3 + \alpha_4\psi_4, \\ \varphi_2 &= \beta_1\psi_1 + \beta_2\psi_2 + \beta_3\psi_3 + \beta_4\psi_4. \end{aligned}$$

Then by Lemma 4, equation (13) is satisfied. We have two cases to consider:

Case 1: The determinants in (13) vanish.

In this case, by Lemma 4, we may write

$$\begin{aligned} \varphi_1 &= \alpha_1\psi_1 + \alpha_2\psi_2, \\ \varphi_2 &= \beta_3\psi_3 + \beta_4\psi_4. \end{aligned}$$

The above equations together with the conditions $\{\varphi_i, \varphi_j\}_a^b = 0, \; i, j = 1, 2$ yield $\{\varphi_i, \varphi_j\}(a) = \{\varphi_i, \varphi_j\}(b) = 0$. Therefore, the domain $\widehat{D}$ is a Type I domain with the separated boundary condition (16).

Case 2: The determinants in (13) do not vanish.

In this case, by Lemma 4 we may write

$$\begin{aligned} \varphi_1 &= \psi_1 + \alpha_3\psi_3 + \alpha_4\psi_4, \\ \varphi_2 &= \psi_2 + \beta_3\psi_3 + \beta_4\psi_4. \end{aligned}$$

The finiteness of $\{\varphi_i, \varphi_j\}(a), \; i, j = 1, 2$ then give that $\{\psi_i, \psi_j\}(a), \; i, j = 1, 2$ are finite. It follows that $\psi_1, \psi_2 \in V$. Furthermore, for any $u \in D_0$ (see Proposition 7),

$$\{\psi_i, u\}(a) = \{\varphi_i, u\}(a) = 0, \; i = 1, 2.$$

Hence,

$$q(\psi_i, u) = \langle L\psi_i, u \rangle, \; i = 1, 2.$$

Consequently $q(\psi_i, \cdot)$ is continuous on D_0 and ψ_1, ψ_2 are actually in $\tilde{D}$. By interchanging the roles of ψ_3, ψ_4 with that of ψ_1, ψ_2 we can similarly show that $\psi_3, \psi_4 \in \tilde{D}$. Therefore $\tilde{D} = D$. We next proceed to show that D contains a Type I domain D_1 with separated boundary condition (16). For this purpose, let $\xi_1 = \psi_1 + \alpha\psi_2$, where α is a real number to be determined so that $\{\xi_1, \xi_1\}(a) = 0$. Therefore, α must satisfy

$$\{\psi_1, \psi_1\}(a) + \alpha\left(\{\psi_1, \psi_2\}(a) + \{\psi_2, \psi_1\}(a)\right) + \alpha^2\{\psi_2, \psi_2\}(a) = 0$$

or

$$\lambda_a - \sigma_a\alpha^2 = 0.$$

Thus, we may take $\alpha = \pm\sqrt{\lambda_a/\sigma_a}$. By a similar argument we can show that $\psi_3, \psi_4 \in \tilde{D}$ and obtain a linear combination $\xi_2 = \psi_3 + \beta\psi_4$ satisfying $\{\xi_2, \xi_2\}(b) = 0$. Using the expressions of ξ_1, ξ_2 we can easily check that $\{\xi_i, \xi_j\}(a) = \{\xi_i, \xi_j\}(b) = 0,\ i = 1, 2$. Thus the domain D_1 defined by

$$D_1 = D_0 \dot{+} \operatorname{span}\{\xi_1, \xi_2\}$$

is a Type I domain with separated boundary condition (16). □

Next we establish the existence of Type I domains with coupled boundary condition (17).

PROPOSITION 16. *The domain $\tilde{D}$ contains a Type I domain with coupled boundary condition (17) if and only if $\delta = 0$.*

Proof. Suppose $\tilde{D}$ contains a Type I domain D_1 with coupled boundary condition (17). Choose $\eta_1, \eta_2 \in D_1 \bmod D_0$. By Lemma 4 we may write

$$\begin{aligned}\eta_1 &= \psi_1 + \alpha_1\psi_3 + \alpha_2\psi_4,\\ \eta_2 &= \psi_2 + \beta_1\psi_3 + \beta_2\psi_4.\end{aligned}$$

Since $\eta_1, \eta_2 \in \tilde{D}$, $\{\eta_i, \eta_i\}(a)$, $i = 1, 2$ are finite. Therefore, $\{\psi_i, \psi_i\}(a)$, $i = 1, 2$ are finite. Also, for $u \in D_0$ the equation $\{\eta_i, u\}(a) = 0$ yields $\{\psi_i, u\}(a) = 0$, $i = 1, 2$. It follows that $q(\psi_i, \cdot)$ is continuous on D_0 and hence $\psi_i \in \tilde{D}$, $i = 1, 2$. By interchanging the roles of ψ_1, ψ_2 with that of ψ_3, ψ_4 we can similarly show that $\psi_3, \psi_4 \in \tilde{D}$. Therefore, $\delta = 0$.

On the other hand suppose $\delta = 0$. Then $\psi_1, \psi_2, \psi_3, \psi_4 \in \tilde{D}$. We are going to demonstrate that $\tilde{D}$ contains a Type I domain with coupled boundary condition (17). By Lemma 4 we try to construct functions φ_1, φ_2 of the form

$$\begin{aligned}\varphi_1 &= \psi_1 + \alpha_1\psi_3 + \alpha_2\psi_4,\\ \varphi_2 &= \psi_2 + \beta_1\psi_3 + \beta_2\psi_4,\end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1$ and β_2 are real parameters to be determined. The equations

$$\{\varphi_i, \varphi_j\}_a^b = 0,\ i, j = 1, 2$$

give rise to the system

$$\begin{aligned}\alpha^t\Lambda_b\alpha &= \lambda_a,\\ \beta^t\Lambda_b\beta &= -\sigma_a,\\ \alpha^t D\beta &= \rho_3,\\ \alpha^t D^t\beta &= \rho_4,\end{aligned}$$

where

$$\begin{aligned}&\alpha^t = [\alpha_1, \alpha_2],\ \beta^t = [\beta_1, \beta_2],\\ &D = \begin{bmatrix}\{\psi_3, \psi_3\}(b) & \{\psi_3, \psi_4\}(b)\\ \{\psi_4, \psi_3\}(b) & \{\psi_4, \psi_4\}(b)\end{bmatrix},\\ &\rho_3 = \{\psi_1, \psi_2\}(a),\ \rho_4 = \{\psi_2, \psi_1\}(a).\end{aligned}$$

By addition and subtraction of the third and fourth equations above, we get the equivalent system

$$\begin{aligned}
\alpha^t \Lambda_b \alpha &= \lambda_a,\\
\beta^t \Lambda_b \beta &= -\sigma_a,\\
\alpha^t \Lambda_b \beta &= \frac{1}{2}(\rho_3 + \rho_4) = 0,\\
\alpha^t R \beta &= -1,
\end{aligned}$$

where

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Explicitly, we have

$$\begin{aligned}
\lambda_b \alpha_1^2 - \sigma_b \alpha_2^2 &= \lambda_a,\\
\lambda_b \beta_1^2 - \sigma_b \beta_2^2 &= -\sigma_a,\\
\lambda_b \alpha_1 \beta_1 - \sigma_b \alpha_2 \beta_2 &= 0,\\
\alpha_1 \beta_2 - \alpha_2 \beta_1 &= 1.
\end{aligned}$$

We are going to demonstrate that one of these equations is deduced from the other three. Specifically, we show that the second equation is deduced from the other three equations. The third and fourth equations may be rewritten in matrix form as

$$\begin{bmatrix} \lambda_b \alpha_1 & -\sigma_b \alpha_2 \\ -\alpha_2 & \alpha_1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solving, we get

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \frac{1}{\lambda_a} \begin{bmatrix} \sigma_b \alpha_2 \\ \lambda_b \alpha_1 \end{bmatrix}, \tag{18}$$

where we used the first equation to write the determinant of the matrix in the left as λ_a. Substituting for β_1, β_2 in the left-hand side of the second equation and using the first equation and (15) we get

$$\begin{aligned}
\lambda_b \left(\frac{\sigma_b}{\lambda_a}\alpha_2\right)^2 - \sigma_b \left(\frac{\lambda_b}{\lambda_a}\alpha_1\right)^2 &= -\frac{\lambda_b \sigma_b}{\lambda_a^2}\left(\lambda_b \alpha_1^2 - \sigma_b \alpha_2^2\right)\\
&= -\frac{\lambda_b \sigma_b}{\lambda_a} = -\frac{\lambda_a \sigma_a}{\lambda_a} = -\sigma_a.
\end{aligned}$$

Thus we only need to solve the system consisting of the first, third and fourth equations. The set of all solutions of the first equation is given parametrically by

$$\alpha_1 = \sqrt{\frac{\lambda_a}{\lambda_b}} \cosh t, \quad \alpha_2 = \sqrt{\frac{\lambda_a}{\sigma_b}} \sinh t.$$

Substituting in (18) we get

$$\beta_1 = \frac{1}{\sqrt{\lambda_a \sigma_b}} \sinh t, \quad \beta_2 = \frac{1}{\sqrt{\lambda_a \lambda_b}} \cosh t.$$

For the particular choice $t = 0$, we obtain that the functions

$$\varphi_1 = \psi_1 + \sqrt{\frac{\lambda_a}{\lambda_b}} \psi_3,$$
$$\varphi_2 = \psi_2 + \frac{1}{\sqrt{\lambda_a \lambda_b}} \psi_4$$

define a Type I domain with coupled boundary condition (17). □

LEMMA 17. *We have*

(1) *If* $\delta = 1$ *then either there is a real linear combination* η_1 *of* ψ_1, ψ_2 *such that* $\eta_1, \psi_3, \psi_4 \in \tilde{D} \bmod D_0$ *or there is a real linear combination* η_2 *of* ψ_3, ψ_4 *such that* $\eta_2, \psi_1, \psi_2 \in \tilde{D} \bmod D_0$. *In the first case, if* η_1, ψ_1 *are linearly independent then* η_1 *can be chosen such that* $[\eta_1, \psi_1](x) = 1$, *for* x *near* a *and if* η_1, ψ_2 *are linearly independent then* η_1 *can be chosen such that* $[\eta_1, \psi_2](x) = -1$, *for* x *near* a. *A similar conclusion holds in the second case.*

(2) *If* $\delta = 2$ *then there is a real linear combination* η_1 *of* ψ_1, ψ_2 *and a real linear combination* η_2 *of* ψ_3, ψ_4 *such that* $\eta_1, \eta_2 \in \tilde{D} \bmod D_0$. *Furthermore, if* η_1, ψ_1 *are linearly independent then* $[\eta_1, \psi_1](x) = 1$, *for* x *near* a *and if* η_1, ψ_2 *are linearly independent then* $[\eta_1, \psi_2](x) = -1$, *for* x *near* a. *A similar conclusion holds for* η_2.

Proof. To show Part 1 suppose $\delta = 1$. Then, since $\tilde{D}$ contains a Type I domain (with separated boundary conditions) we can find real functions $\eta_1, \eta_2 \in \tilde{D} \bmod D_0$ such that

$$\eta_1 = \alpha_1 \psi_1 + \alpha_2 \psi_2,$$
$$\eta_2 = \beta_3 \psi_3 + \beta_4 \psi_4.$$

Since $\dim(\tilde{D} \bmod D_0) = 3$, one more linear combination η_3 of the ψ-functions must belong to $\tilde{D}$. Write

$$\eta_3 = \widehat{\alpha}_1 \psi_1 + \widehat{\alpha}_2 \psi_2 + \widehat{\beta}_3 \psi_3 + \widehat{\beta}_4 \psi_4$$
$$= \zeta_1 + \zeta_2,$$

where $\zeta_1 = \widehat{\alpha}_1 \psi_1 + \widehat{\alpha}_2 \psi_2$ and $\zeta_2 = \widehat{\beta}_3 \psi_3 + \widehat{\beta}_4 \psi_4$. Since $\{\eta_3, \eta_3\}(a)$ and $\{\eta_3, \eta_3\}(b)$ are finite, $\zeta_1, \zeta_2 \in \tilde{D}$. Again, since $\dim(\tilde{D} \bmod D_0) = 3$, the functions $\eta_1, \eta_2, \zeta_1, \zeta_2$ cannot all be linearly independent modulo D_0. Thus, a non-trivial linear combination

$$\theta_1 \eta_1 + \theta_2 \eta_2 + \theta_3 \zeta_1 + \theta_4 \zeta_2 = \zeta_0 \in D_0.$$

https://doi.org/10.1017/S0017089509005060 Published online by Cambridge University Press

Since the functions $\psi_1, \psi_2, \psi_3, \psi_4$ are linearly independent modulo D_0, we must have $\zeta_0 = 0$ and

$$\begin{aligned} \theta_1\alpha_1 + \theta_3\widehat{\alpha}_1 &= 0, \\ \theta_1\alpha_2 + \theta_3\widehat{\alpha}_2 &= 0, \\ \theta_2\beta_3 + \theta_4\widehat{\beta}_3 &= 0, \\ \theta_2\beta_4 + \theta_4\widehat{\beta}_4 &= 0. \end{aligned}$$

This system can be written as

$$\begin{bmatrix} \alpha_1 & \widehat{\alpha}_1 \\ \alpha_2 & \widehat{\alpha}_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_3 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} \beta_3 & \widehat{\beta}_4 \\ \beta_4 & \widehat{\beta}_4 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_4 \end{bmatrix} = 0.$$

For a non-trivial solution, at least one of the coefficient matrices must be singular. Observe that if both matrices are singular, then $\zeta_1 = \gamma_1\eta_1$ and $\zeta_2 = \gamma_2\eta_2$. In this case $\eta_3 = \gamma_1\eta_1 + \gamma_2\eta_2$, which is a contradiction. Therefore, exactly one of the two coefficient matrices given above must be singular. If $\left[\begin{smallmatrix} \alpha_1 & \widehat{\alpha}_1 \\ \alpha_2 & \widehat{\alpha}_2 \end{smallmatrix}\right]$ is singular then $\eta_1, \psi_3, \psi_4 \in \tilde{D} \bmod D_0$ and if $\left[\begin{smallmatrix} \beta_3 & \widehat{\beta}_4 \\ \beta_4 & \widehat{\beta}_4 \end{smallmatrix}\right]$ is singular then $\eta_2, \psi_1, \psi_2 \in \tilde{D} \bmod D_0$. The rest of Part 1 can be proven in the same way as in Lemma 13.

To show Part 2, suppose $\delta = 2$. Then, since $\tilde{D}$ contains a Type I domain (with separated boundary conditions) we can find real functions $\eta_1, \eta_2 \in \tilde{D} \bmod D_0$ such that

$$\begin{aligned} \eta_1 &= \alpha_1\psi_1 + \alpha_2\psi_2, \\ \eta_2 &= \beta_3\psi_3 + \beta_4\psi_4. \end{aligned}$$

The rest of Part 2 can be proven in the same way as in Lemma 13. □

Lemma 17 means that the functions $\psi_1, \psi_2, \psi_3, \psi_4$ in Lemmas 2 and 3 can be chosen such that (a) in the case $\delta = 1$, $\psi_1, \psi_2, \psi_3 \in \tilde{D} \bmod D_0$ or $\psi_2, \psi_3, \psi_4 \in \tilde{D} \bmod D_0$ and (b) in the case $\delta = 2$, $\psi_1, \psi_3 \in \tilde{D} \bmod D_0$. These choices will always be assumed in the sequel.

THEOREM 18. *All Type I domains are characterized as follows.*

(1) *If $\delta = 2$ then there is only one Type I domain, namely*

$$D_1 = D_0 \dotplus \operatorname{span}\{\psi_1, \psi_3\}.$$

(2) *If $\delta = 1$ then Type I domains (necessarily with separated boundary condition (16)) form a one-parameter family given by*

$$D_1(\theta) = D_0 \dotplus \operatorname{span}\left\{\psi_1 + e^{i\theta}\sqrt{\frac{\lambda_a}{\sigma_a}}\psi_2, \psi_3\right\}, \quad \theta \in \{0, \pi\},$$

if $\psi_1, \psi_2, \psi_3 \in \tilde{D} \bmod D_0$ or

$$D_1(\theta) = D_0 \dotplus \operatorname{span}\left\{\psi_2, \psi_3 + e^{i\theta}\sqrt{\frac{\lambda_a}{\sigma_a}}\psi_4\right\}, \quad \theta \in \{0, \pi\},$$

if $\psi_2, \psi_3, \psi_4 \in \tilde{D} \bmod D_0$.

(3) *If* $\delta = 0$ *then*

(a) *Type I domains with separated boundary condition (16) form a two-parameter family given by*

$$D_1(\theta_1, \theta_2) = D_0 \dotplus \operatorname{span}\left\{\psi_1 + e^{i\theta_1}\sqrt{\frac{\lambda_a}{\sigma_a}}\psi_2,\ \psi_3 + e^{i\theta_2}\sqrt{\frac{\lambda_b}{\sigma_b}}\psi_4\right\},\ \theta_1, \theta_2 \in \{0, \pi\}.$$

(b) *Type I domains with coupled boundary condition (17), form a three-parameter family given by*

$$D_1(t, \theta_1, \theta_2) = D_0 \dotplus \operatorname{span}\{\psi_1 + \alpha_1\psi_3 + \alpha_2\psi_4,\ \psi_2 + \beta_1\psi_3 + \beta_2\psi_4\},$$

where

$$\begin{array}{ll} \alpha_1 = e^{i\theta_1}\sqrt{\frac{\lambda_a}{\lambda_b}}\cosh t, & \alpha_2 = e^{i\theta_2}\sqrt{\frac{\lambda_a}{\sigma_b}}\sinh t, \\ \beta_1 = \frac{1}{\sqrt{\lambda_a\sigma_b}}e^{-i\theta_2}\sinh t, & \beta_2 = \frac{1}{\sqrt{\lambda_a\lambda_b}}e^{-i\theta_1}\cosh t, \end{array} \tag{19}$$

$t \geq 0, \theta_1, \theta_2 \in \{0, \pi\}$.

Proof. Part 1 is shown in the same way as in Theorem 14 with minor modifications.

To prove Part 2, assume $\delta = 1$ and that $\psi_1, \psi_2, \psi_3 \in \tilde{D} \bmod D_0$. Let D_1 be a Type I domain and choose $\eta_1, \eta_2 \in D_1 \bmod D_0$. By Proposition 16, D_1 has separated boundary condition (16). Then by Lemma 4 we can write

$$\begin{aligned} \eta_1 &= \alpha_1\psi_1 + \alpha_2\psi_2, \\ \eta_2 &= \psi_3. \end{aligned}$$

The condition $\{\eta_1, \eta_1\}(a) = 0$ gives

$$\lambda_a |\alpha_1|^2 + 2i \operatorname{Im}(\alpha_1\bar{\alpha}_2) - \sigma_a |\alpha_2|^2 = 0,$$

from which we get the one-parameter family of solutions

$$\alpha_2 = e^{i\theta}\sqrt{\frac{\lambda_a}{\sigma_a}}\alpha_1,\ \theta \in \{0, \pi\}.$$

Part 3(a) can be shown in exactly the same way as Part 2.

To show Part 3(b), assume $\delta = 0$. Let D_1 be a Type I domain and choose $\varphi_1, \varphi_2 \in D_1 \bmod D_0$. Then by Lemma 4 we can write

$$\begin{aligned} \varphi_1 &= \psi_1 + \alpha_1\psi_3 + \alpha_2\psi_4, \\ \varphi_2 &= \psi_2 + \beta_1\psi_3 + \beta_2\psi_4. \end{aligned}$$

We proceed as in the proof of Proposition 16 to obtain the system of equations

$$\begin{aligned} &\lambda_b |\alpha_1|^2 + 2i \operatorname{Im}(\alpha_1\bar{\alpha}_2) - \sigma_b |\alpha_2|^2 = \lambda_a, \\ &\beta_1 = \tfrac{\sigma_b}{\lambda_a}\overline{\alpha}_2, \quad \beta_2 = \tfrac{\lambda_b}{\lambda_a}\overline{\alpha}_1. \end{aligned}$$

It is straightforward to see that the solutions of this system are given by equations (19). □

https://doi.org/10.1017/S0017089509005060 Published online by Cambridge University Press

The following simple examples illustrate the various situations of Theorems 14 and 18.

(1) Let $I = (0, \infty)$, $w(x) = 1$, $p(x) = x$ and $g(x) = 0$. In this case

$$\theta(x) = 1,$$
$$\varphi(x) = -\ln x.$$

The end point 0 is LC and the end point ∞ is LP. Forming the functions ψ_1, ψ_2 as in Lemma 3, we see that $\psi_1 \in \tilde{D}$, $\psi_2 \notin \tilde{D}$. Therefore, $\delta = 1$ and by Part 1 of Theorem 14 there is only one Type I domain D_1. Any $u \in D_1$ has the form

$$u = u_0 + c\psi_1, \ u_0 \in D_0, \ c \in \mathbb{R}.$$

Since 0 is a regular point, we know (see [**6**]) that $u_0(0) = u_0'(0) = 0$. Since $\psi_1 \equiv 1$ near 0 we see that $u(0)$ is finite and $u'(0) = 0$. Near ∞ we have $u = u_0$, hence the behaviour of u at ∞ is completely determined by the behaviour of u_0 there. The condition $u_0u_0' \to 0$ at infinity implies that $(u_0^2)' \to 0$ at ∞. Hence, $u_0^2(\infty)$ exists (which could be infinite). Together with the requirement that $u_0 \in L^2(I)$, we must have $u_0(\infty) = 0$. Therefore, $u(\infty) = 0$ for any $u \in D_1$. (This is also true for any $u \in D$.) Nothing can be asserted about $u_0'(\infty)$ since, for example, any function $u_0 \in D_0$ which is identical with $\frac{\sin x^2}{x}$ near ∞ does not have a derivative limit at ∞. Thus, D_1 is described by the boundary condition $u'(0) = 0$.

(2) Let $I = (0, \infty)$, $w(x) = 1$, $p(x) = 1$ and $g(x) = 1$. In this case

$$\theta(x) = \cosh x,$$
$$\varphi(x) = -\sinh x.$$

The end point 0 is regular and the end point ∞ is LP. Forming the functions ψ_1 and ψ_2 as the $\widetilde{\psi}$-versions of those in Lemma 3, we see that $\psi_1, \psi_2 \in \tilde{D}$. Therefore, $\delta = 0$ and Part 2 of Theorem 14 applies. In this case

$$\lambda_a = \sigma_a = \frac{1}{2},$$
$$C_a = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

and

$$\psi_1 \equiv \frac{1}{\sqrt{2}}\{\cosh x - \sinh x\},$$
$$\psi_2 \equiv \frac{1}{\sqrt{2}}\{\cosh x + \sinh x\},$$
$$\psi_1 + \sqrt{\frac{\lambda_a}{\sigma_a}}\psi_2 \equiv \frac{2}{\sqrt{2}}\cosh x,$$
$$\psi_1 - \sqrt{\frac{\lambda_a}{\sigma_a}}\psi_2 \equiv -\frac{2}{\sqrt{2}}\sinh x$$

https://doi.org/10.1017/S0017089509005060 Published online by Cambridge University Press

near 0. Following similar reasoning as in Example 1 we see that the two Type I domains D_1, D_2 of Part 2 of Theorem 14 are described by the boundary conditions $u'(0) = 0$ and $u(0) = 0$, respectively.

(3) Let $I = (-1, 1)$, $w(x) = 1$, $p(x) = (1 - x^2)$ and $g(x) = 0$. In this case

$$\theta(x) = 1,$$
$$\varphi(x) = \frac{1}{2}\log\frac{1-x}{1+x}.$$

Both end points are LC and, forming the functions $\psi_1, \psi_2, \psi_3, \psi_4$ as in Lemma 3, we see that $\psi_1, \psi_3 \in \tilde{D}$ while $\psi_2, \psi_4 \in D \bmod \tilde{D}$. Therefore, $\delta = 2$ and by Part 1 of Theorem 18 there is only one Type I domain $D_1 = D_0 \dot{+} \operatorname{span}\{\psi_1, \psi_3\}$. For any $u \in D_1$, $u = u_0 + c$ near -1 for some $u_0 \in D_0$ and some scalar c. The condition $[u_0, \psi_1]_{-1}^{1} = 0$ yields $u_0^{[1]}(-1) = 0$. Therefore, $u^{[1]}(-1) = 0$. Similarly we conclude that $u^{[1]}(1) = 0$. Therefore, D_1 is described by the boundary conditions $u^{[1]}(-1) = u^{[1]}(1) = 0$.

(4) If in Example 3 we take $I = (0, 1)$, then (the $\widetilde{\psi}$-versions) of $\psi_1, \psi_2, \psi_3 \in \tilde{D}$ and $\psi_4 \notin \tilde{D}$. Therefore, $\delta = 1$ and Part 2 of Theorem 18 applies. Here we have

$$\lambda_a = \sigma_a = \frac{1}{2}$$

and

$$\psi_1 \equiv \frac{1}{\sqrt{2}}\left\{1 + \frac{1}{2}\log\frac{1-x}{1+x}\right\},$$
$$\psi_2 \equiv \frac{1}{\sqrt{2}}\left\{1 - \frac{1}{2}\log\frac{1-x}{1+x}\right\},$$

near 0. For $u \in D(\theta)$, $u = u_0 + c(\psi_1 + e^{i\theta}\sqrt{\lambda_a/\sigma_a}\,\psi_2)$ for some $u_0 \in D_0$ and some scalar c. Using the fact that 0 is a regular point and a straightforward calculation we can show that

$$i\sin\frac{\theta}{2}\,u(0) - \cos\frac{\theta}{2}\,u^{[1]}(0) = 0.$$

Also, using the reasoning as in Example 3 we can show that

$$u^{[1]}(1) = 0.$$

These are the two boundary conditions determining $D(\theta)$. Observe that there are two real Type I domains (corresponding to $\theta = 0$ and $\theta = \pi$) described by the boundary conditions

$$u^{[1]}(0) = u^{[1]}(1) = 0$$

and

$$u(0) = u^{[1]}(1) = 0.$$

(5) Let $I = (0, 1)$, $w(x) = 1$, $p(x) = \sqrt{x}$ and $g(x) = 0$. In this case

$$\theta(x) = 1,$$
$$\varphi(x) = 2 - 2\sqrt{x}.$$

Both end points are regular and, forming the functions $\psi_1, \psi_2, \psi_3, \psi_4$ as in Lemma 3, we see that $\psi_1, \psi_2, \psi_3, \psi_4 \in \tilde{D}$. Therefore, $\delta = 0$ and Part 3 of Theorem 18 applies. There are four real Type I operators with separated boundary conditions corresponding to the values of $\theta_1, \theta_2 \in \{0, \pi\}$. The expressions in Part 3(a) of Theorem 18 reduce to the functions $1, \sqrt{x}$ near 0 and 1 in this case. A straightforward computation then shows that the domains of these operators are described by the boundary conditions

$$u^{[j]}(0) = 0, \ u^{[k]}(1) = 0, \ j, k \in \{0, 1\}.$$

Observe that the case $j = k = 0$ gives the Friedrichs extension. To discuss Part 3(b) we found it more convenient to start with the solutions

$$\theta(x) = 1,$$
$$\varphi(x) = 2 - 2\sqrt{x}.$$

Here we have

$$C_a = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad C_b = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -2 \end{bmatrix},$$
$$\lambda_a = \sigma_a = \frac{1}{2},$$
$$\lambda_b = \frac{-2 + \sqrt{5}}{2}, \quad \sigma_b = \frac{2 + \sqrt{5}}{2},$$

and

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{2}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sqrt{x},$$

near 0 and

$$\begin{bmatrix} \psi_3 \\ \psi_4 \end{bmatrix} = \frac{1}{\sqrt{10 - 4\sqrt{5}}} \begin{bmatrix} 1 \\ -2 + \sqrt{5} \end{bmatrix} - \frac{2}{\sqrt{10 + 4\sqrt{5}}} \begin{bmatrix} 1 \\ 2 + \sqrt{5} \end{bmatrix} \sqrt{x},$$

near 1. As an illustration we consider the case $\theta_1 = \theta_2 = t = 0$. In this case

$$\alpha_1 = \sqrt{2 + \sqrt{5}}, \ \beta_2 = 2\sqrt{2 + \sqrt{5}}, \ \alpha_2 = \beta_1 = 0.$$

A tedious but straightforward calculation gives the following boundary conditions description of the domain D_1 corresponding to this case:

$$\begin{bmatrix} u(1) \\ u^{[1]}(1) \end{bmatrix} = \sqrt{\frac{3}{2}} \begin{bmatrix} -3\sqrt{5} - 6 & -5\sqrt{5} - 6 \\ 3 + \sqrt{5} & 1 + \sqrt{5} \end{bmatrix} \begin{bmatrix} u(0) \\ u^{[1]}(0) \end{bmatrix}.$$

ACKNOWLEDGEMENT. The first author wishes to acknowledge the excellent research facilities provided by King Fahd University of Petroleum & Minerals.

REFERENCES

1. M. A. El-Gebeily and K. M. Furati, Real self-adjoint Sturm–Liouville problems, *Appl. Anal.* **83**(4) (2004), 377–387.

2. W. N. Everitt, M. Giertz and J. B. McLeod, On the strong and weak limit-point classification of second-order differential expressions, *Proc. London Math. Soc.* **29** (1974), 142–158.

3. T. Kato, Perturbation in *Theory for linear operators* (Springer Verlag, Heidelberg, Berlin, 1995).

4. A. M. Krall and A. Zettl, Singular self-adjoint Sturm–Liouville problems, *Diff. Integral Eq.* **1**(4) (1988), 423–432.

5. M. Marletta and A. Zettl, The Friedrichs extension of singular differential operators, *JDE* **160** (2001), 404–421.

6. M. A. Naimark, *Linear differential operators*: Part II (Ungar, New York, 1968).

7. H.-D. Niessen and A. Zettl, The Friedrichs extension of regular ordinary differential operators, *Proc. Roy. Soc. Edinburgh Sect. A* **114** (1990), 229–236.

8. D. O'Regan and M. El-Gebeily, Existence, upper and lower solutions and quasilinearization for singular differential equations, *IMA J. Appl. Math.* **73** (2008), 323–344.

9. J. Weidmann, *Spectral theory of ordinary differential operators*, Lecture Notes in Mathematics, vol. 1258 (Springer, Heidelberg, Germany, 1987).

10. A. Zettl, Sturm–Liouville problems, in *Spectral theory and computational methods of Sturm–Liouville problems* (D. Hinton and P. Schaefer, Editors), Pure and Applied Mathematics (Marcel Dekker, New York, 1997), 1–104.