BULL. AUSTRAL. MATH. SOC. 53C40, 53C42
VOL. 73 (2006) [9–16]

COMPLETE SPACELIKE HYPERSURFACES IN A DE SITTER SPACE

SHU SHICHANG

In this paper, we characterise the n-dimensional ($n \geqslant 3$) complete spacelike hypersurfaces M^n in a de Sitter space S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures. We show that if the multiplicities of such principal curvatures are greater than 1, then M^n is isometric to $H^k(\sinh r) \times S^{n-k}(\cosh r)$, $1 < k < n-1$. In particular, when M^n is the complete spacelike hypersurfaces in S_1^{n+1} with the scalar curvature and the mean curvature being linearly related, we also obtain a characteristic Theorem of such hypersurfaces.

1. INTRODUCTION

Let R_1^{n+2} be the $(n+2)$-dimensional Lorentz-Minkowski space, that is, the real vector space R^{n+2} endowed with Lorentzian metric $\langle , \rangle$ given by

$$\langle p, q\rangle = -p_0q_0 + p_1q_1 + \cdots + p_{n+1}q_{n+1},$$

for $p, q \in R^{n+2}$. We define the de Sitter space by

$$S_1^{n+1} = \{p \in R_1^{n+2} \mid \langle p, p\rangle = 1\}.$$

In this way, S_1^{n+1} is a Lorentz manifold with constant sectional curvature 1. A hypersurface M^n of S_1^{n+1} is said to be spacelike if the induced metric on M^n from that of ambient space is positive definite. There are many interesting results in the study of spacelike hypersurfaces with constant mean curvature, one can see [**6, 2, 13, 9**]. It is well-known that the investigation on spacelike hypersurfaces with constant scalar curvature is also important and interesting. Zheng [**14, 15**] proved that when M^n is an n-dimensional compact spacelike hypersurface with constant scalar curvature immersed in a de Sitter space S_1^{n+1}, if $K \geqslant 0$ and $R \leqslant 1$, then M^n is totally umbilical and is isometric to a sphere, where K and $n(n-1)R$ are the sectional curvature and the scalar curvature of M^n, respectively. If M^n is the complete spacelike hypersurface with constant scalar curvature, Liu [**8**] recently obtained a characteristic Theorem of such hypersurfaces in terms of the squared norm of the second fundamental form of M^n. In [**10**], Nomizu classified

Received 21st April, 2005
Project supported by NNSF of China(No.69972036) and NSF of Shaanxi Province(No.2003A02)

https://doi.org/10.1017/S0004972700038570 Published online by Cambridge University Press

the isoparametric hypersurfaces in a de Sitter space. He proved that these hypersurfaces are totally umbilical or have two distinct principal curvatures. Hence, it is natural and important to investigate the spacelike hypersurfaces with two distinct principal curvatures. In this paper, we characterise all complete spacelike hypersurfaces of constant scalar curvature with two distinct principal curvatures whose multiplicities are greater than 1.

In order to represent our theorems, we need some notations (see [7] or [8]). The well-known complete spacelike hypersurfaces with constant mean curvature are given by

$$M^n = \{p \in S_1^{n+1} \mid p_{k+1}^2 + \cdots + p_{n+1}^2 = \cosh^2 r\},$$

with $r \in R^1$ and $1 \leqslant k \leqslant n$, where R^1 is the set of all real numbers. We can prove that M^n is isometric to the Riemannian product $H^k(\sinh r) \times S^{n-k}(\cosh r)$ of a k-dimensional hyperbolic space and a $(n-k)$-dimensional sphere of radii $\sinh r$ and $\cosh r$, respectively. M^n has k principal curvatures equal to $\coth r$ and $(n-k)$ principal curvatures equal to $\tanh r$, so the mean curvature is given by

$$nH = k \coth r + (n-k)\tanh r.$$

If $k = 1$, the Riemannian product $H^1(\sinh r) \times S^{n-1}(\cosh r)$ is called a hyperbolic cylinder.We obtain the following:

Theorem 1.1. *Let M^n be an n-dimensional complete spacelike hypersurface in S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures. If the multiplicities of these two distinct principal curvatures are greater than 1, then M^n is isometric to the Riemannian product $H^k(\sinh r) \times S^{n-k}(\cosh r), 1 < k < n-1$.*

Let M^n be the complete spacelike hypersurfaces in S_1^{n+1} with scalar curvature $n(n-1)R$ and the mean curvature H being linearly related, that is, we may assume $n(n-1)R = k'H (k' =$ constant $\geqslant 0)$. Cheng [4] proved that if the sectional curvatures are nonnegative and H obtains its maximum on M, then M^n is isometric to an Euclidean space R^n or a sphere $S^n(c_1), 0 < c_1 < 1$. In this paper, we also obtain a characteristic Theorem of such hypersurfaces in terms of H. We have the following:

Theorem 1.2. *Let M^n be an n-dimensional $(n \geqslant 3)$ complete spacelike hypersurface with $n(n-1)R = k'H$ in a de Sitter space S_1^{n+1}, where k' is a positive constant. If the mean curvature H is non-negative and obtains its maximum on M^n, then*

(1) *If $H^2 < \big(4(n-1)\big)/n^2$ on M^n, then M^n is totally umbilical.*
(2) *If $H^2 = \big(4(n-1)\big)/n^2$ on M^n, then M^n is totally umbilical, or M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$.*
(3) *If $\big(4(n-1)/n^2\big) < H^2 \leqslant 1$ on M^n and the squared norm of the second fundamental form $|h|^2$ satisfies $|h|^2 \leqslant nH^2 + (B_H^-)^2$ or $|h|^2 \geqslant nH^2 + (B_H^+)^2$ on M^n, then M^n is totally umbilical, or M^n is isometric to a hyperbolic*

cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$, *where* $B_H^{\pm}$ *are the two real roots of the polynomial*

$$P_H(x) = x^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}x + n(1-H^2).$$

2. Preliminaries

Let M^n be an n-dimensional spacelike hypersurfaces in S_1^{n+1}. We choose a local field of semi-Riemannian orthonormal frames $e_1, \cdots, e_{n+1}$ in S_1^{n+1} such that at each point of $M^n, e_1, \cdots, e_n$ span the tangent space of M^n and form an othonormal frame there. We use the following convention on the range of indices:

$$1 \leqslant A, B, C, \cdots \leqslant n+1; \quad 1 \leqslant i, j, k, \cdots \leqslant n.$$

Let $\omega_1, \cdots, \omega_{n+1}$ be the dual frame field so that the semi-Riemannian metric of S_1^{n+1} is given by $d\overline{s}^2 = \sum_i \omega_i^2 - \omega_{n+1}^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1$ and $\varepsilon_{n+1} = -1$.

The structure equations of S_1^{n+1} are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \tag{2.2}$$

where

$$\Omega_{AB} = -\frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.3}$$

$$K_{ABCD} = \varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.4}$$

Restricting these forms to M^n, we have

$$\omega_{n+1} = 0. \tag{2.5}$$

Cartan's Lemma implies that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.6}$$

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.8}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.9}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \tag{2.10}$$

is the second fundamental form of M^n.
From the above equation, we have

$$n(n-1)R = n(n-1) - n^2H^2 + |h|^2, \tag{2.11}$$

where $n(n-1)R$ is the scalar curvature of M^n, H is the mean curvature, and $|h|^2 = \sum_{i,j} h_{ij}^2$ is the squared norm of the second fundamental form of M^n.

The Codazzi equation and the Ricci identity are

$$h_{ijk} = h_{ikj}, \tag{2.12}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{2.13}$$

where h_{ijk} and h_{ijkl} denote the first and the second covariant derivatives of h_{ij}.

3. Proof of Theorems

In order to prove Theorem 1.1,we firstly need the following Proposition 3.1 from [**7**], (originally due to Otsuki [**12**]) for Riemannian space forms.

Proposition 3.1. ([**7**]) *Let M^n be a spacelike hypersurface in S_1^{n+1} such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of principal vectors corresponding to each principal curvature λ is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

From Proposition 3.1, it is easy to prove Theorem 1.1.

Proof of Theorem 1.1: Let λ, μ be the principal curvatures of multiplicities k and $n-k$ respectively, where $1 < k < n-1$. By (2.11) we have

$$n(n-1)(1-R) = k(k-1)\lambda^2 + 2k(n-k)\lambda\mu + (n-k)(n-k-1)\mu^2. \tag{3.1}$$

Denote by D_λ and D_μ the integral submanifolds of the corresponding distribution of the space of principal vectors corresponding to the principal curvature λ and μ, respectively. From Proposition 3.1, we know that λ is constant on D_λ. Since the scalar curvature is constant, (3.1) implies that μ is constant on D_λ. Making use of Proposition 3.1 again, we have μ is constant on D_μ. Therefore, we know that μ is constant on M^n. By the same assertion we know that λ is constant on M^n. Therefore M^n is isoparametric. By the congruence theorem in [1], M^n is isometric to $H^k(\sinh r) \times S^{n-k}(\cosh r), 1 < k < n-1$. This completes the proof of Theorem 1.1.

In order to prove Theorem 1.2, we introduce an operator $\Box$ due to Cheng and Yau [5] by

$$\Box f = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}, \tag{3.2}$$

where f is a C^2-function on M^n, the gradient and Hessian (f_{ij}) are defined by

$$df = \sum_i f_i\omega_i, \quad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}. \tag{3.3}$$

The Laplacian of f is defined by $\Delta f = \sum_i f_{ii}$.

We choose a local frame field $e_1, \cdots, e_n$ at each point of M^n, such that $h_{ij} = \lambda_i\delta_{ij}$. From (3.2) and (2.11), we have

$$\begin{aligned} \Box(nH) &= nH\Delta(nH) - \sum_i \lambda_i(nH)_{ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i(nH)_{ii} \\ &= -\frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta|h|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{ii}. \end{aligned} \tag{3.4}$$

From (2.12) and (2.13), by a standard and direct calculation, we have

$$\frac{1}{2}\Delta|h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i(nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2, \tag{3.5}$$

where $R_{ijij} = 1 - \lambda_i\lambda_j (i \neq j)$ denotes the sectional curvature of the section spanned by $\{e_i, e_j\}$.

From (3.4) and (3.5), we get

$$\Box(nH) = -\frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j}(1-\lambda_i\lambda_j)(\lambda_i - \lambda_j)^2. \tag{3.6}$$

We need the following Lemma 3.2 original due to Cheng [4].

LEMMA 3.2. *Let M^n be an n-dimensional spacelike hypersurface in a de Sitter space S_1^{n+1} with $n(n-1)R = k'H$ ($k' =$ constant > 0). If the mean curvature $H \geqslant 0$, then the operator*

$$L = \Box + (k'/2n)\Delta$$

is elliptic and $R > 0, H > 0$.

The proof of Lemma 3.2 is similar to that of Proposition 1 in [4], thus will be omitted.

We also need the following algebraic Lemma due to [**11**] and [**3**].

LEMMA 3.3. *Let $\mu_i, i = 1, \cdots, n$ be real numbers, with $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2 \geqslant 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leqslant \sum_i \mu_i^3 \leqslant \frac{n-2}{\sqrt{n(n-1)}}\beta^3. \tag{3.7}$$

and equality holds if and only if either $(n-1)$ of the numbers μ_i are equal to $\beta/\sqrt{n(n-1)}$ or $(n-1)$ of the numbers μ_i are equal to $-\beta/\sqrt{n(n-1)}$.

□

PROOF OF THEOREM 1.2: From (3.6) we have

$$\begin{aligned} nLH &= n[\Box H + (k'/2n)\Delta H] \\ &= \Box(nH) + (1/2)\Delta[n(n-1)R] \\ &= |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j}(1-\lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 \\ &= |\nabla h|^2 - n^2|\nabla H|^2 + n|h|^2 - n^2H^2 + |h|^4 - nH\sum_i \lambda_i^3. \end{aligned} \tag{3.8}$$

We choose a orthonormal frame field $e_1, e_2, \cdots, e_n$ at each point in M^n such that $h_{ij} = \lambda_i\delta_{ij}$. Then we have $|h|^2 = \sum_{i,j} h_{ij}^2 \neq 0$. In fact, if $|h|^2 = \sum_i \lambda_i^2 = 0$ at a point of M^n, then $\lambda_i = 0 (i = 1, 2, \cdots, n)$ at this point. Therefore $H = 0$ and $R = 0$ at this point. From (2.11) we have $n(n-1) = 0$. This is impossible.

From (2.11) and $n(n-1)R = k'H$, we have

$$k' \nabla_i H = -2n^2 H \nabla_i H + 2\sum_{j,k} h_{kj}h_{kji},$$

$$\left(\frac{k'}{2} + n^2H\right)^2 |\nabla H|^2 = \sum_i \left(\sum_{j,k} h_{kj}h_{kji}\right)^2 \leqslant \sum_{i,j} h_{ij}^2 \sum_{i,j,k} h_{ijk}^2 = |h|^2|\nabla h|^2.$$

Therefore, we have

$$\begin{aligned} |\nabla h|^2 - n^2|\nabla H|^2 &\geqslant \left[\left(\frac{k'}{2} + n^2H\right)^2 - n^2|h|^2\right]|\nabla H|^2\frac{1}{|h|^2} \\ &= \left[\frac{(k')^2}{4} + n^3(n-1)\right]|\nabla H|^2\frac{1}{|h|^2} \geqslant 0. \end{aligned} \tag{3.9}$$

Let $|g|^2$ be a nonnegative C^2-function on M^n defined by $|g|^2 = |h|^2 - nH^2$. Since $\sum_i (H - \lambda_i) = 0, \sum_i (H-\lambda_i)^2 = |h|^2 - nH^2 = |g|^2$, by Lemma 3.3 we get

$$\begin{aligned} -nH\sum_i \lambda_i^3 &= -3nH^2|h|^2 + 2n^2H^4 + nH\sum_i (H-\lambda_i)^3 \\ &\geqslant -3nH^2|g|^2 - n^2H^4 - nH\frac{n-2}{\sqrt{n(n-1)}}|g|^3. \end{aligned} \tag{3.10}$$

Therefore, from (3.8), (3.9) and (3.10), we have

$$\begin{aligned} nLH &\geqslant |g|^2\Big\{|g|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|g| + n - nH^2\Big\} \\ &= |g|^2 P_H(|g|), \end{aligned} \tag{3.11}$$

where

$$P_H(|g|) = |g|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|g| + n(1-H^2). \tag{3.12}$$

The discriminant of $P_H(|g|)$ is $\big(n/(n-1)\big)\big(n^2H^2 - 4(n-1)\big)$.

(1) If $H^2 < \big(4(n-1)\big)/n^2$ on M^n, then $P_H(|g|) > 0$ on M^n and the right-hand side of (3.11) is nonnegative. Since the operator L is elliptic and H obtains its maximum on M^n, from (3.11) we know that $H = const.$ on M^n. From (3.11) again, we get $|g|^2P_H(|g|) = 0$, so $|g|^2 = 0$ and M^n is totally umbilical.

(2) If $H^2 = \big(4(n-1)\big)/n^2$ on M^n, then $P_H(|g|) = \big(|g| - (n-2)/\sqrt{n}\big)^2 \geqslant 0$ on M^n. We have from (3.11), $|g|^2P_H(|g|) = 0$. Hence, we know that $|g|^2 = 0$ and M^n is totally umbilical; or $P_H(|g|) = 0$.

If $P_H(|g|) = 0$, then $|g| = (n-2)/\sqrt{n}$. From (3.11) the equality holds in Lemma 3.3. Therefore we know that $(n-1)$ of the numbers $H - \lambda_i$ are equal to

$$\frac{|g|}{\sqrt{n(n-1)}} = \frac{n-2}{n\sqrt{n-1}},$$

or equal to the negative of this last expression. This implies that M^n has $(n-1)$ principal curvatures equal and constant. As H is constant, the other principal curvature is constant as well, so M^n is isoparametric. Therefore, we know that M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$ from the congruence theorem in [**1**].

(3) If $\big(4(n-1)/n^2\big) < H^2 \leqslant 1$ on M^n, then $P_H(|g|)$ has two real roots B_H^- and B_H^+ given by

$$B_H^{\pm} = \sqrt{\frac{n}{4(n-1)}}\Big[(n-2)H \pm \sqrt{n^2H^2 - 4(n-1)}\Big]. \tag{3.13}$$

Clearly, B_H^+ is always positive, and $B_H^- \geqslant 0$ if and only if $\big(4(n-1)\big)/n^2 < H^2 \leqslant 1$. Since we suppose $|h|^2 \leqslant nH^2 + (B_H^-)^2$, or $|h|^2 \geqslant nH^2 + (B_H^+)^2$ on M^n, which means $|g| \leqslant B_H^-$ or $|g| \geqslant B_H^+$ on M^n. Therefore we know that $P_H(|g|) \geqslant 0$ on M^n. Since L is elliptic and H obtains its maximum on M^n, we know that $H = const.$ on M^n from (3.11). Thus we get $|g|^2P_H(|g|) = 0$, and so $|g|^2 = 0$ and M^n is totally umbilical; or $P_H(|g|) = 0$. If $P_H(|g|) = 0$, then we have

$$|g| = B_H^-, \quad \text{or} \quad |g| = B_H^+, \tag{3.14}$$

on M^n. If $B_H^- = 0$, then we get $|g| = 0$ and M^n is totally umbilical. If $B_H^- > 0$, by (3.14) and (3.11), the equality holds in Lemma 3.3. By making use of the same assertion as in the proof of (2) above, we infer that M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$. This completes the proof of Theorem 1.2. $\square$

References

[1] N. Abe, N. Koike and S. Yamaguchi, 'Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form', *Yokohama Math. J.* **35** (1987), 123–136.

[2] K. Akutagawa, 'On spacelike hypersurfaces with constant mean curvature in a de Sitter space', *Math. Z.* **196** (1987), 13-19.

[3] H. Alencar and M.P. do Carmo, 'Hypersurfaces with constant mean curvature in spheres', *Proc. Amer. Math. Soc.* **120** (1994), 1223–1229.

[4] Q.M. Cheng, 'Complete spacelike hypersurfaces of a de Sitter space with $r = kH$', *Mem. Fac. Sci. Kyushu Univ.* **44** (1990), 67–77.

[5] S.Y. Cheng and S.T. Yau, 'Hypersurfaces with constant scalar curvature', *Math. Ann.* **225** (1977), 195–204.

[6] A.J. Goddard, 'Some remarks on the existence of spacelike hypersurfaces of constant mean curvature', *Math. Proc. Cambridge Phil. Soc.* **82** (1977), 489–495.

[7] A.B. Jr, A.G. Colares and O. Palmas, 'Complete spacelike hypersurfaces with constant mean curvature in the de Sitter space: A gap Theorem', *Illinois J. Maths.* **47** (2003), 847–866.

[8] X. Liu, 'Complete spacelike hypersurfaces with constant scalar curvature', *Manuscripta Math.* **105** (2001), 367–377.

[9] S. Montiel, 'A characterization of hyperbolic cylinders in the de Sitter space', *Tôhoku Math. J.* **48** (1996), 23–31.

[10] K. Nomizu, 'On isoparametric hypersurfaces in the Lorentzian space forms', *Japan. J. Math.* **7** (1981), 217–226.

[11] M. Okumura, 'Hypersurfaces and a pinching problem on the second fundamental tensor', *Amer. J.Math.* **96** (1974), 207–213.

[12] T. Otsuki, 'Minimal hypersurfaces in a Riemannian manifold of constant curvature', *Amer. J.Math.* **92** (1970), 145–173.

[13] J. Ramanathan, 'Complete spacelike hypersurfaces of constant mean curvature in the de Sitter space', *Indiana University Math. J.* **36** (1987), 349–359.

[14] Y. Zheng, 'Spacelike hypersurfaces with constant scalar curvature in the de Sitter spaces', *Differential Geom. Appl.* **6** (1996), 51–54.

[15] Y. Zheng, 'On spacelike hypersurfaces in the de Sitter spaces', *Ann. Global Anal. Geom.* **13** (1995), 317–321.

Department of Mathematics
Xianyang Teachers' University
Xianyang 712000
Shaanxi
Peoples Republic of China
e-mail: shushichang@126.com