ON THE SEMIGROUP OF DIFFERENTIABLE MAPPINGS

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To Bernhard Hermann Neumann on his 60th birthday

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The purpose of this paper is to improve a result in [2] on the automorphisms of the semigroup $\mathscr{D} = \mathscr{D}(E)$ of all (Fréchet)-differentiable mappings of a real Banach space E into itself.

We denote the derivative of $f \in \mathscr{D}$ at $a \in E$ by $f'(a)$. This means that $f'(a) \in \mathscr{L} = \mathscr{L}(E)$ (the Banach algebra of all continuous linear mappings of E into itself with the usual upper bound norm) and

$$\lim_{||x|| \to 0} ||x||^{-1} ||r(f; a, x)|| = 0,$$

where

$$r(f; a, x) = f(a+x) - f(a) - f'(a)(x) \qquad \text{for } x \in E.$$

It is well-known that, for fg which is defined by

$$(fg)(x) = f(g(x)) \qquad \text{for every } x \in E,$$

we have $fg \in \mathscr{D}$ whenever $f \in \mathscr{D}$ and $g \in \mathscr{D}$, and

$$(fg)'(a) = f'(g(a))g'(a).$$

This product defines a semigroup structure in $\mathscr{D}$. An *automorphism* ϕ of $\mathscr{D}$ is a bijection of $\mathscr{D}$ such that

$$\phi(fg) = \phi(f)\,\phi(g) \qquad \text{for every } f \in \mathscr{D} \text{ and } g \in \mathscr{D}.$$

An automorphism ϕ is said to be *inner* if there exists a bijection $h \in \mathscr{D}$ such that $h^{-1} \in \mathscr{D}$ and

$$(*) \qquad \phi(f) = hfh^{-1} \qquad \text{for every } f \in \mathscr{D}.$$

We denote the set of real numbers by $\mathscr{R}$. For $\alpha \in \mathscr{R}$, the mapping $x \to \alpha x$ of E into itself is obviously continuous and linear. We denote this mapping by α. Since $\alpha \in \mathscr{D}$, for an automorphism ϕ of $\mathscr{D}$, we can consider $\{\phi(\alpha) | \alpha \in \mathscr{R}\}$ which is a one-parameter group of diffeomorphisms (i.e. bijective and bi-differentiable mappings).

For $a = \phi(0)(0)$ and the translation $t_a : x \to x+a$, the mapping $\phi_0 : \mathscr{D} \to \mathscr{D}$ defined by

$$\phi_0(f) = t_a^{-1} \phi (f) t_a$$

is an automorphism which satisfies $\phi_0(0) = 0$.

DEFINITION. An automorphism ϕ of $\mathscr{D}$ is said to be uniform if, for any positive $\varepsilon \in \mathscr{R}$ and every $\{\alpha_n\} \subset \mathscr{R}$ such that $\alpha_n \to 0$, there exists a positive $\delta \in \mathscr{R}$ such that $||x|| < \delta$ implies

$$\sup_{u \geqq 1} ||\alpha_n^{-1} \phi_0(\alpha_n)(x) - x|| \leqq \varepsilon ||x||.$$

The main result of this paper is the following theorem.

THEOREM. *An automorphism of $\mathscr{D}$ is inner if and only if it is uniform.*

If $\phi(\alpha) \in \mathscr{L}$ for every $\alpha \in \mathscr{R}$, $\{\phi(\alpha)\}$ is a one-parameter group of topological linear isomorphisms of E into itself. The continuity with respect to the parameter (see (2) below) leads to the conclusion that $\phi(\alpha) = \alpha$ for every $\alpha \in \mathscr{R}$, from which the uniformity immediately follows and, therefore, ϕ is inner. This is the result obtained in [2].

If we take the sum $f+g$ as well as product fg into consideration, the set $\mathscr{D}$ is a near-ring. If ϕ is a near-ring automorphism, then it is easy to see that $\phi(\alpha) = \alpha$ for every $\alpha \in \mathscr{R}$, which implies that ϕ is uniform. This implies that *the near-rings $\mathscr{D}(E_1)$ and $\mathscr{D}(E_2)$ are isomorphic if and only if the Banach spaces E_1 and E_2 are diffeomorphic.* On the other hand, from our theorem it follows that *the semigroups $\mathscr{D}(E_1)$ and $\mathscr{D}(E_2)$ are isomorphic by a uniform isomorphism if and only if E_1 and E_2 are diffeomorphic.*

We believe that the answer to the following problem is affirmative.

PROBLEM. *Is every automorphism of $\mathscr{D}$ uniform?*

Therefore, in the proof of sufficiency, we shall avoid using the uniformity wherever possible, which sometimes makes the proof unnecessarily long.

Proof of the necessity

We assume that ϕ is an inner automorphism of the semigroup $\mathscr{D}$. Therefore, there exists a diffeomorphism $h : E \to E$ such that $(*)$ is true. Then, since $\phi(1) = 1$, we have $(h_0^{-1})(0) = h_0'(0)^{-1}$ and $h_0(0) = 0$ where $h_0 = t_a^{-1} h$ with $a = h(0)$.

Let ε be an arbitrary positive number. There exists $\varepsilon_1 > 0$ such that

$$||h_0'(0)||\varepsilon_1 + (||h_0'(0)^{-1}|| + \varepsilon_1)\varepsilon_1 < \varepsilon.$$

Putting $r_1(x) = r(h_0; 0, x)$ and $r_2(x) = r(h_0^{-1}; 0, x)$, we can take $\delta_1 > 0$

such that $0 < ||x|| < \delta_1$ implies $||r_i(x)|| < \varepsilon_1 ||x||$ $(i = 1, 2)$. Since$_0^{-1}$ h is continuous, there exists $\delta > 0$ such that

$$0 < \delta < \delta_1 \text{ and } ||h_0^{-1}(x)|| < \delta_1 \text{ if } ||x|| < \delta.$$

Then, for $\alpha \in \mathscr{R}$ such that $0 < |\alpha| < 1$, if $||x|| < \delta$,

$$\begin{aligned} ||\alpha^{-1} r_1(\alpha h_0^{-1}(x))|| &\leqq ||h_0^{-1}(x)|| \, (||\alpha h_0^{-1}(x)||)^{-1} ||r_1(\alpha h_0^{-1}(x))|| \\ &< (||h_0'(0)^{-1}|| \, ||x|| + ||r_2(x)||)\varepsilon_1. \end{aligned}$$

Therefore, since

$$\alpha^{-1}\phi_0(\alpha)(x) - x = h_0'(0) r_2(x) + \alpha^{-1} r_1(\alpha h_0^{-1}(x)),$$

we have, if $||x|| < \delta$,

$$\begin{aligned} ||\alpha^{-1}\phi_0(\alpha)(x) - x|| &\leqq ||h_0'(0)|| \, ||r_2(x)|| + ||\alpha^{-1} r_1(\alpha h_0^{-1}(x))|| \\ &< ||h_0'(0)|| \varepsilon_1 ||x|| + (||h_0'(0)^{-1}|| \, ||x|| + ||r_2(x)||)\varepsilon_1 \\ &\leqq \{||h_0'(0)||\varepsilon_1 + (||h_0'(0)^{-1}|| + \varepsilon_1)\varepsilon_1\} ||x|| \\ &< \varepsilon \, ||x||. \end{aligned}$$

Proof of the sufficiency

Let ϕ be an automorphism of $\mathscr{D}$. The following fact has been proved by K. D. Magill, Jr. [1].

There exists a bijection $h : E \to E$ which satisfies $(*)$.

All we know about this h at this stage is that it is a bijection (i.e., one-to-one and onto). We are going to prove that $h \in \mathscr{D}$ and $h^{-1} \in \mathscr{D}$.

Since $\phi^{-1}(f) = h^{-1} f h$ and ϕ^{-1} is also an automorphism, any statement about h can be replaced by the same statement about h^{-1}. We shall use this fact freely.

Moreover, we can assume that $h(0) = 0$, because, if $h(0) = a \neq 0$, we have only to consider the bijection $h_0 = t_a^{-1} h$, which corresponds to the automorphism ϕ_0.

For the sake of convenience, we denote the set of all sequences $\{\varepsilon_n\} \subset \mathscr{R}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ by (c_0).

(1) $\inf_{n\geqq 1} ||\varepsilon_n^{-1} h(\varepsilon_n a)|| > 0$ *for every* $a \in E$ *and any* $\{\varepsilon_n\} \in (c_0)$.

Assume that there exist $a \in E$ and $\{\varepsilon_n\} \in (c_0)$ such that

$$\lim_{n\to\infty} ||\varepsilon_n^{-1} h(\varepsilon_n a)|| = 0.$$

For any $\{\delta_n\} \in (c_0)$, taking one of its subsequences if necessary, we can assume that $\delta_n \varepsilon_n^{-1} \to 0$. Then,

$$\delta_n^{-1} h(\delta_n a) = \delta_n^{-1} h(\delta_n \varepsilon_n^{-1} \varepsilon_n a) = \delta_n^{-1} \phi(\delta_n \varepsilon_n^{-1}) h(\varepsilon_n a).$$

On the other hand, the uniformity implies that there exists $\delta > 0$ such that $||x|| < \delta$ implies

$$\sup_{n \geqq 1} ||\delta_n^{-1} \varepsilon_n \phi(\delta_n \varepsilon_n^{-1})(x)|| \leqq ||x||.$$

Since $\lim_{n\to\infty} h(\varepsilon_n a) = 0$, we get

$$\begin{aligned} ||\delta_n^{-1} h(\delta_n a)|| &= |\varepsilon_n^{-1}\, ||\delta_n^{-1} \varepsilon_n \phi(\delta_n \varepsilon_n^{-1}) h(\varepsilon_n a)|| \\ &\leqq ||\varepsilon_n^{-1} h(\varepsilon_n a)||, \end{aligned}$$

which implies

$$\lim_{n\to\infty} \delta_n^{-1} h(\delta_n a) = 0.$$

Therefore,

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} h(\varepsilon\, a) = 0,$$

which means that h is Gateaux-differentiable at 0, because, for any x, if we take $\chi \in \mathscr{L}$ such that $\chi(a) = x$,

$$\varepsilon^{-1} h(\varepsilon x) = \varepsilon^{-1} h(\varepsilon \chi(a)) = \varepsilon^{-1} \phi(\chi) h(\varepsilon a),$$

from which it follows that $\lim_{\varepsilon\to 0} \varepsilon^{-1} h(\varepsilon x) = 0$. Moreover, h is Gateaux-differentiable at every point, because, for $t_x : z \to x+z$, we have $t_x \in \mathscr{D}$ and

$$\varepsilon^{-1}[h(x+\varepsilon z) - h(x)] = \varepsilon^{-1}[\phi(t_x) h(\varepsilon z) - \phi(t_x) h(0)],$$

from which it follows that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[h(x+\varepsilon z) - h(x)] = \phi(t_x)'(0) \big(\lim_{\varepsilon \to 0} \varepsilon^{-1} h(\varepsilon z)\big).$$

If we denote the Gateaux-derivative of h at x by $h^*(x)$, then we have $h^*(x) = 0$ for every $x \in E$. The mean value theorem then implies that $h = 0$, which is a contradiction.

For the conjugate space $\bar{E}$, the value of $\bar{a} \in \bar{E}$ at $x \in E$ is denoted by $\langle x, \bar{a} \rangle$.

(2) *For any $\bar{a} \in \bar{E}$, $\langle h(x), \bar{a} \rangle$ is continuous with respect to x.*

To prove the continuity at $a \in E$, we use the method used by K. D. Magill, Jr. [1]. We take positive $\varepsilon \in \mathscr{R}$ and non-zero $b \in E$ and consider the mapping $g \in \mathscr{D}$ such that

$$g(x) = \beta(\langle x - h(a), \bar{a} \rangle) b + h(a),$$

where $\beta : \mathscr{R} \to \mathscr{R}$ is a differentiable function such that

$$\beta(\xi) = 0 \text{ if } |\xi| \geqq \varepsilon; = 1 \text{ if } \xi = 0.$$

We take $f \in \mathscr{D}$ such that $\phi(f) = g$. Then, $f(a) \neq a$, because, if $f(a) = a$, we have

$$h(a) = hf(a) = \phi(f) h(a) = gh(a) = b + h(a),$$

which is a contradiction. Since f is continuous, there exists $\delta > 0$ such that $||x-a|| < \delta$ implies $f(x) \neq a$. Therefore, if $||x-a|| < \delta$, we have $gh(x) = hf(x) \neq h(a)$, which means that $\beta(\langle h(x)-h(a), \bar{a}\rangle) \neq 0$. By the definition of β, we have $\langle h(x)-h(a), \bar{a}\rangle < \varepsilon$.

(3) $\sup_{n\geqq 1} ||\varepsilon_n^{-1}h(\varepsilon_n a)|| < \infty$ *for any* $a \in E$ *and any* $\{\varepsilon_n\} \in (c_0)$. *As a special case,* $\lim_{n\to\infty} h(\varepsilon_n a) = 0$.

Let us suppose that there exist $a \in E$ and $\{\varepsilon_n\} \in (c_0)$ such that

$$\lim_{n\to\infty} ||\varepsilon_n^{-1}h^{-1}(\varepsilon_n a)|| = \infty.$$

Then, for some $\bar{a} \in \bar{E}$, we have

$$\lim_{n\to\infty} \langle \varepsilon_n^{-1}h^{-1}(\varepsilon_n a), \bar{a}\rangle = \infty.$$

For these $a \in E$ and $\bar{a} \in \bar{E}$, we consider the mapping $a \otimes \bar{a} \in \mathscr{L}$ that is defined by

$$a \otimes \bar{a}(x) = \langle x, \bar{a}\rangle\, a.$$

Then,

$$\begin{aligned}\phi(a \otimes \bar{a})'(0)(a) &= \lim_{n\to\infty} \varepsilon_n^{-1}\phi(a \otimes \bar{a})(\varepsilon_n a)\\ &= \lim_{n\to\infty} \varepsilon_n^{-1}h[\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle a]\\ &= \lim_{n\to\infty} \left(\varepsilon_n^{-1}\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle\right)\left(\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle\right)^{-1}\\ &\qquad \times h[\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle a],\end{aligned}$$

from which it follows that

$$\lim_{n\to\infty} \left(\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle\right)^{-1}h\,[\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle a] = 0,$$

which contradicts the facts proved in (1) and (2).

(4) *For any* $a \in E$ *and any* $\{\varepsilon_n\} \in (c_0)$, *there exists a subsequence* $\{\varepsilon_{n_k}\}$ *such that*

$$\{\varepsilon_{n_k}^{-1}h(\varepsilon_{n_k}a)\}$$

is convergent.

Since a can be supposed to be non-zero, we can take $\bar{a} \in \bar{E}$ such that $\langle a, \bar{a}\rangle \neq 0$ and $\phi(a \otimes \bar{a})'(0)(a) \neq 0$. For this $a \otimes \bar{a}$, we take $\{\delta_n\} \in (c_0)$ such that

$$\langle h^{-1}(\delta_n a), \bar{a}\rangle = \varepsilon_n,$$

which is possible because of (2). Since the sequence of real numbers

$$\{\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{a}\rangle\}$$

is bounded, it contains a convergent subsequence

$$\{\delta_{n_k}^{-1}\langle h^{-1}(\delta_{n_k}a), \bar{a}\rangle\}.$$

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Then,

$$0 \neq \phi(a \otimes \bar{a})'(0)(a) = \lim_{k\to\infty} \delta_{n_k}^{-1}\phi(a \otimes \bar{a})(\delta_{n_k}a)$$
$$= \lim_{k\to\infty} \delta_{n_k}^{-1}\langle h^{-1}(\delta_{n_k}a),\ \bar{a}\rangle \varepsilon_{n_k}^{-1}h(\varepsilon_{n_k}a),$$

which implies that

$$\lim_{k\to\infty} \delta_{n_k}^{-1}\langle h(\delta_{n_k}a),\ \bar{a}\rangle \neq 0.$$

Therefore, we have the limit

$$\lim_{k\to\infty} \varepsilon_{n_k}^{-1}h(\varepsilon_{n_k}a) = \left(\lim_{k\to\infty} \delta_{n_k}^{-1}\langle h^{-1}(\delta_{n_k}a),\ \bar{a}\rangle\right)^{-1}\phi(a \otimes \bar{a})'(0)(a).$$

(5) *The limit* $\lim_{\varepsilon\to 0}\varepsilon^{-1}h(\varepsilon a)$ *exists.*

We have only to show that, if the limits

$$\lim_{n\to\infty} \varepsilon_n^{-1}h(\varepsilon_n a) = a_1 \text{ and } \lim_{n\to\infty} \delta_n^{-1}h(\delta_n a) = a_2$$

exist for $\{\varepsilon_n\} \in (c_0)$ and $\{\delta_n\} \in (c_0)$, then we have $a_1 = a_2$.

We can assume, taking a subsequence of $\{\delta_n\}$ if necessary, that

$$\lim_{n\to\infty} \delta_n\varepsilon_n^{-1} = 0.$$

Then,

$$\delta_n^{-1}h(\delta_n a) = \delta_n^{-1}h(\delta_n\varepsilon_n^{-1}\varepsilon_n a) = \delta_n^{-1}\phi(\delta_n\varepsilon_n^{-1})h(\varepsilon_n a)$$
$$= \varepsilon_n^{-1}[\delta_n^{-1}\varepsilon_n\phi(\delta_n\varepsilon_n^{-1})h(\varepsilon_n a) - h(\varepsilon_n a)] + \varepsilon_n^{-1}h(\varepsilon_n a).$$

The uniformity then implies that

$$||a_2 - a_1|| = \lim_{n\to\infty} ||\delta_n^{-1}h(\delta_n a) - \varepsilon_n^{-1}h(\varepsilon_n a)||$$
$$= \lim_{n\to\infty} ||\varepsilon_n^{-1}[\delta_n^{-1}\varepsilon_n\phi(\delta_n\varepsilon_n^{-1})h(\varepsilon_n a) - h(\varepsilon_n a)]|| = 0.$$

We denote the limit $\lim_{\varepsilon\to 0}\varepsilon^{-1}h(\varepsilon a)$ *by* $h^*(0)(a)$.

(6) *h is differentiable at every point in all directions.*

Let a be an arbitrary point and consider the mapping $t_a : x \to x+a$. Then, $t_a \in \mathscr{D}$ and

$$\varepsilon^{-1}[h(a+\varepsilon x) - h(a)] = \varepsilon^{-1}[\phi(t_a)h(\varepsilon x) - \phi(t_a)h(0)]$$
$$= \varepsilon^{-1}[\phi(t_a)'(0)h(\varepsilon x) + r(\phi(t_a);\ 0,\ h(\varepsilon x))].$$

Therefore,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[h(a+\varepsilon x) - h(a)] = \phi(t_a)'(0)h^*(0)(x).$$

We denote this limit by $h^*(a)(x)$. Obviously,

$$h^*(a)(\alpha x) = \alpha h^*(a)(x).$$

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(7) *For any* $a \otimes \bar{a}$, $h(a \otimes \bar{a}) \in \mathscr{D}$ *and*

$$(h(a \otimes \bar{a}))'(x)(y) = \langle y, \bar{a} \rangle h^*(\langle x, \bar{a} \rangle\, a)(a).$$

Since

$$\varepsilon^{-1}[h(a \otimes \bar{a})(x+\varepsilon y) - h(a \otimes \bar{a})(x)] = \varepsilon^{-1}[h(\langle x, \bar{a}\rangle a + \varepsilon \langle y, \bar{a}\rangle a) - h(\langle x, \bar{a}\rangle a)],$$

it follows from (6) that the limit as $\varepsilon \to 0$ exists and it is

$$\langle y, \bar{a}\rangle h^*(\langle x, \bar{a}\rangle a)(a),$$

which is obviously continuous and linear with respect to y. Moreover,

$$\begin{aligned}\overline{\lim_{||y|| \to 0}}\, ||y||^{-1} ||h(a \otimes \bar{a})(x+y) - h(a \otimes \bar{a})(x) - (h(a \otimes \bar{a})) * (x)(y)|| \\ \leqq ||\bar{a}|| \overline{\lim_{||y|| \to 0}}\, ||(\langle y, \bar{a}\rangle)^{-1}[h(\langle x, \bar{a}\rangle a + \langle y, \bar{a}\rangle a) - h(\langle x, \bar{a}\rangle a] \\ - h^*(\langle x, \bar{a}\rangle a)(a)|| \\ = 0,\end{aligned}$$

which means that $h(a \otimes \bar{a}) \in \mathscr{D}$.

(8) *For any* $a \otimes \bar{a}$, $(a \otimes \bar{a})h \in \mathscr{D}$ *and*

$$((a \otimes \bar{a})h)'(x)(y) = \langle h^*(x)(y), \bar{a}\rangle a.$$

By (7), we have

$$(a \otimes \bar{a})h = \phi^{-1}(h(a \otimes \bar{a})) \in \mathscr{D}.$$

The formula for $((a \otimes \bar{a})h)'(x)(y)$ is obvious.

(9) $h^*(a) \in \mathscr{L}$ *for every* $a \in E$.

The linearity follows immediately from (8). To prove the continuity, let us take an arbitrary non-zero $b \otimes \bar{b}$. Then

$$\begin{aligned}|\langle h^*(a)(x), \bar{b}\rangle| &= ||b||^{-1}\, ||((b \otimes \bar{b})h)'(a)(x)|| \\ &\leqq ||b||^{-1}\, ||((b \otimes \bar{b})h)'(a)||\; ||x||,\end{aligned}$$

which means the set

$$\{h^*(a)(x) |\; ||x|| \leqq 1\}$$

is weakly bounded. Therefore, $h^*(a)$ is continuous.

We define $r_1(x)$ *and* $r_2(x)$ *by*

$$h(x) - h^*(0)(x) = r_1(x) \text{ and } h^{-1}(x) - (h^{-1})*(0)(x) = r_2(x).$$

(10) *For any sequence* $\{x_n\}$ *such that* $\lim_{n\to\infty} x_n = 0$, *the sequence* $\{||x_n||^{-1} r_i(x_n)\}$ *converges weakly to* 0 *for* $i = 1, 2$. *Therefore, the sequence* $\{||x_n||^{-1}\, h(x_n)\}$ *is bounded, which implies that* $\lim_{n\to\infty} h(x_n) = 0$.

From (8) it follows that $(a \otimes \bar{a})r_1(x)$ is the remainder of $(a \otimes \bar{a})h$ at 0. Therefore,

$$\lim_{n\to\infty} ||x_n||^{-1} (a \otimes \bar{a})r_1(x_n) = 0,$$

which implies that

$$\lim_{n\to\infty} \langle ||x_n||^{-1} r_1(x_n), \bar{a}\rangle = 0$$

for every $\bar{a} \in \bar{E}$.

(11) $\lim_{||x||\to 0} ||x||^{-1} r_i(x) = 0$ $(i = 1, 2)$. *Therefore*, $h \in \mathscr{D}$ and $h^{-1} \in \mathscr{D}$. Assume that there exists a sequence $\{x_n\} \subset E$ such that

$$\lim_{n\to\infty} x_n = 0 \text{ and } ||x_n||^{-1} ||r_1(x_n)|| \geqq \gamma > 0 \qquad (n = 1, 2, \cdots)$$

for some positive $\gamma \in \mathscr{R}$. By (5), we can take $\{\varepsilon_n\} \in (c_0)$ such that

$$||\varepsilon_n^{-1} r_1(\varepsilon_n x_n)|| \leqq ||x_n||^2 \qquad (n = 1, 2, \cdots).$$

Then, for large n, we have

$$\begin{aligned} ||\varepsilon_n^{-1}\phi(\varepsilon_n)h(x_n)-h(x_n)|| &= ||\varepsilon_n^{-1}h(\varepsilon_n x_n)-h(x_n)|| \\ &= ||\varepsilon_n^{-1}r_1(\varepsilon_n x_n)-r_1(x_n)|| \geqq ||r_1(x_n)||-||\varepsilon_n^{-1}r_1(\varepsilon_n x_n)|| \\ &\geqq (\gamma-||x_n||)||x_n|| \geqq (\gamma-||x_n||)(\inf_{n\geqq 1} ||x_n|| \, ||h(x_n)||^{-1})||h(x_n)||. \end{aligned}$$

Since, by (10), $\inf_{n\geqq 1} ||x_n|| \, ||h(x_n)||^{-1} > 0$ which implies that $\lim_{n\to\infty} h(x_n) = 0$, this contradicts the uniformity.

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